Homework 8

Math 461: Probability Theory, Spring 2022
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Due date: Apr 1, 2022

Instruction

1. Each problem is worth 10 points and only five randomly chosen problems will be graded.
2. Convert a photocopy of your solutions to **one single pdf file** and upload it on Moodle.
3. Please indicate whom you worked with, it will not affect your grade in any way.

1. (a) Let X be the Gamma random variable with $\lambda > 0$ and $\alpha > 0$. Show that $\mathrm{Var}(X) = \frac{\alpha}{\lambda^2}$.
   (b) Let Y be the exponential random variable with parameter $\lambda > 0$. Show that
$$\mathbb{E}\, Y^k = \frac{k!}{\lambda^k} \qquad k = 1, 2, \cdots .$$

2. Suppose that 3 balls are chosen without replacement from an urn consisting of 5 white and 8 red balls. Let X_i equal 1 if the i-th ball selected is white, and let it equal 0 otherwise. Give the joint probability mass function of
   (a) X_1, X_2;
   (b) X_1, X_2, X_3.

3. Consider a sequence of independent Bernoulli trials, each of which is a success with probability p. Let X_1 be the number of failures preceding the first success, and let X_2 be the number of failures between the first two successes. Find the joint mass function of X_1 and X_2.

4. The joint probability density function of X and Y is given by
$$f(x, y) = \frac{6}{7}\left(x^2 + \frac{xy}{2}\right), 0 < x < 1, 0 < y < 2$$
   and 0 otherwise.
   (a) Verify that this is indeed a joint density function.
   (b) Compute the density function of X.
   (c) Find $\mathbb{P}(X > Y)$.
   (d) Find $\mathbb{P}(Y > 1 \mid X < 1/2)$.
   (e) Find $\mathbb{E}\, X$.
   (f) Find $\mathbb{E}\, Y$.

5. Let X, Y be jointly distributed with density function $f(x, y) = e^{-(x+y)}$ for $0 \leqslant x < \infty$, $0 \leqslant y < \infty$. Find (a) $\mathbb{P}(X < Y)$ and (b) $\mathbb{P}(X < a)$ for $a \in \mathbb{R}$.

6. A man and a woman agree to meet at a certain location about 12:30PM. If the man arrives at a time uniformly distributed between 12:15 and 12:45, and if the woman independently arrives at a time uniformly distributed between 12:00 and 1PM, find the probability that the first to arrive waits no longer than 5 minutes. What is the probability that the man arrives first?

7. The joint density of X and Y is given by

$$f(x,y) = \begin{cases} xe^{-(x+y)}, & x > 0, y > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Determine whether X and Y are independent.

8. The joint density function of X and Y is $f(x,y) = \begin{cases} x+y & \text{if } 0 < x < 1, 0 < y < 1, \\ 0 & \text{otherwise.} \end{cases}$

   (a) Are X and Y independent?
   (b) Find the density function of X.
   (c) Find $\mathbb{P}(X + Y < 1)$.
   (d) Find $\mathbb{E}\, X$.
   (e) Find $\mathrm{Var}(X)$.

9. Let X and Y be jointly distributed with density function

$$f(x,y) = \begin{cases} 12xy(1-x), & 0 < x < 1, 0 < y < 1, \\ 0, & \text{otherwise.} \end{cases}$$

   (a) Are X and Y independent?
   (b) Find $\mathbb{E}\, X$.
   (c) Find $\mathbb{E}\, Y$.
   (d) Find $\mathrm{Var}(X)$.
   (e) Find $\mathrm{Var}(Y)$.

10. If X_1 and X_2 are independent exponential random variables with respective parameters λ_1 and λ_2, find the distribution of $Z = X_1/X_2$. Also compute $\mathbb{P}(X_1 < X_2)$.