Chapter 1. Probability

Math 3215 Spring 2024

Georgia Institute of Technology

Section 1.
Properties of Probability

Why Probability and Statistics?

Two main reasons are uncertainty and complexity.

Uncertainty is all around us and is usually modeled as randomness: it appears in call centers, electronic circuits, quantum mechanics, medical treatment, epidemics, financial investments, insurance, games (both sports and gambling), online search engines, for starters.

Probability is a good way of quantifying and discussing what we know about uncertain things, and then making decisions or estimating outcomes.

Why Probability and Statistics?

Some things are too complex to be analyzed exactly (like weather, the brain, social science), and probability is a useful way of reducing the complexity and providing approximations.

Definition: Experiments, Sample spaces, Events

We consider experiments for which the outcome cannot be predicted with certainty.

Such experiments are called random experiments. ex) Toss a coin

The collection of all possible outcomes $\{H, T\}$ is denoted by S and is called the sample space.

Given a sample space S, let A be a part of the collection of outcomes in S.

The subset A is called an event. $\{H\}$, $\{T\}$

events.

S : sample space.

$A, B, C, \cdots, A_1, A_2, \cdots, B_1, B_2, \cdots$: subsets of S "events"

Algebra of sets

Empty set (Null set): ϕ $S = \{1, 2, 3, 4, 5\}$

A is a subset of B: Every outcome in A is in B, $A \subset B$

The union of A and B = the set of outcomes in A $\boxed{\text{or}}$ in B = $A \cup B$ = $\{1,2,3\}$
$A = \{1,2\}$, $B = \{2,3\}$

The intersection of A and B = the set of outcomes in A $\boxed{\text{and}}$ in B = $A \cap B$ = $\{2\}$

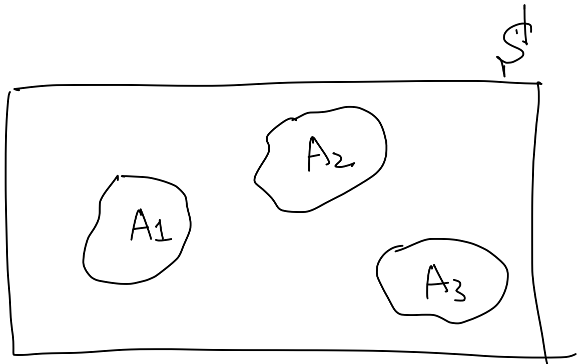
The complements of A = the set of outcomes $\boxed{\text{not in}}$ A = A^c = $\{3, 4, 5\}$

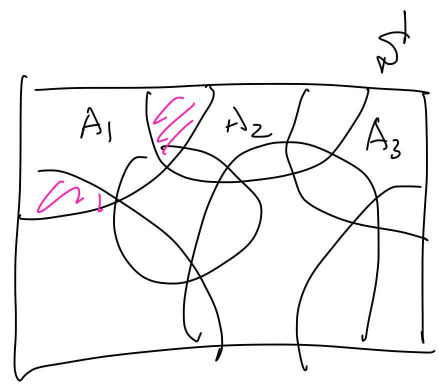
$A_1, A_2, \cdots, A_k$ are mutually exclusive events: $A_i \cap A_j = \phi$
no overlaps

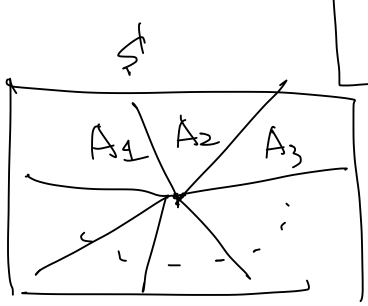
$A_1, A_2, \cdots, A_k$ are exhaustive events: $A_1 \cup A_2 \cup \cdots \cup A_k = S$
covers S

$A_1, A_2, \cdots, A_k$ are mutually exclusive and exhaustive events:

a partition of S.







Algebra of sets

Commutative Laws Order doesn't matter.

$A \cup B = B \cup A$ and $A \cap B = B \cap A$

Associative Laws For 3 subsets, the same 2 operations

$(A \cup B) \cup C = A \cup (B \cup C)$ and $(A \cap B) \cap C = A \cap (B \cap C)$

Distributive Laws 3 subsets, mixed operations.

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ and $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

De Morgan's Laws $\cup$ & comp, $\cap$ & comp.

$(A \cup B)^c = A^c \cap B^c$ and $(A \cap B)^c = A^c \cup B^c$

Definition of Probability

$S = \{H, T\}$

Consider repeating the experiment a number of times, say, n times. We call these repetitions trials.

$= \{H\}$

Count the number of times that event A actually occurred throughout these n trials; this number is called the frequency of event A and is denoted by $N(A)$.

The ratio $N(A)/n$ is called the relative frequency of event A in these n repetitions of the experiment. $\frac{48}{100} = 0.45$

As n increase, one can expect that the relative frequency tends to stabilize, close to some number p.

This p is called the **probability of A**.

Definition of Probability

$\mathbb{P}$ {sets} $\longrightarrow$ [0,1]

events, S

Definition

Probability is a real-valued set function $\mathbb{P}$ that assigns, to each event A in the sample space S, a number $\mathbb{P}(A)$, called the probability of the event A, such that the following properties are satisfied:

1. $\mathbb{P}(A) \geq 0$ for all events A
2. $\mathbb{P}(S) = 1$
3. For mutually exclusive events $A_1, A_2, \cdots$, $\mathbb{P}(\overbrace{A_1 \cup A_2 \cup \cdots}^{\text{union}}) = \overbrace{\mathbb{P}(A_1) + \mathbb{P}(A_2) + \cdots}^{\text{sum}}$

No overlaps.

$$\begin{cases} \mathbb{P}(A) \geq 0 \\ \mathbb{P}(\Omega) = 1 \leftarrow 100\% \\ \text{Mutually Exclusive} \end{cases}$$

$\mathbb{P}(\text{Union}) = \text{Sum of } \mathbb{P}(\ )$

Definition of Probability

Theorem

Let A, B be events.

