Homework 10

Math 461: Probability Theory, Spring 2022
Daesung Kim

Due date: Apr 22, 2022

Instruction

1. Each problem is worth 10 points and only five randomly chosen problems will be graded.
2. Convert a photocopy of your solutions to **one single pdf file** and upload it on Moodle.
3. Please indicate whom you worked with, it will not affect your grade in any way.

1. If X and Y are independent and identically distributed with mean μ and variance σ^2, find $\mathbb{E}[(X-Y)^2]$.
2. If $\mathbb{E}[X] = 1$ and $\mathrm{Var}(X) = 4$, find (a) $\mathbb{E}[(2+X)^2]$ and (b) $\mathrm{Var}(4+2X)$.
3. The random variables X and Y have a joint density function given by
$$f(x,y) = \begin{cases} 2e^{-2x}/x & \text{if } 0 < x < \infty, 0 < y < x \\ 0 & \text{otherwise.} \end{cases}$$
Compute $\mathrm{Cov}(X,Y)$.
4. A total of n balls, numbered 1 through n, are put into n urns, also numbered 1 through n in such a way that ball i is equally likely to go into any of the urns $1, 2, \ldots, i$. Find
(a) the expected number of urns that are empty;
(b) the probability that none of the urns is empty.
5. Consider n independent flips of a coin having probability p of landing on heads. Say that a changeover occurs whenever an outcome differs from the one preceding it. For instance, if $n = 5$ and the outcome is $HHTHT$, then there are 3 changeovers. Find the expected number of changeovers.
6. A group of 20 people consisting of 10 married couples is randomly arranged into 10 pairs of 2 each. Compute the mean and variance of the number of married couples that are paired together.
7. Let $X_1, X_2, \ldots$ be independent random variables with common mean μ and common variance σ^2. Set $Y_n = X_n + X_{n+1} + X_{n+2}$, $n \geqslant 1$. For $j \geqslant 0$, find $\mathrm{Cov}(Y_n, Y_{n+j})$.
8. The joint density of X and Y is given by
$$f(x,y) = \begin{cases} e^{-x/y-y}/y & \text{if } 0 < x < \infty, 0 < y < \infty \\ 0 & \text{otherwise.} \end{cases}$$
Compute $\mathbb{E}(X^2|Y=y)$.
9. The joint density of X and Y is given by
$$f(x,y) = \begin{cases} e^{-y}/y & \text{if } 0 < x < y, 0 < y < \infty \\ 0 & \text{otherwise.} \end{cases}$$
Compute $\mathbb{E}(X^3|Y=y)$.

10. The number of people who enter an elevator on the ground floor is a Geometric random variable with mean 10. If there are N floors above the ground floor, and if each person is equally likely to get off at any one of the N floors, independently of where the others get off, compute the expected number of stops that the elevator will make before discharging all of its passengers.