# Section 4.9 : Applications to Markov Chains

Chapter 4 : Vector Spaces

Math 1554 Linear Algebra

# Topics and Objectives

**Topics**

We will cover these topics in this section.

1. Markov chains
2. Steady-state vectors
3. Convergence

**Objectives**

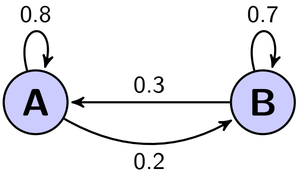
For the topics covered in this section, students are expected to be able to do the following.

1. Construct stochastic matrices and probability vectors.
2. Model and solve real-world problems using Markov chains (e.g. - find a steady-state vector for a Markov chain)
3. Determine whether a stochastic matrix is regular.

# Example 1

- A small town has two libraries, $A$ and $B$.
- After 1 month, among the books checked out of $A$,
  - 80% returned to $A$
  - 20% returned to $B$
- After 1 month, among the books checked out of $B$,
  - 30% returned to $A$
  - 70% returned to $B$

If both libraries have 1000 books today, how many books does each library have after 1 month? After one year? After $n$ months? A place to simulate this is `http://setosa.io/markov/index.html`



# Example 1 Continued

The books are equally divided by between the two branches, denoted by $\vec{x}_0 = \begin{bmatrix} .5 \\ .5 \end{bmatrix}$. What is the distribution after 1 month, call it $\vec{x}_1$? After two months?

After $k$ months, the distribution is $\vec{x}_k$, which is what in terms of $\vec{x}_0$?

# Markov Chains

A few definitions:

- A **probability vector** is a vector, $\vec{x}$, with non-negative elements that sum to 1.
- A **stochastic matrix** is a square matrix, $P$, whose columns are probability vectors.
- A **Markov chain** is a sequence of probability vectors $\vec{x}_k$, and a stochastic matrix $P$, such that:

$$\vec{x}_{k+1} = P\vec{x}_k, \quad k = 0, 1, 2, \ldots$$

- A **steady-state vector** for $P$ is a vector $\vec{q}$ such that $P\vec{q} = \vec{q}$.

# Example 2

Determine a steady-state vector for the stochastic matrix

$$\begin{pmatrix} .8 & .3 \\ .2 & .7 \end{pmatrix}$$

# Convergence

We often want to know what happens to a process,

$$\vec{x}_{k+1} = P\vec{x}_k, \quad k = 0, 1, 2, \ldots$$

as $k \to \infty$.

**Definition**: a stochastic matrix $P$ is **regular** if there is some $k$ such that $P^k$ only contains strictly positive entries.

**Theorem**

If $P$ is a regular stochastic matrix, then $P$ has a unique steady-state vector $\vec{q}$, and $\vec{x}_{k+1} = P\vec{x}_k$ converges to $\vec{q}$ as $k \to \infty$.

# Example 3

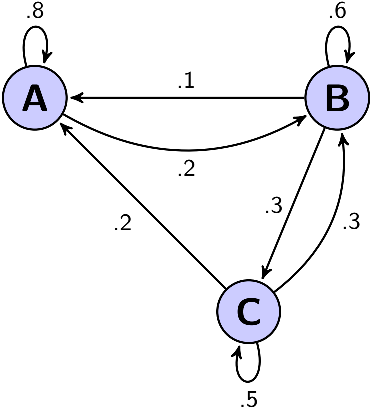
A car rental company has 3 rental locations, A, B, and C. Cars can be returned at any location. The table below gives the pattern of rental and returns for a given week.

| | | rented from | | |
|---|---|---|---|---|
| | | A | B | C |
| | A | .8 | .1 | .2 |
| returned to | B | .2 | .6 | .3 |
| | C | .0 | .3 | .5 |

There are 10 cars at each location today.

a) Construct a stochastic matrix, $P$, for this problem.

b) What happens to the distribution of cars after a long time? You may assume that $P$ is regular.




$$P = \begin{bmatrix} .8 & .1 & .2 \\ .2 & .6 & .3 \\ .0 & .3 & .5 \end{bmatrix}$$

# Section 5.1 : Eigenvectors and Eigenvalues

Chapter 5 : Eigenvalues and Eigenvectors

Math 1554 Linear Algebra

# Topics and Objectives

**Topics**

We will cover these topics in this section.