1. $\mathbb{P}(A) = 1 - \mathbb{P}(A^c)$
2. $\mathbb{P}(\varnothing) = 0$
3. If $A \subset B$, then $\mathbb{P}(A) \leq \mathbb{P}(B)$.
4. $\mathbb{P}(A) \leq 1$ for all events A.

① A, A^c : mutually exclusive & exhaustive

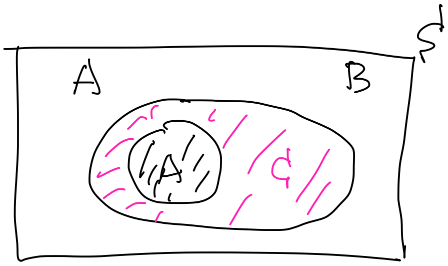
$A \cap A^c = \phi$ $\qquad A \cup A^c = \Omega$

$\Rightarrow 1 = \mathbb{P}(\Omega) = \mathbb{P}(A \cup A^c) = \mathbb{P}(A) + \mathbb{P}(A^c)$

$\mathbb{P}(A) = 1 - \mathbb{P}(A^c)$

② $\underbrace{\Omega, \phi}_{\text{events}}$: mutually exclusive & exhaustive

③ $A \subset B$



$B = A \cup C$

no overlap.

Definition of Probability

0.33 , 0.5 ,

Example

A fair coin is flipped successively until the same face is observed on successive flips.

What is the probability that it will take three or more flips of the coin to observe the same face on two consecutive flips?

A $\qquad \mathbb{P}(A)$

$$A^c \begin{cases} H\ H & \leftarrow \frac{1}{2}\cdot\frac{1}{2} \\ T\ T & \leftarrow \frac{1}{2}\cdot\frac{1}{2} \end{cases}$$

$$A \begin{cases} H\ T\ T & \leftarrow \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2} = \frac{1}{8} \\ T\ H\ H & \leftarrow \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2} = \frac{1}{8} \\ H\ T\ H\ H \\ T\ H\ T\ T \\ H\ T\ H\ T\ T \\ \vdots \end{cases}$$

infinitely many situations

$$\{\underbrace{HH}_{\frac{1}{2}\cdot\frac{1}{2}}, \underbrace{TT}_{\frac{1}{2}\cdot\frac{1}{2}}, HT, TH\} \quad H,H$$

$$\mathbb{P}(A^c) = \frac{2}{4} = 0.5 = \frac{1}{2}\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{1}{2}$$

TT

$$\mathbb{P}(A) = 1 - \mathbb{P}(A^c) = 1 - 0.5 = 0.5$$

"mutually exclusive" $\mathbb{P}(\text{union}) = \text{sum of } \mathbb{P}(\ )$

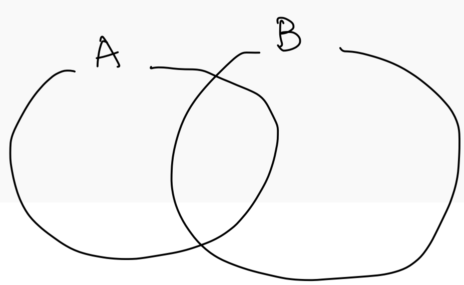
Definition of Probability

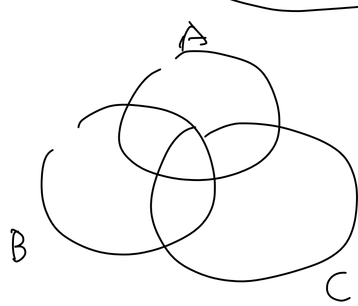
Theorem

For events A, B, C,

sum

$$\begin{aligned}
\mathbb{P}(A \cup B) &= \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B) \\
\mathbb{P}(A \cup B \cup C) &= \mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(C) \quad \text{sum} \\
&\quad - \mathbb{P}(A \cap B) - \mathbb{P}(B \cap C) - \mathbb{P}(C \cap A) \\
&\quad + \mathbb{P}(A \cap B \cap C)
\end{aligned}$$





Definition of Probability

Example

Among a certain population of men, 30% are smokers, 40% are obese, and 25% are both smokers and obese.

Suppose we select a man at random from this population.

What is the probability that the selected man is either a smoker or obese?

$A = \{ \text{smoker} \}$

$B = \{ \text{obese} \}$

$\mathbb{P}(\text{Smoker } \underline{\underline{\text{or}}} \text{ obese})$

$= \mathbb{P}(A \cup B)$

$= \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$

$= \mathbb{P}(\text{Smoker}) + \mathbb{P}(\text{Obese}) - \mathbb{P}(\text{both})$

$= 0.3 + 0.4 - 0.25 = \underbrace{0.45}$

45%

Probability with Equally likely outcomes

sample space

Let $S = (e_1, e_2, \ldots, e_m)$. & finite sample space

If each of these outcomes has the same probability of occurring, we say that the m outcomes are equally likely.

In this case, $\mathbb{P}(A)$ is equal to

$$
\begin{aligned}
1 = \mathbb{P}(S) &= \mathbb{P}(\{e_1, \ldots, e_m\}) \\
&= \mathbb{P}(\{e_1\} \cup \{e_2\} \cup \cdots \cup \{e_m\}) \quad \text{mutually exclusive} \\
&= \mathbb{P}(\{e_1\}) + \mathbb{P}(\{e_2\}) \cdots + \mathbb{P}(\{e_m\})
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{P}(\{e_1\}) = \mathbb{P}(\{e_2\}) = \cdots = \mathbb{P}(\{e_m\}) \\
= \frac{1}{m} = \frac{1}{\#\text{ of all outcomes}}
\end{aligned}
$$

A : an event

$$\Rightarrow \quad \mathbb{P}(A) = \frac{\#\text{ of outcomes in } A}{\#\text{ of all outcomes}}.$$

Computing $\mathbb{P}(A)$ = "Counting" # of outcomes.

Probability with Equally likely outcomes

Example

Let a card be drawn at random from an ordinary deck of 52 playing cards.

