MATH 461 LECTURE NOTE
WEEK 11

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1. Independent Random Variables (Sec 6.2)

Definition
Two random variables X and Y are independent if for any sets A and B

$$\mathbb{P}(X \in A, Y \in B) = \mathbb{P}(X \in A)\mathbb{P}(Y \in B).$$

It is equivalent to the following:

(i) $F(a, b) = F_X(a)F_Y(b)$ for all $a, b \in \mathbb{R}$;
(ii) (Discrete case) $p(x, y) = p_X(x)p_Y(y)$ for all $x, y \in \mathbb{R}$;
(iii) (Discrete case) $p(x, y) = h(x)g(y)$ for all $x, y \in \mathbb{R}$, for some h and g;
(iv) (Jointly continuous case) $f(x, y) = f_X(x)f_Y(y)$ for all $x, y \in \mathbb{R}$;
(v) (Jointly continuous case) $f(x, y) = h(x)g(y)$ for all $x, y \in \mathbb{R}$, for some h and g.

Otherwise, we say that X and Y are dependent.

Remark 1. Let E, F be events on a sample space S. Recall that E, F are independent if $\mathbb{P}(E \cup F) = \mathbb{P}(E)\mathbb{P}(F)$. Define

$$X = I_E = \begin{cases} 1, & E \text{ occurs,} \\ 0, & \text{otherwise,} \end{cases} \qquad Y = I_F = \begin{cases} 1, & F \text{ occurs,} \\ 0, & \text{otherwise.} \end{cases}$$

Then, X and Y are independent if and only if E and F are independent.

Example 2. If the joint density function of X and Y is

$$f(x, y) = \begin{cases} 6e^{-2x}e^{-3y}, & 0 < x < \infty, 0 < y < \infty \\ 0, & \text{otherwise,} \end{cases}$$

are the random variables independent? Find the marginal densities f_X and f_Y.

Example 3. If the joint density function of X and Y is

$$f(x, y) = \begin{cases} 24xy, & 0 < x < 1, 0 < y < 1, 0 < x + y < 1 \\ 0, & \text{otherwise,} \end{cases}$$

are the random variables independent? Find the marginal densities f_X and f_Y.

Example 4. Two points are selected randomly on a line of length L so as to be on opposite sides of the midpoint of the line. In other words, the two points X and Y are independent random variables such that X is uniformly distributed over $(0, L/2)$ and Y is uniformly distributed over $(L/2, L)$. Find the probability that the distance between the two points is greater than $L/3$.

Definition
Random variables $X_1, X_2, \cdots, X_n$ are independent if, for any sets $A_1, A_2, \cdots, A_n$

$$\mathbb{P}(X_1 \in A_1, X_2 \in A_2, \cdots, X_n \in A_n) = \mathbb{P}(X_1 \in A_1)\mathbb{P}(X_2 \in A_2) \cdots \mathbb{P}(X_n \in A_n).$$

If the random variables are jointly continuous, it is equivalent to

$$f(x_1, x_2, \cdots, x_n) = f_{X_1}(x_1)f_{X_2}(x_2) \cdots f_{X_n}(x_n)$$

for all $x_1, x_2, \cdots, x_n$.

Example 5. If X_1, X_2, X_3 are independent random variables that are uniformly distributed over $(0,1)$, compute the probability that the largest of the three is greater than the sum of the other two.

Remark 6. Let X_1, X_2, X_3, X_4 be independent uniform random variables on $[0,1]$. Define $X^{(i)}$ be the i-th smallest random variable between X_1, X_2, X_3, X_4 for $i = 1,2,3,4$. Let $Y = X^{(2)}$ and $Z = 1 - X^{(3)}$, then one can see that the joint density of Y and Z is

$$f(y,z) = \begin{cases} 24yz, & 0 < y < 1, 0 < z < 1, 0 < y + z < 1, \\ 0, & \text{otherwise.} \end{cases}$$

Further examples.

Example 7. Suppose that the number of people who enter a post office on a given day is a Poisson random variable with parameter λ. Each person who enters the post office is a male with probability p and a female with probability $1-p$. Show that the number of males and females entering the post office are independent Poisson random variables with respective parameters λp and $\lambda(1-p)$.

Example 8. Buffon's needle problem A table is ruled with equidistant parallel lines a distance D apart. A needle of length L, where $L < D$, is randomly thrown on the table. What is the probability that the needle will intersect one of the lines (the other possibility being that the needle will be completely contained in the strip between two lines)? Let us determine the position of the needle by specifying

(i) the distance X from the middle point of the needle to the nearest parallel line and
(ii) the angle θ between the needle and the projected line of length X.

