Math 3215: Intro to Probability and Statistics

Exam 2 Solution, Summer 2023

1. Let X and Y be two discrete random variables with joint pmf

$$f_{X,Y}(1,1) = f_{X,Y}(2,1) = \frac{1}{8}, \quad f_{X,Y}(1,2) = \frac{1}{4}, \quad f_{X,Y}(2,2) = \frac{1}{2}.$$

   (a) (5 points) Find $\mathbb{E}[XY]$.

   (b) (5 points) Find the conditional expectation of X given $Y = 1$.

   (c) (5 points) Find the conditional expectation $\mathbb{E}[X|Y]$.

Solution:

(a) $\mathbb{E}[XY] = 1/8 + 2/8 + 2/4 + 2 = 23/8$.

(b) Since $f_{X|Y}(x|1) = 1/2$ for $x = 1, 2$, $\mathbb{E}[X|Y = 1] = 3/2$.

(c) Similarly, $\mathbb{E}[X|Y = 2] = 1/3 + 4/3 = 5/3$. Thus, $\mathbb{E}[X|Y] = 3/2$ with probability $\mathbb{P}(Y = 1) = 1/4$ and $\mathbb{E}[X|Y] = 5/3$ with probability $\mathbb{P}(Y = 2) = 3/4$.

2. Let X and Y be continuous random variables with joint probabilitydensity function

$$f(x,y) = \frac{x}{5} + cy$$

for $0 < x < 1$ and $1 < y < 5$, and otherwise 0.

   (a) (5 points) Find the constant c.

   (b) (5 points) Find the marginal pdfs of X and Y.

   (c) (5 points) Are they independent?

Solution:

(a) Since $\int_0^1 \int_1^5 (\frac{x}{5} + cy)\, dydx = \frac{4}{10} + 12c = 1$, $c = \frac{1}{20}$.

(b) $f_X(x) = \int_1^5 (\frac{x}{5} + \frac{y}{20})\, dy = \frac{4x+3}{5}$ and $f_Y(y) = \int_0^1 (\frac{x}{5} + \frac{y}{20})\, dx = \frac{y+2}{20}$.

(c) They are dependent.

3. Let X be a random variable with cdf given by

$$F_X(x) = 1 - e^{-e^x}$$

for $-\infty < x < \infty$. Let $Y = e^X$.

   (a) (7 points) Find the pdf of X.

   (b) (8 points) Find the cdf and pdf of Y.

Solution:

(a) $f_X(x) = e^{x-e^x}$

(b) For $t > 0$, $F_Y(t) = \mathbb{P}(Y \le t) = \mathbb{P}(e^X \le t) = \mathbb{P}(X \le \log t) = 1 - e^{-e^{\log t}} = 1 - e^{-t}$ and $f_Y(t) = e^{-t}$. Since $Y > 0$, $F_Y(t) = 0 = f_Y(t)$ for $t \le 0$.

4. Let X be a uniform random variable on $(-1, 1)$ and $Y = |X|$.
   (a) (5 points) Find the pdf of Y.
   (b) (5 points) Compute $\mathrm{Cov}(X, Y)$.
   (c) (5 points) Compute $\mathbb{P}(X \le \frac{1}{2})$, $\mathbb{P}(Y \le \frac{1}{2})$, and $\mathbb{P}(X \le \frac{1}{2}, Y \le \frac{1}{2})$.