What is the probability that a king is drawn?

$52 = 13 \times 4$

H : A 1 2 ---- 10 J Q K

D :

S :

C :

Assume Equal Probability.

$P($ a card drawn $= K)$

$= \dfrac{\#\text{ of outcomes in the event} \overset{\shortparallel}{} 4}{\#\text{ of total outcomes} \underset{52}{\shortparallel}} = \dfrac{4}{52}$

$= \dfrac{1}{13}$.

Section 2.
Methods of Enumeration

Multiplication Principle

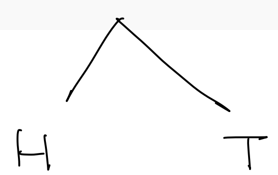
Suppose that an experiment E_1 has n_1 outcomes and, for each of these possible outcomes, an experiment E_2 has n_2 possible outcomes.

Then the composite experiment E_1E_2 that consists of performing first E_1 and then E_2 has $n_1 n_2$ possible outcomes.

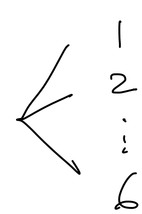
The **multiplication principle** can be extended to a sequence of more than two experiments or procedures.

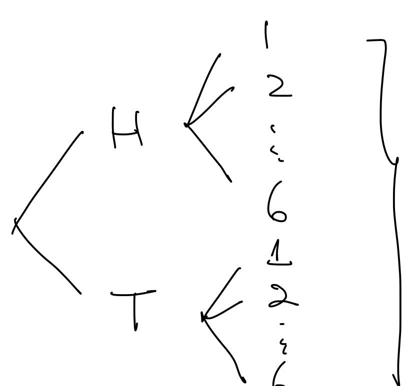
Example

E_1 = Toss a fair coin



E_2 = Roll a die





$\leftarrow 12 = 6 + 6 = 2 \cdot 6$

Multiplication Principle

ANS : $6 \cdot 5 \cdot 5 \cdot 2^{12}$.

Example

A cafe lets you order a deli sandwich your way.

There are: E_1, six choices for bread; E_2, four choices for meat; E_3, four choices for cheese; and E_4, 12 different garnishes (condiments).

What is the number of different sandwich possibilities, if you may choose one bread, 0 or 1 meat, 0 or 1 cheese, and from 0 to 12 condiments?

E_1 : bread , $n_1 = 6$,

E_2 : meat , $n_2 = 4 + 1 = 5$

0 , 1

E_3 : cheese , $n_3 = 5$,

0 or 1

$\boxed{E_4}$: condiments , $n_4 = 2^{12}$ (or 13 ↑ 0, 1)

2^{13}

13 !

Tomato w / w.o 2

olives w / w.o 2

Lettuce : :

: : 2

12 subexperiments!

Permutation

Example

What is the number of the arrangements of four letters a, b, c, d?

Definition

Each of the arrangements (in a row) of n different objects is called a **permutation** of the n objects. $(n = 4)$

abcd ←

abdc ←

acbd

acdb

adbc

adcb

⋮

permutation of a, b, c, d

$E_1 \quad E_2 \quad E_3 \quad E_4$

$4 \times 3 \times 2 \times 1$

$= 4 \cdot 3 \cdot 2 \cdot 1 = 4!$

Factorial

In general, # of permutations of n objects

$$= \underbrace{\overset{n}{\square}\ \overset{n-1}{\square}\ \cdots\ \overset{1}{\square}}_{n \text{ spot}} = n\cdot(n-1)\cdot(n-2)\cdots 2\cdot 1$$

$$= n!$$

Permutation

If only r positions are to be filled with objects selected from n different objects, $r \leq n$, then the number of possible ordered arrangements is

$n\cdot(n-1)\cdot\cdots\cdot(n-r+1)$
$= {}_nP_r$

Definition

Each of the ${}_nP_r$, arrangements is called a permutation of n objects taken r at a time.

$$\underbrace{\overset{n}{\square}\ \overset{(n-1)}{\square}\ \cdots\ \overset{n-r+1}{\square}}_{r \text{ positions}} \qquad n-r+1 = n-(r-1)$$

$n = 6$

$$\underbrace{\overset{6}{\square}\ \overset{5}{\square}\ \overset{4}{\square}}_{r=3} \qquad 4 \neq (n-r)?$$

$n - r = 6 - 3 = 3.$

$${}_nP_r = \overbrace{n\cdot(n-1)\cdots\cdot(n-r+1)}^{r \text{ integers}}$$

$$= \frac{n\cdot(n-1)\cdots(n-r+1)\,(n-r)\,(n-r-1)\cdots 1}{(n-r)\,(n-r-1)\cdots 1}$$

$$= \frac{n!}{(n-r)!}$$

Permutation

Example

What is the number of ways of selecting a president, a vice president, a secretary, and a treasurer in a club consisting of ten persons?

$n = 10$

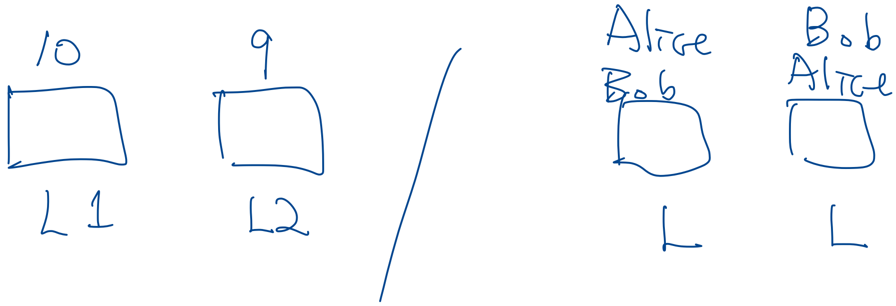
| 10 | 9 | 8 | 7 |
| P | VP | S | T |

ordered arrangement

why?

