Practice Problems for Final

MATH 3215, Spring 2024

1. Let $X_1, X_2, \cdots, X_7$ be an i.i.d. sequence of Poisson random variables with parameter $\lambda = 2$. Let $W = \sum_{i=1}^{7} X_i$. Find the MGF of W. How is W distributed? Find the probabilities $\mathbb{P}(W = 6)$ and $\mathbb{P}(W = 5 | X_1 = 2)$.

2. Suppose $X \sim N(1, 4)$ and $Y \sim N(2, 12)$ are independent normal random variables. Let $W = X + Y$. Find the MGF of W. Find the probability $\mathbb{P}(3 \leq W \leq 9)$.

3. An instructor has 50 exams that will be graded in sequence. The times required to grade the 50 exams are independent, with a common distribution that has mean 20 minutes and standard deviation 4 minutes. Approximate the probability that the instructor will grade at least 25 of the exams in the first 450 minutes of work.

4. A certain type of electrical motors is defective with probability $1/100$. Pick 1000 motors and let X be the number of defective ones among these 1000 motors.

   (a) What is the probability that among the 1000 motors 13 or less are defective, i.e., what is $\mathbb{P}(X \leq 13)$.

   (b) Using a normal approximation, with mid-point correction, write down an expression for the probability that among the 1000 mortors 13 or less are defective. Use the corresponding tables (and $\sqrt{\frac{99}{10}} \approx 3.15$) to find an approximate value for thsi probability

5. A fair die will be rolled 720 times independently.

   (a) What is the probability that among the 720 rolls the number 6 will appear between 135 and 150 times inclusively? That is, what is $\mathbb{P}(135 \leq X \leq 150)$? Write down the probability without using the tables and approximations.

   (b) Using a normal approximation, **without mid-point correction**, write down an expression for the probability that among the 720 rolls the number 6 will appear between 135 and 150 times inclusively. Use the corresponding tables to find an approximate value for this probability.

   (c) Using a normal approximation, **with mid-point correction**, write down an expression for the probability that among the 720 rolls the number 6 will appear between 135 and 150 times inclusively. Use the corresponding tables to find an approximate value for this probability.

6. If X is a random variable with mean 33 and variance 16, use Chebyshev's inequality to find

   (a) A lower bound for $\mathbb{P}(23 < X < 43)$.

   (b) An upper bound for $\mathbb{P}(|X - 33| \geq 14)$.

7. Let $\overline{X}$ be the mean of a random sample of size $n = 15$ from a distribution with mean $\mu = 80$ and variance $\sigma^2 = 60$. Use Chebvshev's inequality to find a lower bound for $\mathbb{P}(75 < \overline{X} < 85)$.

8. Let $W_1 < W_2 < \cdots < W_{10}$ be the order statistics of n independent observations from a $U(0, 1)$ distribution.

   (a) Find the PDFs of W_1 and W_{10}.

   (b) Find $\mathbb{E}[W_1]$ and $\mathbb{E}[W_{10}]$.

9. Let $Y_1 < Y_2 < \cdots < Y_5$ be the order statistics of a random sample of size 5 from a distribution with PDF $f(x) = e^{-x}$ for $0 < x < \infty$.

(a) Find the PDF of Y_3.

(b) Find the PDF of $U = e^{-Y_3}$.

10. Suppose that X is a discrete random variable with pmf

$$f(x) = \frac{2 + \theta(2 - x)}{6}, \qquad x = 1, 2, 3,$$

where the unknown parameter θ belongs to the parameter space $\Omega = \{-1, 0, 1\}$. Suppose further that a random sample X_1, X_2, X_3, X_4 is taken from this distribution, and the four observed values are $(x_1, x_2, x_3, x_4) = (3, 2, 3, 1)$. Find the maximum likelihood estimate of θ.

11. A random sample of size 16 from the normal distribution $N(\mu, 25)$ yielded $\overline{X} = 73.8$. Find a 95% confidence interval for μ.