Fact: W : a subspace in $\mathbb{R}^n$

$\dim(W) + \dim(W^{\perp}) = n$

Midterm 3 Lecture Review Activity, Math 1554

1. Indicate **true** if the statement is true, otherwise, indicate **false**.

	true	false
a) If S is a two-dimensional subspace of $\mathbb{R}^{50}$, then the dimension of $S^{\perp}$ is 48.	⊗	○
b) An eigenspace $= \text{Null}(A - \lambda I)$ is a subspace spanned by ~~a single eigenvector~~. eigenvectors corr. to a single eigenvalue λ.	○	⊗
c) The $n \times n$ zero matrix can be diagonalized.	⊗	○
d) A least-squares line that best fits the data points $(0, y_1), (1, y_2), (2, y_3)$ is unique for any values y_1, y_2, y_3.	⊗	○

2. If possible, give an example of the following.

2.1) A matrix, A, that is in echelon form, and $\dim\left((\text{Row}\,A)^{\perp}\right) = 2$, $\dim\left((\text{Col}\,A)^{\perp}\right) = 1$

2.2) A singular 2×2 matrix whose eigenspace corresponding to eigenvalue $\lambda = 2$ is the line $x_1 = 2x_2$. The other eigenspace of the matrix is the x_2 axis.

2.3) A subspace S, of $\mathbb{R}^4$, that satisfies $\dim(S) = \dim(S^{\perp}) = 3$.

2.4) A 2×3 matrix, A, that is in RREF. $(\text{Row}\,A)^{\perp}$ is spanned by $\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$.

Least square line

Model: $\underline{y = \beta_0 + \beta_1 x}$

Data: $(0, y_1)$ $(1, y_2)$ $(2, y_3)$

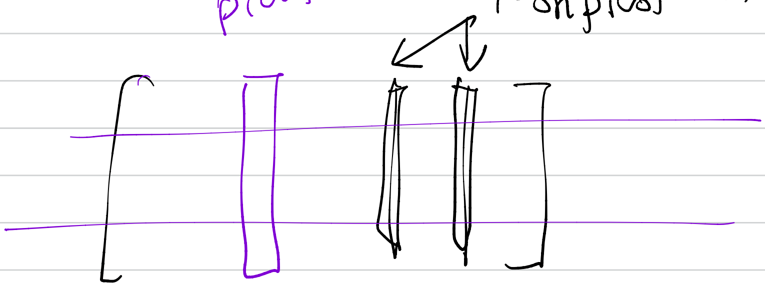
$$\begin{cases} y_1 = \beta_0 + \beta_1 \cdot 0 \\ y_2 = \beta_0 + \beta_1 \cdot 1 \\ y_3 = \beta_0 + \beta_1 \cdot 2 \end{cases}$$

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \boxed{\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}} \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix}$$

- $\mathrm{Row}(A)^{\perp} = \mathrm{Col}(A^T)^{\perp} = \mathrm{Nul}((A^T)^T) = \mathrm{Nul}(A)$.

$\dim(\mathrm{Row}(A)^{\perp}) = 2 = \underline{\dim(\mathrm{Nul}(A))}$

pivot Nonpivot = # of Non-pivot



2

$$\begin{bmatrix} 1 & * & * \\ 0 & 0 & 0 \end{bmatrix}$$

- $\dim(\mathrm{Col}(A)^{\perp}) = 1$

$+$

Assume $\dim(\mathrm{Col}(A)) = 1 \Rightarrow$ $\mathrm{Col}(A)$ in $\mathbb{R}^2$

2

3. Circle **possible** if the set of conditions are create a situation that is possible, otherwise, circle **impossible**. For the situations that are possible give an example.

   3.1) A is $n \times n$, $A\vec{x} = A\vec{y}$ for a particular $\vec{x} \neq \vec{y}$, $\vec{x}$ and $\vec{y}$ are in $\mathbb{R}^n$, and $\dim((\text{Row}\, A)^{\perp}) \neq 0$.

   possible **impossible**

   3.2) A is $n \times n$, $\lambda \in \mathbb{R}$ is an eigenvalue of A, and $\dim((\text{Col}(A - \lambda I))^{\perp}) = 0$.

   possible **impossible**

   3.3) $\text{proj}_{\vec{v}}\vec{u} = \text{proj}_{\vec{u}}\vec{v}$, $\vec{v} \neq \vec{u}$, and $\vec{u} \neq \vec{0}$, $\vec{v} \neq \vec{0}$.

   possible **impossible**

4. Consider the matrix A.

$$A = \begin{pmatrix} 1 & -3 & 0 & 2 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

   Construct a basis for the following subspaces and state the dimension of each space.