4 "distinct" positions

$$10 \cdot 9 \cdot 8 \cdot 7 = {}_{10}P_4 = \frac{10!}{6!}$$

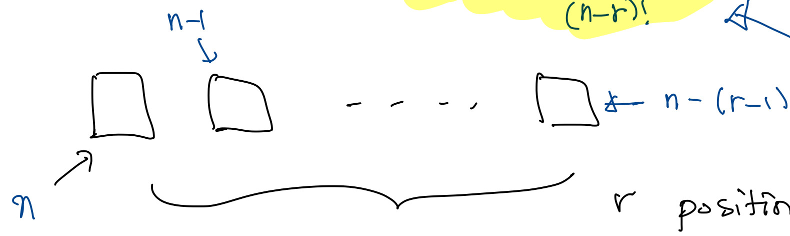


$\neq 10 \cdot 9$

n different object. r objects sample without replacement.

$\Rightarrow$ $r!$? $\quad {}_nC_r = \dfrac{n!}{r!\,(n-r)!}$? (order)

$${}_nP_r = \frac{n!}{(n-r)!} \text{ ?}$$



r positions $\Rightarrow$ $n\cdot(n-1)\cdots(n-r+1)$

$${}_nC_r = \frac{n!}{r!\,(n-r)!}$$

ordering of the chosen r objects

Sampling

Suppose that a set contains n objects. Consider the problem of selecting r objects from this set.

- If r objects are selected from a set of n objects, and if the order of selection is noted, then the selected set of r objects is called an ordered sample of size r.
- Sampling with replacement occurs when an object is selected and then replaced before the next object is selected.
- Sampling without replacement occurs when an object is not replaced after it has been selected.

Sampling

$n = 52 \qquad r = 5$

Example

What are the number of "ordered" samples of five cards that can be drawn with/without replacement?

With Replacement

$$= \boxed{}_{52} \; \boxed{}_{52} \; \boxed{}_{52} \; \boxed{}_{52} \; \boxed{}_{52}$$

$$= 52^5$$

Without Replacement

$$= \boxed{}_{52} \; \boxed{}_{51} \; \boxed{}_{50} \; \boxed{}_{49} \; \boxed{}_{48}$$

$$= 52 \cdot 51 \cdot 50 \cdot 49 \cdot 48 = {}_{52}P_5 = \frac{52!}{47!}$$

Combination

Ex) 2 leader out of 10 people

Definition

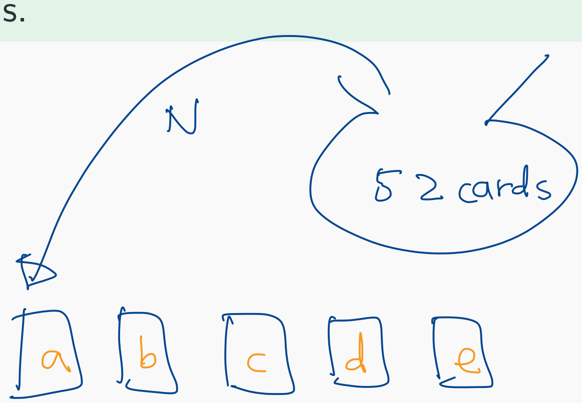
Each of the unordered subsets of $\{1, 2, \cdots, n\}$ is called a **combination** of n objects taken r at a time.

Combination

Example

$N=$ The number of possible five-card hands (in five-card poker) drawn from a deck of 52 playing cards.

E_1



no order

E_2 $\downarrow$ give them order $= 5!$ # of ways

Ordered arrangement of 5 cards out of 52 cards

$$N \cdot 5! = 52 \cdot 51 \cdot 50 \cdot 49 \cdot 48 = {}_{52}P_5 = \frac{52!}{47!}$$

$$N = \frac{52!}{5!\,47!}$$

In general,

$\longrightarrow$ r objects without order

$$\binom{n}{r} = {}_nC_r = \frac{{}_nP_r}{r!} = \frac{n!}{r!\,(n-r)!}$$

"n choose r"

Binomial Theorem

$$(a+b)^n = (a+b)\cdot(a+b)\ \cdots\ (a+b)$$

$$= 1\cdot a^n + n\cdot a^{n-1}\cdot b + \cdots + \binom{n}{k}a^k\, b^{n-k} + \cdots + \underbrace{1}_{\binom{n}{n}} b^n$$

Binomial Theorem

$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$

Suppose that a set contains n objects of two types: r of one type and $n-r$ of the other type.

The number of distinguishable arrangements is # of way of choosing r spots among n spots

= unorder arrangement $= \binom{n}{r}$

r Green balls $\qquad$ n−r Red balls

n balls

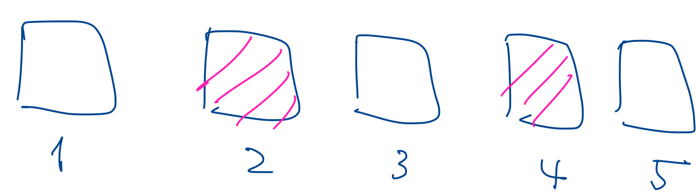
Ex

G G G R R

G G R G R

G R R G G

⋮



Binomial Theorem

Example

A coin is flipped ten times and the sequence of heads and tails is observed.

Find the number of possible 10-tuples that result in four heads and six tails.

$E_1 \quad E_2 \quad \cdots \quad E_{10}$

$\{H, T\} \quad \{H, T\} \quad \cdots \quad \{H, T\}$

Total outcomes $= 2^{10}$

4 H , 6 T

H ... H T ... T

$\square \cdots \square$ 10 spots

Choose 4 spots $\to$ H.

$$\binom{10}{4} = \frac{10!}{4!\,6!}.$$

$6 \to$ " $\binom{10}{6} = \frac{10!}{6!\,4!}$

Binomial Theorem

Multinomial coefficients

The coefficient of $a_1^{r_1} a_2^{r_2} \cdots a_s^{r_s}$ in the expansion of $(a_1 + \cdots + a_s)^n$ is

Section 3.
Conditional Probability

Conditional Probability

Example

Suppose that we are given 20 tulip bulbs that are similar in appearance and told that eight will bloom early, 12 will bloom late, 13 will be red, and seven will be yellow.