2. Sums of Independent Random Variables (Sec 6.3)

In this section, we consider the sum of two independent random variables X and Y. If X and Y are jointly continuous and independent, then the joint density is $f(x,y) = f_X(x)f_Y(y)$ where f_X and f_Y are the densities for X and Y respectively. Then, the cdf of $X+Y$ is

$$F_{X+Y}(t) = \mathbb{P}(X+Y \le t) = \iint_{x+y\le t} f_X(x)f_Y(y)\,dxdy = \int_{\mathbb{R}} F_X(t-y)f_Y(y)\,dy.$$

The cdf of $X+Y$ is called the convolution of F_X and F_Y. Taking derivative with respect to t, we get

$$f_{X+Y}(t) = \int_{\mathbb{R}} f_X(t-y)f_Y(y)\,dy.$$

Example 9. If X and Y are independent uniform random variables on $(0,1)$, find the density of $X+Y$.

Sum of independent random variables
Suppose X and Y are independent. Let $Z = X+Y$.

(i) If $X \sim N(\mu_1, \sigma_1^2)$ and $Y \sim N(\mu_2, \sigma_2^2)$, then $Z \sim N(\mu_1+\mu_2, \sigma_1^2+\sigma_2^2)$.
(ii) If $X \sim \Gamma(s,\lambda)$ and $Y \sim \Gamma(t,\lambda)$, then $Z \sim \Gamma(s+t,\lambda)$.
(iii) If $X \sim \text{Bin}(n,p)$ and $Y \sim \text{Bin}(m,p)$, then $Z \sim \text{Bin}(n+m,p)$.
(iv) If $X \sim \text{Poisson}(\lambda_1)$ and $Y \sim \text{Poisson}(\lambda_2)$, then $Z \sim \text{Poisson}(\lambda_1+\lambda_2)$.
(v) If $X \sim \text{NegBin}(r,p)$ and $Y \sim \text{NegBin}(s,p)$, then $Z \sim \text{NegBin}(r+s,p)$.

Example 10. If $X \sim N(0,\frac{1}{2})$ and $Y \sim N(0,\frac{1}{2})$ are independent, then what is $f_{X+Y}(t)$?

Example 11. If $X \sim \text{Poisson}(\lambda_1)$ and $Y \sim \text{Poisson}(\lambda_2)$, then what is $\mathbb{P}(X+Y=n)$?

Further examples.

Example 12. The gross weekly sales at a certain restaurant is a normal random variable with mean \$2200 and standard deviation \$230. What is the probability that the total gross sales over the next 2 weeks exceeds \$5000?

Example 13. Let $X \sim U(0,1)$ and $Y \sim \text{Exp}(1)$ be independent. Find the distribution of $Z = X+Y$.

3. Conditional Distribution (Sec 6.4-6)

Suppose X and Y are discrete with the joint pmf $p(x,y)$, that is $\mathbb{P}(X=x,Y=y)=p(x,y)$. Let y satisfy $p_Y(y)=\sum_x p(x,y)>0$. The conditional pmf of X given $Y=y$ is defined by

$$p_{X|Y}(x|y)=\mathbb{P}(X=x|Y=y)=\frac{\mathbb{P}(X=x,Y=y)}{\mathbb{P}(Y=y)}=\frac{p(x,y)}{p_Y(y)}.$$

Note that if X and Y are independent, then $p_{X|Y}(x|y)=p_X(x)$. The conditional cdf of X given $Y=y$ is

$$F_{X|Y}(t|y)=\mathbb{P}(X\le t|Y=y)=\sum_{x\le t}p_{X|Y}(x|y).$$

Example 14. If X and Y are independent Poisson random variables with respective parameters λ_1 and λ_2, calculate the conditional distribution of X given that $X+Y=n$.

Suppose X and Y are jointly continuous with joint density $f(x,y)$. For y with $f_Y(y)>0$, the conditional density of X given $Y=y$ is defined by

$$f_{X|Y}(x|y)=\frac{f(x,y)}{f_Y(y)}.$$

If X and Y are independent, then $f_{X|Y}(x|y)=f_X(x)$. Then, the conditional probability and the conditional cdf of X given $Y=y$ can be written as

$$\mathbb{P}(X\in A|Y=y)=\int_A f_{X|Y}(x|y)\,dx$$
$$F_{X|Y}(t|y)=\mathbb{P}(X\le t|Y=y)=\int_{-\infty}^{t} f_{X|Y}(x|y)\,dx.$$

Example 15. Suppose that the joint density of X and Y is given by

$$f(x,y)=\begin{cases}\frac{e^{-\frac{x}{y}}e^{-y}}{y}, & 0<x,y<\infty,\\ 0, & \text{otherwise.}\end{cases}$$

Find $f_{X|Y}(x|y)$ and $\mathbb{P}(X>1|Y=y)$.