1. Eigenvectors, eigenvalues, eigenspaces
2. Eigenvalue theorems

**Objectives**

For the topics covered in this section, students are expected to be able to do the following.

1. Verify that a given vector is an eigenvector of a matrix.
2. Verify that a scalar is an eigenvalue of a matrix.
3. Construct an eigenspace for a matrix.
4. Apply theorems related to eigenvalues (for example, to characterize the invertibility of a matrix).

# Eigenvectors and Eigenvalues

If $A \in \mathbb{R}^{n \times n}$, and there is a $\vec{v} \neq \vec{0}$ in $\mathbb{R}^n$, and

$$A\vec{v} = \lambda\vec{v}$$

then $\vec{v}$ is an **eigenvector** for $A$, and $\lambda \in \mathbb{C}$ is the corresponding **eigenvalue**.

Note that

- We will only consider square matrices.
- If $\lambda \in \mathbb{R}$, then
  - when $\lambda > 0$, $A\vec{v}$ and $\vec{v}$ point in the same direction
  - when $\lambda < 0$, $A\vec{v}$ and $\vec{v}$ point in opposite directions
- Even when all entries of $A$ and $\vec{v}$ are real, $\lambda$ can be complex (a rotation of the plane has no **real** eigenvalues.)
- We explore complex eigenvalues in Section 5.5.

# Example 1

Which of the following are eigenvectors of $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$? What are the corresponding eigenvalues?

a) $\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

b) $\vec{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

c) $\vec{v}_3 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

# Example 2

Confirm that $\lambda = 3$ is an eigenvalue of $A = \begin{pmatrix} 2 & -4 \\ -1 & -1 \end{pmatrix}$.

# Eigenspace

**Definition**

Suppose $A \in \mathbb{R}^{n\times n}$. The eigenvectors for a given $\lambda$ span a subspace of $\mathbb{R}^n$ called the $\lambda$-**eigenspace** of $A$.

**Note:** the $\lambda$-eigenspace for matrix $A$ is $\mathrm{Nul}(A - \lambda I)$.

**Example 3**

Construct a basis for the eigenspaces for the matrix whose eigenvalues are given, and sketch the eigenvectors.

$$\begin{pmatrix} 5 & -6 \\ 3 & -4 \end{pmatrix}, \quad \lambda = -1, 2$$

# Theorems

Proofs for the most these theorems are in Section 5.1. If time permits, we will explain or prove all/most of these theorems in lecture.

1. The diagonal elements of a triangular matrix are its eigenvalues.

2. $A$ invertible $\Leftrightarrow 0$ is not an eigenvalue of $A$.

3. Stochastic matrices have an eigenvalue equal to 1.

4. If $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ are eigenvectors that correspond to distinct eigenvalues, then $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ are linearly independent.

# Warning!

We can't determine the eigenvalues of a matrix from its reduced form.

Row reductions change the eigenvalues of a matrix.

**Example**: suppose $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. The eigenvalues are $\lambda = 2, 0$, because

$$A \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} =$$

$$A \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} =$$

- But the reduced echelon form of $A$ is:
- The reduced echelon form is triangular, and its eigenvalues are:

# Section 5.2 : The Characteristic Equation

Chapter 5 : Eigenvalues and Eigenvectors

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# Topics and Objectives

**Topics**

We will cover these topics in this section.

1. The characteristic polynomial of a matrix
2. Algebraic and geometric multiplicity of eigenvalues
3. Similar matrices

**Objectives**

For the topics covered in this section, students are expected to be able to do the following.

1. Construct the characteristic polynomial of a matrix and use it to identify eigenvalues and their multiplicities.
2. Characterize the long-term behaviour of dynamical systems using eigenvalue decompositions.