If one bulb is selected at random, the probability that it will produce a red tulip is

The probability that it will produce a red tulip given that it will bloom early is

	Early	Late	
Red	a	b	13
Yellow	c	d	7
	8	12	

$\mathbb{P}(\text{Red}) = \frac{13}{20}$ $\qquad$ $\mathbb{P}(\text{Yellow}) = \frac{7}{20}$

$\mathbb{P}(\text{Early}) = \frac{8}{20}$ $\qquad$ $\mathbb{P}(\text{Late}) = \frac{12}{20}$

Choose one at Random, Assume we know it blooms early.

$\frac{a}{8} = \mathbb{P}(\text{Red Given Early}) = \frac{13}{20}$? $\quad \frac{\#\text{ of Red \& Early}}{\#\text{ of Early}}$

Conditional Probability

New sample space.

Definition

The conditional probability of an event A, given that event B has occurred, is defined by

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$$

provided that $\mathbb{P}(B) > 0$.

If every outcome is equally likely.

$$\mathbb{P}(A | B) = \frac{\#\text{ of outcomes in } A \cap B \,/\, \text{total outcomes in } S}{\#\text{ of outcomes in } B \,/\, \text{total outcomes in } S}$$

$$= \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}.$$

Conditional Probability

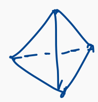
Prob. of A given B

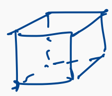
Example

If $\mathbb{P}(A) = 0.4$, $\mathbb{P}(B) = 0.5$, and $\mathbb{P}(A \cap B) = 0.3$, then $\mathbb{P}(A|B) = \dfrac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$

$= \dfrac{0.3}{0.5} = \dfrac{3}{5} = \underline{\underline{0.6}}$

Conditional Probability





16.

Example $S = \{ (1,1), (1,2), (1,3), \dots \}$

Two fair four-sided dice are rolled and the sum is determined. Let A be the event that a sum of 3 is rolled, and let B be the event that a sum of 3 or a sum of 5 is rolled. The conditional probability that $\underbrace{\text{a sum of 3 is rolled}}_{A}$, given that $\underbrace{\text{a sum of 3 or 5 is rolled}}_{B}$, is

$$A = \{ \text{Sum} = 3 \} = \{ (1,2), (2,1) \} = A \cap B$$

$$B = \{ \text{Sum} = 5 \text{ or } 3 \} = \{ (1,4), (2,3), (3,2), (4,1), (1,2), (2,1) \}$$

$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} = \frac{\# \text{ of } A \cap B}{\# \text{ of } B}$$

Equally Likely Case.

$$= \frac{2}{6} = \frac{1}{3}.$$

$\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$

$\mathbb{P}(A) \geq 0$

$\mathbb{P}(S) = 1$

$A_1, A_2, \cdots$ mutually exclusive $\Rightarrow$ $\mathbb{P}(\text{Union}) = \mathbb{P}(A_1) + \mathbb{P}(A_2) + \cdots$

$A_1 \cup A_2 \cup \cdots$ sum of $\mathbb{P}(\cdot)$

Properties of Conditional probabilities

$\mathbb{P}(A|B) = \dfrac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$

Theorem

Suppose $\mathbb{P}(B) > 0$.

1. $\mathbb{P}(A|B) \geq 0$.
2. $\mathbb{P}(B|B) = 1$.
3. If $A_1, A_2, \cdots, A_k$ are mutually exclusive events, then

$$\mathbb{P}(A_1 \cup A_2 \cup \cdots \cup A_k|B) = \mathbb{P}(A_1|B) + \cdots + \mathbb{P}(A_k|B).$$

4. $\mathbb{P}(A^c|B) = 1 - \mathbb{P}(A|B)$.

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$$

From $\mathbb{P}(A|B) \longrightarrow$ Find $\mathbb{P}(A \cap B)$.

The multiplication rule

The multiplication rule

The probability that two events, A and B, both occur is given by

$$\mathbb{P}(A \cap B) = \mathbb{P}(A|B)\mathbb{P}(B) = \mathbb{P}(B|A)\mathbb{P}(A).$$

$\mathbb{P}(B) \cdot \mathbb{P}(A) = \mathbb{P}(A \cap B)$

True sometimes . . .

The multiplication rule

Example

At a county fair carnival game there are 25 balloons on a board, of which ten balloons are yellow, eight are red, and seven are green.

A player throws darts at the balloons to win a prize and randomly hits one of them.

Suppose the player throws darts twice.

What is the probability that the both balloons hit are yellow?

~~A = { 1st is Yellow }~~

$B = \{ 2^{nd} \text{ is Yellow} \}$

$\mathbb{P}(A) = \frac{10}{25} = \frac{2}{5}$

$\mathbb{P}(B) = ? \quad \frac{9}{24}? \quad \frac{2}{5}? \quad \frac{9}{25}?$

$\frac{9}{24} = \mathbb{P}(B|A)$

$\mathbb{P}(A \cap B) = \mathbb{P}(B|A)\,\mathbb{P}(A)$

$= \frac{9}{24} \cdot \frac{10}{25} = \frac{3}{8} \cdot \frac{2}{5} = \frac{3}{20}.$

Q: $\mathbb{P}(B) = ?$ $= \dfrac{\mathbb{P}(A \cap B)}{\mathbb{P}(A|B)}$, $\mathbb{P}(A|B) = ?$

$\mathbb{P}(B) = \mathbb{P}(B \cap A) + \mathbb{P}(B \cap A^c)$

$= \mathbb{P}(B|A) \cdot \mathbb{P}(A) + \underline{\mathbb{P}(B|A^c)} \cdot \underline{\mathbb{P}(A^c)}$

$= \frac{9}{24} \cdot \frac{10}{25} + \frac{10}{24} \cdot \frac{15}{25}$

The multiplication rule

Example

A bowl contains seven blue chips and three red chips.

Two chips are to be drawn successively at random and without replacement.

Compute the probability that the first draw results in a red chip and the second draw results in a blue chip.

