In-Class Midterm 1 Review, Math 1554

1. Consider the matrix A and vectors $\vec{b}_1$ and $\vec{b}_2$.

$$A = \begin{pmatrix} 1 & 4 \\ 2 & 8 \end{pmatrix}, \quad \vec{b}_1 = \begin{pmatrix} -2 \\ -4 \end{pmatrix}, \quad \vec{b}_2 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

$x_1 + 4x_2 = -2$

$$\left[\begin{array}{cc|c} 1 & 4 & -2 \\ \cancel{2} & \cancel{8} & \cancel{-4} \\ 0 & 0 & 0 \end{array}\right]$$

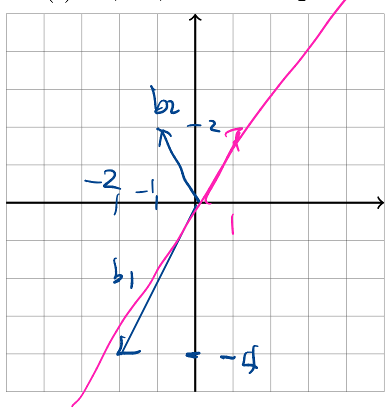
If possible, on the grids below, draw

(i) the two vectors and the span of the columns of A,

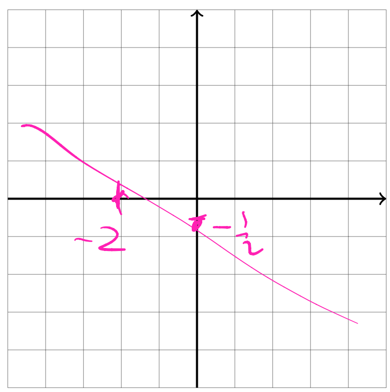
(ii) the solution set of $A\vec{x} = \vec{b}_1$.

(iii) the solution set of $A\vec{x} = \vec{b}_2$.

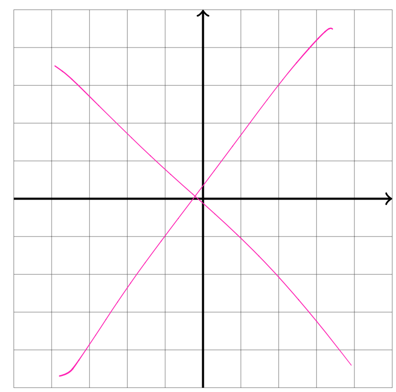
(i) $\vec{b}_1$, $\vec{b}_2$, column span



ii) solution set $Ax = \vec{b}_1$

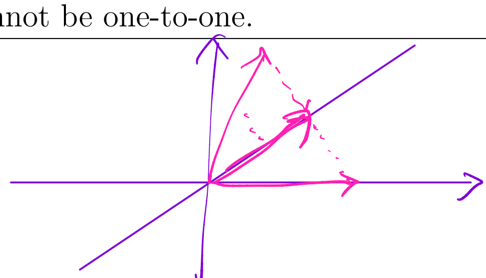


iii) solution set $Ax = \vec{b}_2$



2. Indicate **true** if the statement is true, otherwise, indicate **false**. For the statements that are false, give a counterexample.

	true	false	counterexample
a) If $A \in \mathbb{R}^{M\times N}$ has linearly dependent columns, then the columns of A cannot span $\mathbb{R}^M$.	◯	◯	
b) If there are some vectors $\vec{b} \in \mathbb{R}^M$ that are not in the range of $T(\vec{x}) = A\vec{x}$, then there cannot be a pivot in every row of A.	◯	◯	
c) If the transform $\vec{x} \to A\vec{x}$ projects points in $\mathbb{R}^2$ onto a line that passes through the origin, then the transform cannot be one-to-one.	◯	◯	



3. If possible, write down an example of a matrix with the following properties. If it is not possible to do so, write *not possible*.

   (a) A linear system that is homogeneous and has no solutions.