   4.1) $(\text{Row} A)^{\perp}$: $\mathcal{B} = \{ [\ \ ], [\ \ ] \}$ in $\mathbb{R}^4$

   4.2) $\text{Col}\, A$: $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

   4.3) $(\text{Col} A)^{\perp}$ $= \text{Nul}\,(A^T)$

$\text{Row}\,(A)^{\perp} = \text{Col}\,(A^T)^{\perp} = \text{Nul}\,(A)$ Parametric vector form.

$\mathcal{B} = \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

$A \in \mathbb{R}^{m \times n}$

$$\left\{\begin{aligned}
\mathrm{Col}(A)^{\perp} &= \mathrm{Nul}(A^T) \\
\mathrm{Nul}(A)^{\perp} &= \mathrm{Col}(A^T) \\
\mathrm{Row}(A) &= \mathrm{Col}(A^T) \\
\dim\left(\underline{\mathrm{Row}(A)}\right) &= \dim\left(\mathrm{Col}(A)\right) \\
\dim(\mathrm{Nul}(A)) + \dim(\mathrm{Col}(A)) &= n
\end{aligned}\right.$$

$$\dim(W) + \dim(W^{\perp}) = n \qquad W \text{ in } \mathbb{R}^n,$$

11/6/24

$A = [\, a_1 \quad a_2 \quad \cdots \quad a_n \,]$

$A^T A = 2 \cdot I_n \quad \Rightarrow$

$$= \begin{bmatrix} — a_1 — \\ — a_2 — \\ \vdots \\ — a_n — \end{bmatrix} \begin{bmatrix} | & | & & | \\ a_1 & a_2 & \cdots & a_n \\ | & | & & | \end{bmatrix} = \begin{bmatrix} a_1 \cdot a_1 & a_1 \cdot a_2 & a_1 \cdot a_3 & \cdots \\ & a_2 \cdot a_2 & & \\ & & \ddots & \\ & & & a_n \cdot a_n \end{bmatrix}$$