$A = \{1^{st} = \text{Red}\}$ $\quad B = \{2^{nd} = \text{Blue}\}$

$\mathbb{P}(A \cap B) = ?$ $\quad \frac{7}{30}, \frac{3}{10},$

$(\mathbb{P}(B) = ?$ $\quad \mathbb{P}(A \mid B) = \; = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} = \frac{1}{3}$

$\mathbb{P}(A) = \frac{3}{10}$, $\quad \mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B \mid A)$

$= \frac{3}{10} \cdot \frac{7}{9_3} = \frac{7}{30}.$

$\mathbb{P}(B) = \mathbb{P}(A \cap B) + \mathbb{P}(A^c \cap B)$

$= \mathbb{P}(A) \cdot \mathbb{P}(B \mid A) + \mathbb{P}(A^c) \cdot \mathbb{P}(B \mid A^c) = \frac{3}{10} \cdot \frac{7}{9_3} + \frac{7}{10} \cdot \frac{6^2}{9_3}$

$= \frac{7 + 14}{30} = \frac{21}{30}.$

$= \frac{7}{10}.$

The multiplication rule

Multiplication rule for three events

$$\mathbb{P}(A \cap B \cap C) = \mathbb{P}(A)\mathbb{P}(B|A)\mathbb{P}(C|A \cap B).$$

$$= \mathbb{P}(C)\,\mathbb{P}(A \mid C)\,\mathbb{P}(B \mid A \cap C)$$

The multiplication rule

Example

Four cards are to be dealt successively at random and without replacement from an ordinary deck of playing cards.

The probability of receiving, in order, a spade, a heart, a diamond, and a club is

1st 2nd 3rd 4th

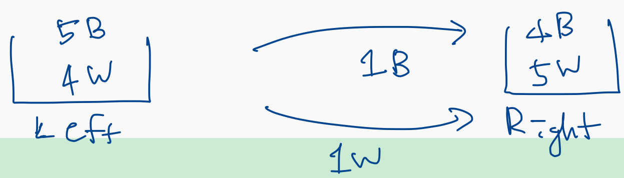
A B C D

$$\mathbb{P}(A \cap B \cap C \cap D)$$

$$= \mathbb{P}(A) \cdot \mathbb{P}(B \mid A) \cdot \mathbb{P}(C \mid A \cap B) \cdot \mathbb{P}(D \mid A \cap B \cap C)$$

$$= \frac{13}{52} \cdot \frac{13}{51} \cdot \frac{13}{50} \cdot \frac{13}{49}$$

The multiplication rule



Example

A boy has five blue and four white marbles in his left pocket and four blue and five white marbles in his right pocket.

If he transfers one marble at random from his left to his right pocket, what is the probability of his then drawing a blue marble from his right pocket?

$A = \{ \text{Blue from Right} \}$

$B = \{ \text{Blue from Left} \}$

$\mathbb{P}(A) = \mathbb{P}(\text{Choose Blue from Left, Put it into Right, Choose Blue from Right})$

$+ \mathbb{P}(\text{White from Left, Put it into Right, Blue from Right})$

$= \mathbb{P}(A \cap B) + \mathbb{P}(A \cap B^c)$

$= \mathbb{P}(A | B) \mathbb{P}(B) + \mathbb{P}(A | B^c) \mathbb{P}(B^c)$

$= \frac{5}{10} \frac{5}{9} + \frac{4}{10} \cdot \frac{4}{9} = \frac{25 + 16}{90} = \frac{41}{90}.$

Section 4.
Independent Events

Independent Events

For certain pairs of events, the occurrence of one of them may or may not change the probability of the occurrence of the other.

In the latter case, they are said to be independent events.

Independent Events

$S = \{ (H,H), (H,T), (T,H), (T,T) \}$

Equally likely outcomes

Example

Flip a coin twice.

Let $A = \{$heads on the first flip$\}$ and $B = \{$tails on the second flip.$\}$

Compute $\mathbb{P}(B|A)$ and $\mathbb{P}(B)$.

$A \cap B = \{ (H,T) \}$

$A = \{ (H,H), (H,T) \}$

$B = \{ (H,T), (T,T) \}$

$$\mathbb{P}(B \mid A) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)} = \frac{\# \text{ in } A\cap B / \# \text{ in } S}{\# \text{ in } A / \# \text{ in } S} = \frac{1}{2}$$

$$\mathbb{P}(B) = \frac{\# \text{ in } B}{\# \text{ in } S} = \frac{2}{4} = \frac{1}{2}$$

& $\mathbb{P}(A) = \frac{1}{2}$.

A and B indep. if

$$\frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} = \mathbb{P}(A \mid B) = \mathbb{P}(A) \Rightarrow \mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$$

or

$$\frac{\mathbb{P}(B \cap A)}{\mathbb{P}(A)} = \mathbb{P}(B \mid A) = \mathbb{P}(B) \Rightarrow \mathbb{P}(B \cap A) = \mathbb{P}(B)\mathbb{P}(A)$$

Recall $\mathbb{P}(A|B)$ only makes sense when $\mathbb{P}(B) \neq 0$

Independent Events

$\mathbb{P}(A|B) = \mathbb{P}(A)$

Definition: Independence

Events A and B are **independent** if and only if $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$.

Otherwise, A and B are called **dependent events**.

$\frac{1}{2} = \mathbb{P}(B \mid red = 1)$

$= \mathbb{P}(B \mid red = 2)$

$\vdots$

Independent Events

$S = \{(1,1)\ (1,2),\ (1,3),\ \ldots\}$ 36 outcomes

$B = \{(1,2)\ (1,4),\ (1,6),\ (2,1 \text{ or } 3 \text{ or } 5)$

$(3,2 \text{ or } 4 \text{ or } 6),\ \ldots\}$ 18 outcomes

Example

A red die and a white die are rolled.

Let $A = \{\text{red is } 4\}$ and $B = \{\text{sum is odd}\}$.

Are they independent?