Bivariate normal random variable. Jointly continuous random variables X and Y are bivariate normal if their density is given by

$$f(x,y)=\frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}}e^{-\frac{1}{2(1-\rho^2)}\left((\frac{x-\mu_X}{\sigma_X})^2+(\frac{y-\mu_Y}{\sigma_Y})^2-2\rho\frac{(x-\mu_X)(y-\mu_Y)}{\sigma_X\sigma_Y}\right)}$$

where $\sigma_X,\sigma_Y>0$, $\rho\in(-1,1)$, and $\mu_X,\mu_Y\in\mathbb{R}$. We denote by

$$\begin{pmatrix}X\\Y\end{pmatrix}\sim N\left(\begin{pmatrix}\mu_X\\ \mu_Y\end{pmatrix},\begin{pmatrix}\sigma_X^2 & \rho\sigma_X\sigma_Y\\ \rho\sigma_X\sigma_Y & \sigma_Y^2\end{pmatrix}\right).$$

Proposition 16. (i) *$X\sim N(\mu_X,\sigma_X^2)$ and $Y\sim N(\mu_Y,\sigma_Y^2)$. In particular, $\mathbb{E}[X]=\mu_X$, $\mathbb{E}[Y]=\mu_Y$, $\mathrm{Var}(X)=\sigma_X^2$, and $\mathrm{Var}(Y)=\sigma_Y^2$.*
(ii) *The random variable X given $Y=y$ is normal with mean $\mu_X+\rho\frac{\sigma_X}{\sigma_Y}(y-\mu_Y)$ and variance $\sigma_X^2(1-\rho^2)$.*

Proof. Let $\overline{x}=\frac{x-\mu_X}{\sigma_X}$ and $\overline{y}=\frac{y-\mu_Y}{\sigma_Y}$, then

$$f(x,y)=\frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}}e^{-\frac{1}{2(1-\rho^2)}\left(\overline{x}^2+\overline{y}^2-2\rho\overline{x}\overline{y}\right)}$$
$$=\frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}}e^{-\frac{1}{2(1-\rho^2)}(\overline{x}-\rho\overline{y})^2}e^{-\frac{1}{2}\overline{y}^2}.$$

Since

$$\begin{aligned}\int_{\mathbb{R}} e^{-\frac{1}{2(1-\rho^2)}(\overline{x}-\rho\overline{y})^2}\,dx &= \int_{\mathbb{R}} e^{-\frac{1}{2\sigma_X^2(1-\rho^2)}\left(x-\left(\mu_X+\rho\frac{\sigma_X}{\sigma_Y}(y-\mu_Y)\right)\right)^2}\,dx \\ &= \int_{\mathbb{R}} e^{-\frac{1}{2\sigma_X^2(1-\rho^2)}x^2}\,dx \\ &= \sqrt{2\pi\sigma_X^2(1-\rho^2)},\end{aligned}$$

we have

$$f_Y(y) = \int_{\mathbb{R}} f(x,y)\,dx = \frac{1}{\sqrt{2\pi\sigma_Y^2}}e^{-\frac{1}{2}\overline{y}^2} = \frac{1}{\sqrt{2\pi\sigma_Y^2}}e^{-\frac{(y-\mu_Y)^2}{2\sigma_Y^2}}$$

and so $Y \sim N(\mu_Y, \sigma_Y^2)$. The same argument for X yields $X \sim N(\mu_X, \sigma_X^2)$. A direct computation leads to

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{1}{\sqrt{2\pi\sigma_X^2(1-\rho^2)}}e^{-\frac{1}{2\sigma_X^2(1-\rho^2)}\left(x-\left(\mu_X+\rho\frac{\sigma_X}{\sigma_Y}(y-\mu_Y)\right)\right)^2}$$

as desired. □

Remark 17. The parameter ρ represents how X and Y correlated.

Joint distribution of maximum and minimum. Let $X_1, X_2, \cdots, X_n$ be independent jointly continuous random variables with the common cdf $F(t)$. Let $U = \max\{X_1, X_2, \cdots, X_n\}$ and $V = \min\{X_1, X_2, \cdots, X_n\}$.

Proposition 18. *The joint density of U and V is*

$$f_{U,V}(u,v) = n(n-1)(F(u)-F(v))^{n-2}f(u)f(v).$$

REFERENCES

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