   (b) A standard matrix A associated to a linear transform, T. Matrix A is in RREF, and $T_A : \mathbb{R}^3 \to \mathbb{R}^4$ is one-to-one.

   (c) A 3×7 matrix A, in RREF, with exactly 2 pivot columns, such that $A\vec{x} = \vec{b}$ has exactly 5 free variables.

4. Consider the linear system $A\vec{x} = \vec{b}$, where

$$A = \begin{pmatrix} 1 & 0 & 7 & 0 & -5 \\ 0 & 1 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$$

   (a) Express the augmented matrix $(\, A \,|\, \vec{b}\,)$ in RREF.

   (b) Write the set of solutions to $A\vec{x} = \vec{b}$ in parametric vector form. Your answer must be expressed as a vector equation.

- $\{ \overset{e_1}{\overset{\shortparallel}{v_1}}, \overset{e_2}{\overset{\shortparallel}{v_2}}, \overset{e_3}{\overset{\shortparallel}{v_3}}, \underset{v_1}{\underset{\shortparallel}{v_4}} \}$ lin. dep.

- $\{ v_1, \cdots, v_4 \}$ lin. indep. $\Longrightarrow$ $\{ v_1, v_2, v_3 \}$ lin. ind.

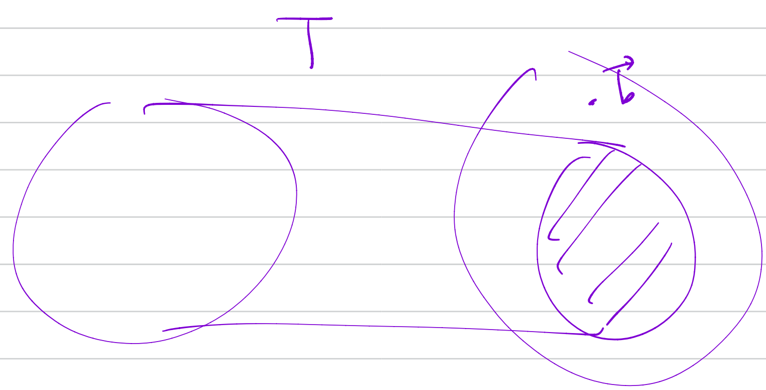
$T : \mathbb{R}^3 \to \mathbb{R}^3$

$$A = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 1 \\ 0 & 2 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