$\mathbb{P}(A \cap B) = \frac{3}{36} = \frac{1}{12} = \frac{1}{6} \cdot \frac{1}{2}$

$\mathbb{P}(A) = 1/6$

$\mathbb{P}(B) = 18/36 = \frac{1}{2}$

$A \cap B = \{(4, 1 \text{ or } 3 \text{ or } 5)\}$ 3 outcomes

$\Rightarrow \quad \mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B)$

Independent Events

$\mathbb{P}(A) = \frac{1}{6}$

$\mathbb{P}(B) = \frac{1}{18}$

Example

A red die and a white die are rolled.

$\mathbb{P}(A \cap B) = \frac{1}{36} \neq \frac{1}{6} \cdot \frac{1}{18}$

Let $A = \{\text{red is } 5\}$ and $B = \{\text{sum is } 11\}$.

Are they independent?

(5, 6) (6, 5)

No.

White \ Red	1	2	3	4	5	6
1	(1,1)	(2,1)	(3,1)			
2						
3						
4						
5						
6						

Independent Events

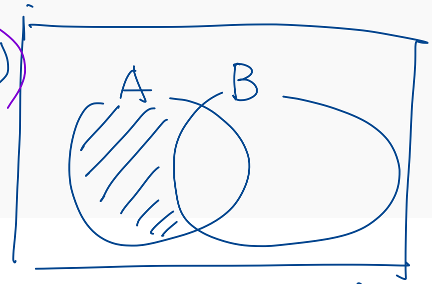
Theorem

If A and B are independent, then the following pairs are independent:

- A and B^c
- A^c and B
- A^c and B^c

Proof

$$\mathbb{P}(A \cap B^c) + \underbrace{\mathbb{P}(A \cap B)}_{= \mathbb{P}(A)\,\mathbb{P}(B)} = \mathbb{P}(A)$$



$$\mathbb{P}(A \cap B^c) = \mathbb{P}(A) - \mathbb{P}(A) \cdot \mathbb{P}(B)$$

$$= \mathbb{P}(A) \underbrace{(1 - \mathbb{P}(B))}_{= \mathbb{P}(B^c)}$$

$$= \mathbb{P}(A) \cdot \mathbb{P}(B^c)$$

$\blacksquare$

Independent Events

Definition: Mutually independence

Events A, B, and C are **mutually independent** if and only if A, B, and C are pairwise independent and

$$\mathbb{P}(A \cap B \cap C) = \mathbb{P}(A)\mathbb{P}(B)\mathbb{P}(C).$$

In other words :

$$\begin{cases} \mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B) \\ \mathbb{P}(B \cap C) = \mathbb{P}(B) \cdot \mathbb{P}(C) \\ \mathbb{P}(C \cap A) = \mathbb{P}(C)\,\mathbb{P}(A) \\ \mathbb{P}(A \cap B \cap C) = \mathbb{P}(A)\mathbb{P}(B)\,\mathbb{P}(C) \end{cases}$$

Note: A, B, C are pairwise indep.
only the first 3 satisfied.

Note: Mutually Indep $\Longrightarrow$ Pairwise Indep.

$\not\Longleftarrow$

Independent Events

Example

Three inspectors look at a critical component of a product.

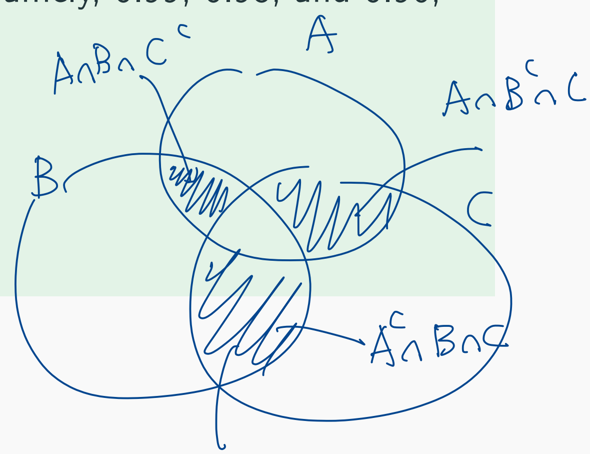
A B C

Their probabilities of detecting a defect are different, namely, 0.99, 0.98, and 0.96, respectively. Assume mutually independence.

Compute the following probabilities:

(a) that exactly two find the defect, and

(b) that all three find the defect.



(a) $\mathbb{P}(\text{Two find Defect})$

$$= \mathbb{P}(A \cap B \cap C^c) + \mathbb{P}(A \cap B^c \cap C) + \mathbb{P}(A^c \cap B \cap C)$$

$$= \mathbb{P}(A) \cdot \mathbb{P}(B) \cdot \mathbb{P}(C^c) + \mathbb{P}(A)\mathbb{P}(B^c)\mathbb{P}(C) + \mathbb{P}(A^c)\mathbb{P}(B)\mathbb{P}(C)$$

$$= 0.99 \cdot 0.98 \cdot (1 - 0.96) + 0.99(1 - 0.98) \cdot 0.96 + (1 - 0.99)\, 0.98 \cdot 0.96$$

(b) $\mathbb{P}(A \cap B \cap C) = \mathbb{P}(A) \cdot \mathbb{P}(B) \cdot \mathbb{P}(C)$

$$= 0.99 \cdot 0.98 \cdot 0.96.$$

Section 5.
Bayes' Theorem

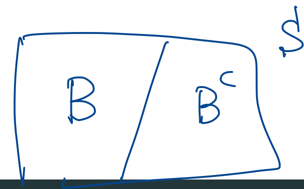
Recall

$$A = (A \cap B) \cup (A \cap B^c)$$

$$\mathbb{P}(A) = \mathbb{P}(A \cap B) + \mathbb{P}(A \cap B^c)$$

$$= \mathbb{P}(A \mid B)\,\mathbb{P}(B) + \mathbb{P}(A \mid B^c)\,\mathbb{P}(B^c)$$