# The Characteristic Polynomial

**Recall:**

$\lambda$ is an eigenvalue of $A \Leftrightarrow (A - \lambda I)$ is not ______________

Therefore, to calculate the eigenvalues of $A$, we can solve

$$\det(A - \lambda I) =$$

The quantity $\det(A - \lambda I)$ is the **characteristic polynomial** of $A$.

The quantity $\det(A - \lambda I) = 0$ is the **characteristic equation** of $A$.

The roots of the characteristic polynomial are the ____________ of $A$.

# Example

The characteristic polynomial of $A = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}$ is:

So the eigenvalues of $A$ are:

# Characteristic Polynomial of $2 \times 2$ Matrices

Express the characteristic equation of

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

in terms of its determinant. What is the equation when $M$ is singular?

# Algebraic Multiplicity

**Definition**

The **algebraic multiplicity** of an eigenvalue is its multiplicity as a root of the characteristic polynomial.

**Example**

Compute the algebraic multiplicities of the eigenvalues for the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

# Geometric Multiplicity

**Definition**

The **geometric multiplicity** of an eigenvalue $\lambda$ is the dimension of $\mathrm{Null}(A - \lambda I)$.

1. Geometric multiplicity is always at least 1. It can be smaller than algebraic multiplicity.
2. Here is the basic example:

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

   $\lambda = 0$ is the only eigenvalue. Its algebraic multiplicity is 2, but the geometric multiplicity is 1.

# Example

Give an example of a $4 \times 4$ matrix with $\lambda = 0$ the only eigenvalue, but the geometric multiplicity of $\lambda = 0$ is one.

# Recall: Long-Term Behavior of Markov Chains

**Recall**:

- We often want to know what happens to a Markov Chain

$$\vec{x}_{k+1} = P\vec{x}_k, \quad k = 0, 1, 2, \dots$$

  as $k \to \infty$.

- If $P$ is regular, then there is a ______________________

**Now lets ask**:

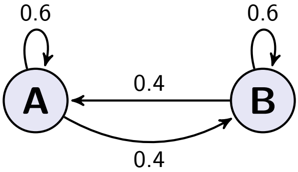
- If we don't know whether $P$ is regular, what else might we do to describe the long-term behavior of the system?
- What can eigenvalues tell us about the behavior of these systems?

# Example: Eigenvalues and Markov Chains

Consider the Markov Chain:

$$\vec{x}_{k+1} = P\vec{x}_k = \begin{pmatrix} 0.6 & 0.4 \\ 0.4 & 0.6 \end{pmatrix} \vec{x}_k, \quad k = 0, 1, 2, 3, \ldots, \quad \vec{x}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

This system can be represented schematically with two nodes, A and B:




**Goal**: use eigenvalues to describe the long-term behavior of our system.

What are the eigenvalues of $P$?

What are the corresponding eigenvectors of $P$?

Use the eigenvalues and eigenvectors of $P$ to analyze the long-term behaviour of the system. In other words, determine what $\vec{x}_k$ tends to as $k \to \infty$.

# Similar Matrices

**Definition**

Two $n \times n$ matrices $A$ and $B$ are **similar** if there is a matrix $P$ so that $A = PBP^{-1}$.

**Theorem**

If $A$ and $B$ similar, then they have the same characteristic polynomial.

If time permits, we will explain or prove this theorem in lecture. Note:

- Our textbook introduces similar matrices in Section 5.2, but doesn't have exercises on this concept until 5.3.
- Two matrices, $A$ and $B$, do not need to be similar to have the same eigenvalues. For example,

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

# Additional Examples (if time permits)

1. True or false.
   a) If $A$ is similar to the identity matrix, then $A$ is equal to the identity matrix.
   b) A row replacement operation on a matrix does not change its eigenvalues.
2. For what values of $k$ does the matrix have one real eigenvalue with algebraic multiplicity 2?

$$\begin{pmatrix} -3 & k \\ 2 & -6 \end{pmatrix}$$