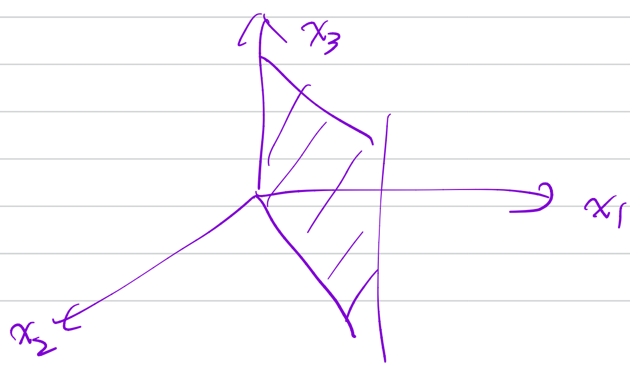
Range of T = Span of Col. of A

$A\vec{x} = 0$

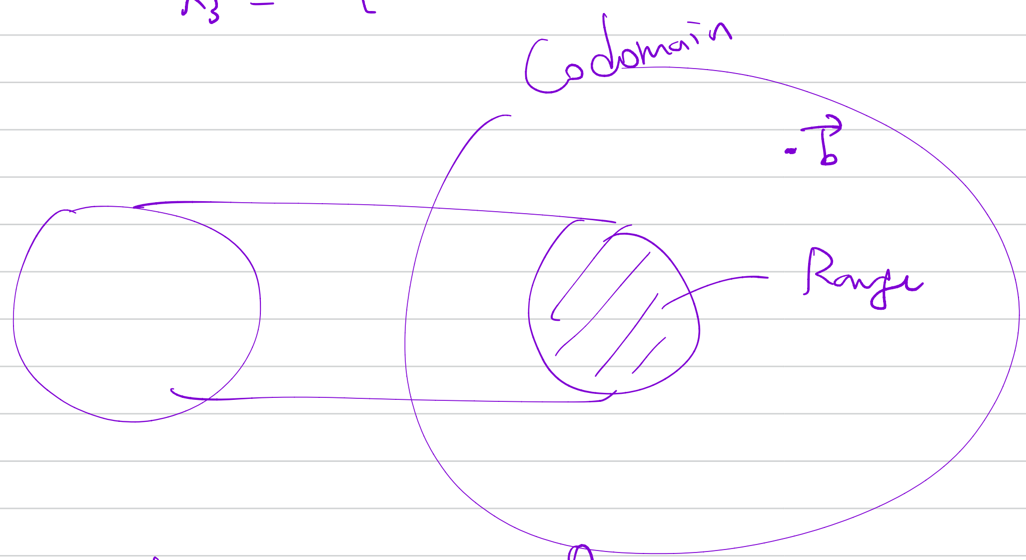
$$2 \begin{bmatrix} 1 & \overset{3}{0} & \\ 0 & 1 & \end{bmatrix}$$



$2x_1 - x_2 + x_3 = 0$ in $\mathbb{R}^3$



$$x_1 = t + s$$
$$x_2 = 2t + 7s$$
$$x_3 = -t$$



$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad b = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad Ax = b \quad \text{inconsistent}$$

$$\begin{array}{cc} & 3 \\ 2 & \left[\qquad\qquad\right] \end{array}$$

$$\begin{bmatrix} -3 & -1 & -1 \\ 1 & 1 & 0 \\ 5 & 1 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{cases} x_1 + x_2 = 0 \\ 2x_2 - x_3 = 0 \end{cases} \Rightarrow$$

$$x_1 = -1, \quad x_2 = 1, \quad x_3 = 2$$

$$c_1\vec{u}_1 + c_2\vec{u}_2 + c_3\vec{u}_3 = \vec{x}$$

$$\left[\begin{array}{ccc|c} \vec{u}_1 & \vec{u}_2 & \vec{u}_3 & \vec{x} \end{array}\right]$$

consistent.

$$\left[\begin{array}{ccc|c} 1 & 5 & -3 & -4 \\ -1 & -4 & 1 & 3 \\ -2 & -7 & 0 & c \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 5 & -3 & -4 \\ 0 & 1 & -2 & -1 \\ 0 & 3 & -6 & c-8 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \end{array}\right]$$

$$c-8 = 3(-1)$$

$$c - 8 = -3 \qquad c = 5$$

$$T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$$

$$A = \begin{array}{c} \\ 3 \end{array} \overset{2}{\begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & -1 \end{bmatrix}}$$

$T(\vec{x}) = \vec{b} = A\vec{x}$ consistent

$$\left[\begin{array}{cc|c} 1 & 1 & b_1 \\ 1 & 0 & b_2 \\ 1 & -1 & b_3 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{cc|c} 1 & 1 & b_1 \\ 0 & -1 & b_2-b_1 \\ 0 & -2 & b_3-b_1 \end{array}\right] \rightarrow \boxed{b_3 - b_1 = 2\cdot(b_2 - b_1)}$$

$$\text{Span}\left(\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}\right)$$

$$\begin{aligned}
\text{Range of } T &= \{ \text{All images} \} \\
&= \{ T(\vec{x}) : \vec{x} \in \mathbb{R}^n \} \\
&= \{ A \cdot \vec{x} : \vec{x} \in \mathbb{R}^n \} \\
&= \{ \text{All lin. comb. of Col. of } A \} \\
&= \text{Span}(\text{Col. of } A)
\end{aligned}$$