B, B^c : Mutually Exclusive + Exhaustive

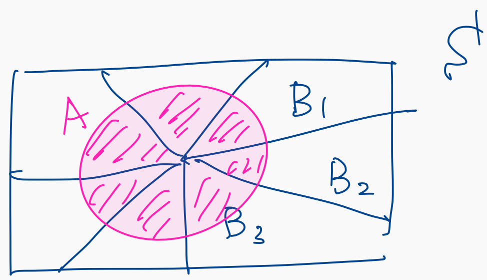


The law of total probabilities

The law of total probabilities (LoTP)

If $B_1, \cdots, B_n$ are mutually exclusive and exhaustive events (partition), then

$$\mathbb{P}(A) = \sum_{k=1}^{n} \mathbb{P}(A \cap B_k) = \sum_{k=1}^{n} \mathbb{P}(A|B_k)\mathbb{P}(B_k).$$



$$\mathbb{P}(A) = \mathbb{P}(A \cap B_1) + \mathbb{P}(A \cap B_2) + \mathbb{P}(A \cap B_3) + \cdots + \mathbb{P}(A \cap B_n)$$

$$= \mathbb{P}(A \mid B_1)\,\mathbb{P}(B_1) + \mathbb{P}(A \mid B_2)\,\mathbb{P}(B_2) + \cdots + \mathbb{P}(A \mid B_n)\,\mathbb{P}(B_n)$$

$\boxed{\text{LoTP}}$

Recall $\mathbb{P}(A \mid B)$ ← Easy to compute

2nd action → ↑ first action

$$\mathbb{P}(B \mid A) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)} = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A \cap B) + \mathbb{P}(A \cap B^c)}$$

$$= \frac{\mathbb{P}(A \mid B)\,\mathbb{P}(B)}{\mathbb{P}(A \mid B)\,\mathbb{P}(B) + \mathbb{P}(A \mid B^c)\,\mathbb{P}(B^c)}$$

Bayes' Theorem

Multiplication Rule

Bayes' Theorem

$$\mathbb{P}(B_k|A) = \frac{\mathbb{P}(A \cap B_k)}{\mathbb{P}(A)} = \frac{\mathbb{P}(B_k)\mathbb{P}(A|B_k)}{\mathbb{P}(A)} = \frac{\mathbb{P}(B_k)\mathbb{P}(A|B_k)}{\sum_{i=1}^{n} \mathbb{P}(A|B_i)\mathbb{P}(B_i)}.$$

LoTP

$$= \frac{\mathbb{P}(A \mid B_k) \cdot \mathbb{P}(B_k)}{\mathbb{P}(A \mid B_1) \cdot \mathbb{P}(B_1) + \mathbb{P}(A \mid B_2)\,\mathbb{P}(B_2) + \cdots + \mathbb{P}(A \mid B_n)\,\mathbb{P}(B_n)}$$

Multiplication Rule : $\mathbb{P}(A \cap B) = \mathbb{P}(B) \cdot \mathbb{P}(A \mid B)$

Def of Conditional Prob

Examples

Example

In a certain factory, machines I, II, and III are all producing springs of the same length.

Of their production, machines I, II, and III respectively produce 2%, 1%, and 3% defective springs.

Of the total production of springs in the factory, machine I produces 35%, machine II produces 25%, and machine III produces 40%.

If the selected spring is defective, what is the conditional probability that it was produced by machine III?

All different senarios

$B_1 = \{\text{From Machine I}\}$ $\quad B_2 = \{\text{From Machine II}\}$

$B_3 = \{\text{From Machine III}\}$ $\quad A = \{\text{Defective}\}$ ← observation

$$\mathbb{P}(B_3 \mid A) \underset{\text{Def}}{=} \frac{\mathbb{P}(B_3 \cap A)}{\mathbb{P}(A)} \underset{\text{LoTP}}{=} \frac{\underset{0.03}{\mathbb{P}(A \mid B_3)}\,\underset{0.40}{\mathbb{P}(B_3)}}{\underset{0.02}{\mathbb{P}(A \mid B_1)}\,\underset{0.35}{\mathbb{P}(B_1)} + \underset{0.01}{\mathbb{P}(A \mid B_2)}\,\underset{0.25}{\mathbb{P}(B_2)} + \underset{0.03}{\mathbb{P}(A \mid B_3)}\,\underset{0.40}{\mathbb{P}(B_3)}}$$

$$= \frac{0.03 \cdot 0.40}{0.02 \cdot 0.35 + 0.01 \cdot 0.25 + 0.03 \cdot 0.40}$$

$$= \frac{3 \cdot 40}{2 \cdot 35 + 1 \cdot 25 + 3 \cdot 40} = \frac{120}{70 + 25 + 120} = \frac{120}{215} = \cdots$$

Examples

Example

Bowl B_1, contains two red and four white chips, bowl B_2 contains one red and two white chips, and bowl B_3 contains five red and four white chips.

Choose one of three bowls with $\mathbb{P}(B_1) = 1/3$, $\mathbb{P}(B_2) = 1/6$, and $\mathbb{P}(B_3) = 1/2$ and draw a chip from the chosen bowl.

Let R be the event that a red chip is chosen.

Compute $\mathbb{P}(R)$ and $\mathbb{P}(B_1|R)$.

$$\mathbb{P}(R) = \mathbb{P}(R \cap B_1) + \mathbb{P}(R \cap B_2) + \mathbb{P}(R \cap B_3)$$

$$= \mathbb{P}(R|B_1)\,\mathbb{P}(B_1) + \mathbb{P}(R|B_2)\,\mathbb{P}(B_2) + \mathbb{P}(R|B_3)\,\mathbb{P}(B_3)$$

$$= \frac{2}{6} \cdot \frac{1}{3} + \frac{1}{3} \cdot \frac{1}{6} + \frac{5}{9} \cdot \frac{1}{2}$$

$$= \frac{\overbrace{2+1+5}^{=8}}{18} = \frac{4}{9}.$$

$$\mathbb{P}(B_1|R) = \frac{\mathbb{P}(B_1 \cap R)}{\mathbb{P}(R)} = \frac{2/\cancel{18}}{8/\cancel{18}} = \frac{2}{8} = \frac{1}{4}.$$