diagonal $\Rightarrow$ orthogonal

length $= \sqrt{2}$

① $Ax \cdot Ay = (Ax)^T \cdot (Ay) = x^T \cdot A^T \cdot A \cdot y$

$= 2 \cdot x^T \cdot y = 2 \cdot x \cdot y$

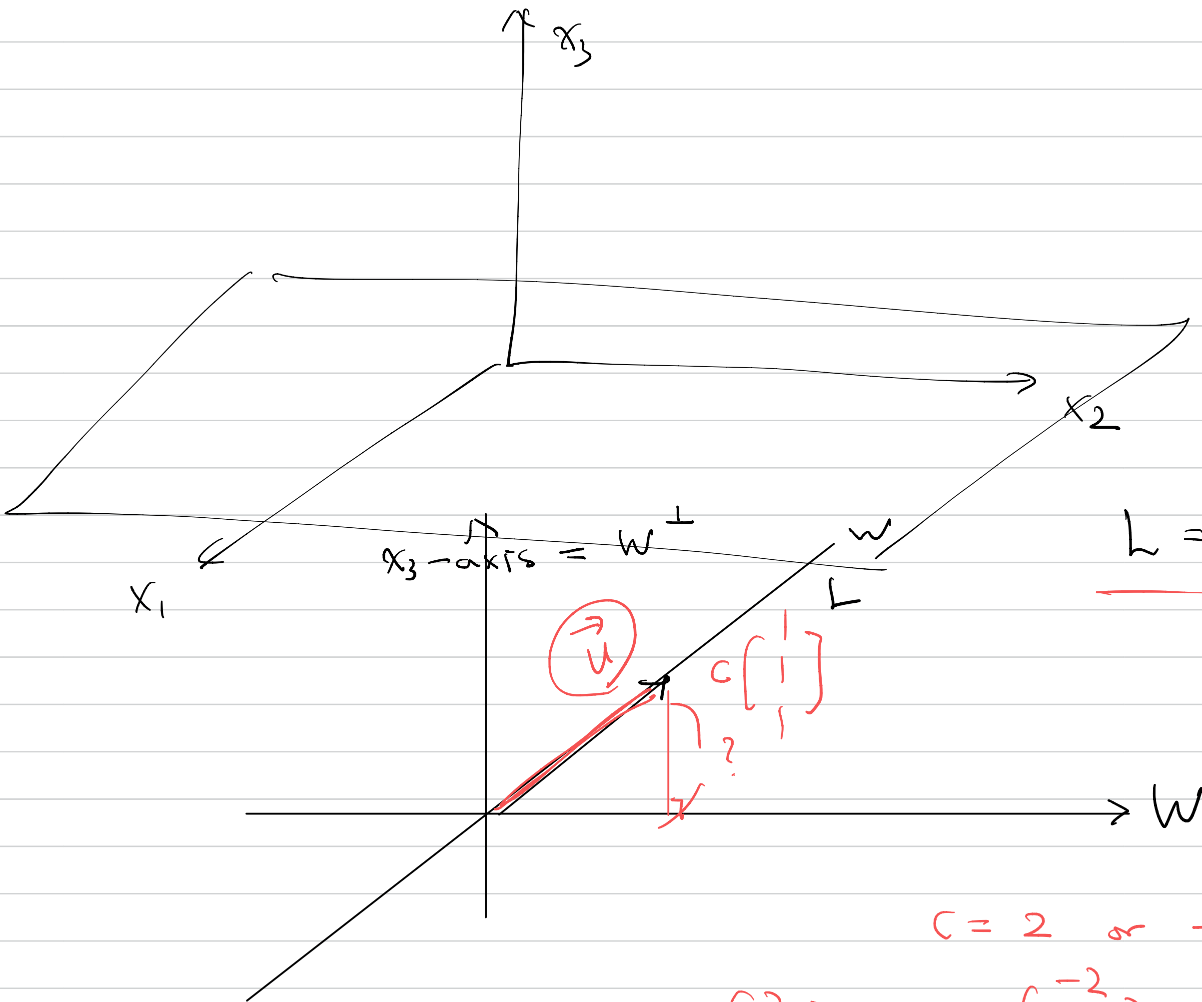
② $n \leq m$?

$\{a_1, a_2, \cdots, a_n\}$ lin. indep. in $\mathbb{R}^m$

$[\, \underset{\frac{1}{\sqrt{2}} a_1}{u_1} \quad \underset{\frac{1}{\sqrt{2}} a_2}{u_2} \cdots \underset{\frac{1}{\sqrt{2}} a_n}{u_n} \,] \quad \Rightarrow \quad n \leq m$

$A = Q \cdot R, \quad R = \sqrt{2} \cdot I_n.$

$$R = Q^T \cdot A = \begin{bmatrix} u_1 \cdot a_1 & u_2 \cdot a_2 & \cdots \\ 0 & & \end{bmatrix} = \sqrt{2}\, I_n.$$



$L = \text{Span}\left(\left\{\begin{bmatrix}1\\1\\1\end{bmatrix}\right\}\right)$

$c = 2$ or -2

$u = \begin{bmatrix}2\\2\\2\end{bmatrix}$ or $\begin{bmatrix}-2\\-2\\-2\end{bmatrix}$

$$\text{Nul}(A) = \text{Nul}(A^TA)$$

① If $x \in \text{Nul}(A)$, then $Ax = 0$

$A^TAx = 0$

$x \in \text{Nul}(A^TA)$

② If $x \in \text{Nul}(A^TA)$, then $A^TAx = 0$

$$\|Ax\|^2 = (Ax)^T \cdot Ax = \underline{x^T A^T A x} = 0$$

$Ax = 0$

$x \in \text{Nul}(A)$

$\lambda = 1$ eigen . $\underline{v_1}$

$v_1 \; , \; 2\cdot v_1 \; , \; 3\cdot v_1 \; , \; \cdots$

$$\boxed{(A)^T} = (P \cdot D \cdot P^{-1})^T = \underbrace{\boxed{(P^{-1})^T}}_{= Q} \cdot \underline{D^T} \cdot \underline{P^T}$$