Solution:

(a) If $a \ge 1$, then $F_Y(a) = \mathbb{P}(Y \le a) = 1$. If $a \le 0$, then $F_Y(a) = \mathbb{P}(Y \le a) = 0$. For $a \in (0, 1)$,

$$F_Y(a) = \mathbb{P}(Y \le a) = \mathbb{P}(-a \le X \le a) = a.$$

By differentiating F_Y with respect to a, we get

$$f_Y(a) = \begin{cases} 1, & a \in (0,1), \\ 0, & \text{otherwise.} \end{cases}$$

(b) Since $\mathbb{E}[XY] = \mathbb{E}[X|X|] = 0$, $\mathbb{E}[X] = 0$, and $\mathbb{E}[Y] = \frac{1}{2}$, we have

$$\mathrm{Cov}(X, Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] = 0.$$

(c) $\mathbb{P}(X \le \frac{1}{2}, Y \le \frac{1}{2}) = \mathbb{P}(-\frac{1}{2} \le X \le \frac{1}{2}) = \frac{1}{2}$, $\mathbb{P}(X \le \frac{1}{2}) = \frac{3}{4}$, and $\mathbb{P}(Y \le \frac{1}{2})\mathbb{P}(-\frac{1}{2} \le X \le \frac{1}{2}) = \frac{1}{2}$.

5. Let (X, Y) have a bivariate normal distribution with common mean 24, common standard deviation $2\sqrt{3}$, and correlation coefficient 0.5. That is, $\mu_X = \mu_Y = 24$, $\sigma_X = \sigma_Y = 2\sqrt{3}$, and $\rho = 0.5$.
   (a) (7 points) Find the expectation of the variance of $X + Y$.
   (b) (8 points) Using the tables, find the conditional probability $\mathbb{P}(X \ge 24.75 | Y = 12)$.

Solution:

(a) $\mathbb{E}[X+Y] = \mathbb{E}[X] + \mathbb{E}[Y] = 24 + 24 = 48$ and $\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y) + 2\,\mathrm{Cov}(X,Y) = 12 + 12 + 2 \cdot 0.5 \cdot 12 = 36$.

(b) Since $X|Y = 12 \sim N(\mu_X + \rho \frac{\sigma_X}{\sigma_Y}(y - \mu_Y), (1-\rho^2)\sigma_X^2) = N(18, 9)$, $\mathbb{P}(X \ge 24.75 | Y = 12) = \mathbb{P}(Z \ge 2.25) = 1 - \mathbb{P}(Z < 2.25) = 1 - 0.9878 = 0.0122$.

6. Let X be a normal random variable with mean 5 and variance 4, that is, $X \sim N(5, 4)$
   (a) (5 points) Find $\mathbb{E}[(3X-2)^2]$.

(b) (5 points) Using the tables, find $\mathbb{P}(X \le 3.5)$.

Solution:

(a) $\mathbb{E}[(3X-2)^2] = 9\mathbb{E}[X^2] - 12\mathbb{E}[X] + 4 = 9(4+25) - 12 \cdot 5 + 4 = 205$.

(b) Since $(X-5)/2 \sim N(0,1)$, $\mathbb{P}(X \le 3.5) = \mathbb{P}(Z \le -1.5/2) = \mathbb{P}(Z \le -0.75) = \mathbb{P}(Z \ge 0.75) \approx 1 - 0.7734 = 0.2266$.

7. Let X, Y be independent exponential random variables with parameters $\lambda_X = 1$ and $\lambda_Y = 2$, that is, their marginal pdfs are $f_X(t) = e^{-t}$ and $f_Y(t) = 2e^{-2t}$ for $t \ge 0$.

(a) (7 points) Let $Z = \max\{X, Y\}$. Find $\mathbb{P}(Z \le 6)$.

(b) (8 points) Let $W = \min\{X, Y\}$. Find the cdf and the pdf of W.

Solution:

(a) $\mathbb{P}(Z \le 6) = \mathbb{P}(X \le 6, Y \le 6) = (1 - \mathbb{P}(X > 6))(1 - \mathbb{P}(Y > 6)) = (1 - e^{-6})(1 - e^{-12})$

(b) $F_W(t) = \mathbb{P}(W \le t) = 1 - \mathbb{P}(W > t) = 1 - \mathbb{P}(X > t, Y > t) = 1 - e^{-t}e^{-2t} = 1 - e^{-3t}$ and $f_W(t) = 3e^{-3t}$ for $t \ge 0$.