MATH 461 LECTURE NOTE
WEEK 8

DAESUNG KIM

1. THE UNIFORM RANDOM VARIABLE (SEC 5.3)

Definition: Uniform random variable
A random variable X is a uniform random variable on (a, b) if its probability density function is

$$f(x) = \begin{cases} \frac{1}{b-a}, & a < x < b, \\ 0, & \text{otherwise.} \end{cases}$$

We denote by $X \sim U(a, b)$.

Example 1. Let X be a uniform random variable on $(0, 10)$. Calculate $\mathbb{P}(X < 3)$, $\mathbb{P}(X > 6)$, and $\mathbb{P}(3 < X < 8)$.

Proposition 2. *If $X \sim U(a, b)$, then $\mathbb{E}[X] = \frac{a+b}{2}$, $\mathrm{Var}(X) = \frac{(a-b)^2}{12}$, and the distribution function is*

$$F(x) = \begin{cases} 0, & x < a, \\ \frac{1}{b-a}(x-a), & a \le x \le b, \\ 1, & x > b. \end{cases}$$

Example 3. A bus travels between the two cities A and B, which are 100 miles apart. If the bus has a breakdown, the distance from the breakdown to city A has a $U(0, 100)$ distribution. There are bus service stations in city A, in B, and in the center of the route between A and B. It is suggested that it would be more efficient to have the three stations located 25, 50, and 75 miles, respectively, from A. Do you agree? Why?

Example 4. Find the density of $Y = \sqrt{U}$ where U is a uniform random variable on $[0, 1]$.

Proposition 5. *Let X be a continuous random variable with cdf F_X. Let $g(x)$ be the inverse of F_X defined on $(0, 1)$, that is, $g(x) = \inf\{t : F_X(t) \ge x\}$. If U is a uniform random variable on $[0, 1]$, then $g(U)$ has the same distribution as X.*

2. NORMAL RANDOM VARIABLES (SEC 5.4-7)

Definition: Normal random variables
X is a normal random variable with parameters μ and σ^2 if its density is

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

for $x \in \mathbb{R}$ and denoted by $X \sim N(\mu, \sigma^2)$. If $\mu = 0$ and $\sigma = 1$, then we call X the standard normal random variable.

In real life, there are a lot of cases where its randomness can be understood by a normal distribution. Later in Section 8, we will see that normal distributions arise in an important result known as Central Limit Theorem. Note that the constant $\frac{1}{\sqrt{2\pi\sigma^2}}$ is chosen so that $\int f(x)\,dx = 1$.

Proposition 6. *Let $X \sim N(\mu, \sigma^2)$.*

(i) *For any $a, b \in \mathbb{R}$, $aX + b \sim N(a\mu + b, a^2\sigma^2)$. In particular, $Z = (X - \mu)/\sigma \sim N(0, 1)$ is standard normal.*

(ii) $\mathbb{E}[X] = \mu$ *and* $\mathrm{Var}(X) = \sigma^2$.

The cumulative distribution function of $N(0,1)$ is

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}}\, dt.$$

Note that the cdf cannot be computed explicitly. Note also that $\Phi(x) = 1 - \Phi(-x)$.

Example 7. If X is a normal random variable with parameters $\mu = 3$ and $\sigma^2 = 9$, find

(i) $\mathbb{P}(2 < X < 5)$,
(ii) $\mathbb{P}(X > 0)$,
(iii) $\mathbb{P}(|X - 3| > 6)$.

Normal approximations to binomial
Let $S_n \sim \mathrm{Bin}(n, p)$ be the number of successes in n independent Bernoulli trials. Then, we have seen that $\mathbb{E}[S_n] = np$, $\mathrm{Var}(S_n) = np(1-p)$. For large n,

$$\frac{S_n - \mathbb{E}[S_n]}{\sqrt{\mathrm{Var}(S_n)}} = \frac{S_n - np}{\sqrt{np(1-p)}} \approx N(0,1).$$

The approximation is good for $np(1-p) \geq 10$. Compared to Poisson approximation, the success probability p needs not to be small.

Example 8. Each item produced by a certain manufacturer is, independently, of acceptable quality with probability .95. Approximate the probability that at most 10 of the next 150 items produced are unacceptable.

Example 9. The ideal size of a first-year class at a particular college is 150 students. The college, knowing from past experience that, on the average, only 30 percent of those accepted for admission will actually attend, uses a policy of approving the applications of 450 students. Compute the probability that more than 150 first-year students attend this college.

3. Exponential Random Variables (Sec 5.4-7)

Definition: Exponential random variable
A random variable X is exponential with parameter $\lambda > 0$ if its density is given by

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0, \\ 0, & x < 0. \end{cases}$$

We denote by $X \sim \exp(\lambda)$.

Proposition 10. *Let* $X \sim \exp(\lambda)$ *for* $\lambda > 0$.

(i) *The cumulative distribution function* $F(x) = 1 - e^{-\lambda x}$.
(ii) $\mathbb{E}[X] = \frac{1}{\lambda}$.
(iii) $\mathrm{Var}(X) = \frac{1}{\lambda^2}$.

Proposition 11 (Memoryless property). *Let* $s, t > 0$ *and* $X \sim \exp(\lambda)$ *for* $\lambda > 0$, *then*

$$\mathbb{P}(X > s + t | X > t) = \mathbb{P}(X > s).$$

Example 12. Suppose that the number of miles that a car can run before its battery wears out is exponentially distributed with an average value of 10,000 miles. If a person desires to take a 5000-mile trip, what is the probability that he or she will be able to complete the trip without having to replace the car battery? What can be said when the distribution is not exponential?

References

[SR] Sheldon Ross, *A First Course in Probability*, 9th Edition, Pearson

Department of Mathematics, University of Illinois at Urbana-Champaign
E-mail address: daesungk@illinois.edu