$$Q \cdot \underline{\underline{D}} \cdot Q^{-1}$$

$$Q^{-1} = ((P^{-1})^T)^{-1} = ((P^T)^{-1})^{-1}$$

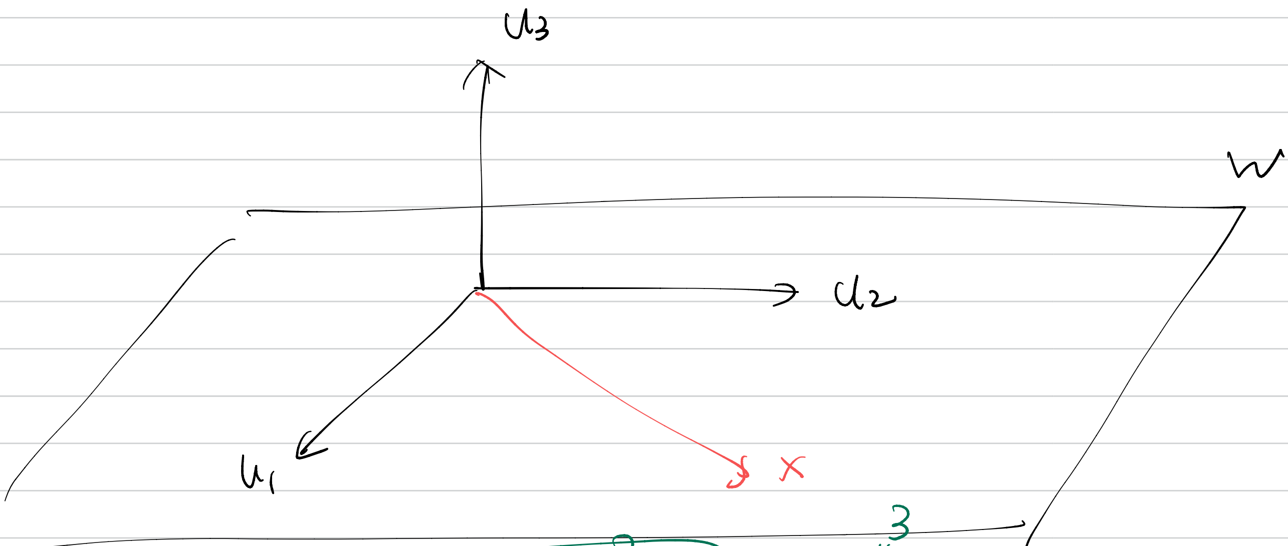
$$= P^T$$

A

$\downarrow$

Q

$$\mathrm{Nul}(A^T) = \mathrm{Col}(A)^{\perp} = \mathrm{Col}(Q)^{\perp} = \mathrm{Nul}(Q^T)$$



$$x = \underbrace{\overbrace{\frac{x \cdot u_1}{u_1 \cdot u_1}}^{=2} u_1}_{= \mathrm{proj}_{u_1} x} + \overbrace{\frac{x \cdot u_2}{u_2 \cdot u_2}}^{=3} u_2$$

$$W^\perp = \mathrm{Nul}\left(\begin{bmatrix}1 & 1 & -1\end{bmatrix}\right)^\perp = \mathrm{Col}\left(\begin{bmatrix}1\\1\\-1\end{bmatrix}\right)$$

$$A = \begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix} = \begin{bmatrix}\sqrt{2} & 0\\ 0 & \sqrt{2}\end{bmatrix} \begin{array}{cc} T(e_1) & T(e_2) \\ \hline \end{array} \hspace{-6em} \overset{A}{\begin{bmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix}}$$

$\lambda = ?$ $\qquad \det(A - \lambda I) = \lambda^2 - 2\lambda + 2 = 0$

$$(\lambda - 1)^2 = -1$$

$$\lambda = 1 \pm i$$

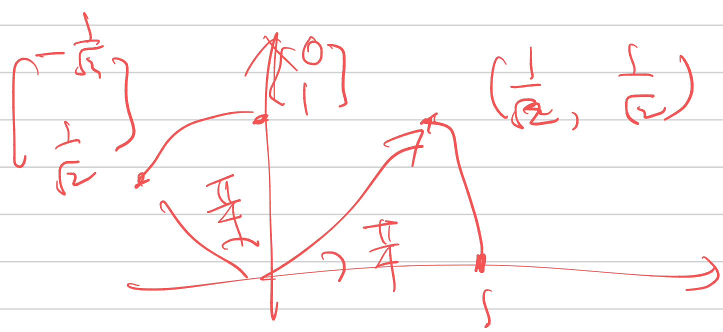
$$= |\lambda| \cdot e^{i\theta} \leftarrow \text{rotation}$$

$|\lambda| = \sqrt{2}$ $\uparrow$ scaling

$$e^{i\theta} = \frac{1}{\sqrt{2}} \pm \frac{1}{\sqrt{2}} i$$

$$= \cos\theta + \sin\theta \cdot i$$

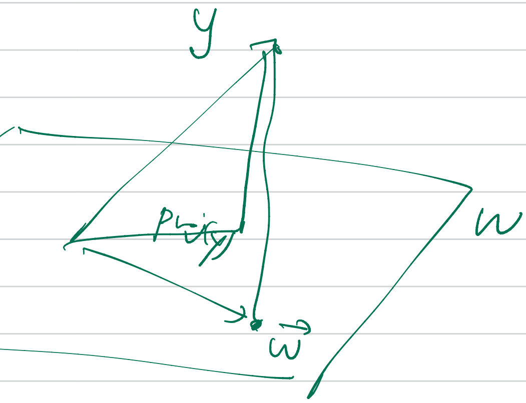
$$\theta = \frac{\pi}{4}, -\frac{\pi}{4}$$



$$\| y - (\underbrace{c_1 u_1 + c_2 u_2}_{\in W}) \| \quad \text{minimize.}$$

$$\geq \| y - \mathrm{proj}_W (y) \|$$

$$= \frac{y \cdot u_1}{u_1 \cdot u_1} u_1 + \frac{y \cdot u_2}{u_2 \cdot u_2} u_2$$

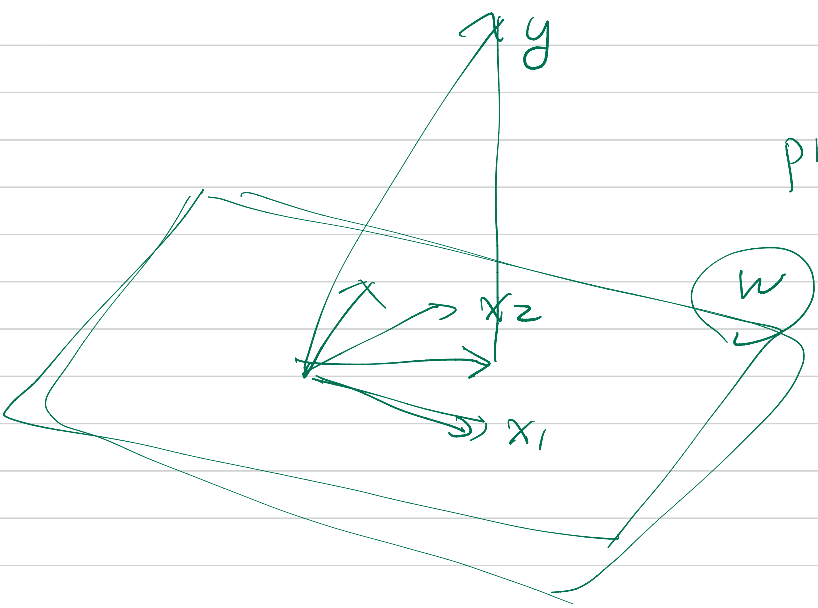


$w_1 \quad v_1, v_2 \in W$

$$\overbrace{y - v_1}^{w_1} \in W^\perp \longrightarrow y = \underset{\in W}{v_1} + \underset{\in W^\perp}{w_1}$$

$$\underbrace{y - v_2}_{w_2} \in W^\perp \longrightarrow y = \underset{\in W}{v_2} + \underset{\in W^\perp}{w_2}$$

$y \in \mathbb{R}^4$, $\quad y = \hat{y} + w^\perp$ unique



$$\mathrm{proj}_W (y) = c_1 x_1 + c_2 x_2$$

?

A : diagonalizable

$\Leftrightarrow \quad A = P \cdot D \cdot P^{-1}$

D : diagonal

P : invertible.

$$A^T = (P^{-1})^T \cdot \boxed{D^T} \cdot P^T$$

$$\square \cdot \square = \square^T \cdot \square$$

Dot product

$$(Ax) \cdot (Ay) = (A \cdot x)^T \cdot (A \cdot x)$$

Matrix Multiplication

$$= x^T \cdot A^T \cdot A \cdot x$$