1. Let X and Y be discrete random variables with joint PMF $p(x, y)$ given by

$$p(0,0) = p(0,1) = p(1,1) = p(1,2) = \frac{1}{4}.$$

(a) Find the conditional expectation $\mathbb{E}[X|Y = 1]$.

(b) Find the covariance $\mathrm{Cov}(X, Y)$.

(c) Are they independent?

(b) $\mathrm{Cov}(X,Y) \underset{\substack{\uparrow \\ \text{Formula}}}{=} \mathbb{E}[X \cdot Y] - \mathbb{E}[X] \cdot \mathbb{E}[Y] = \frac{3}{4} - \frac{1}{2} \cdot 1 = \frac{1}{4}$

$\mathbb{E}[X] = \sum_x \sum_y \boxed{x} \cdot p(x,y) = \sum_x x \cdot p_X(x)$

$p_X(x) = \sum_y p(x,y)$, $p_X(0) = p(0,0) + p(0,1) = \frac{1}{2}$

$p_X(1) = p(1,1) + p(1,2) = \frac{1}{2}$

$\mathbb{E}[X] = 0 \cdot \frac{1}{2} + 1 \cdot \frac{1}{2} = \frac{1}{2}$

$(X \sim \mathrm{Ber}(\frac{1}{2}))$

$p_Y(y) = \sum_x p(x,y)$

$p_Y(0) = \frac{1}{4} = p_Y(2)$, $p_Y(1) = \frac{2}{4} = \frac{1}{2}$

$\mathbb{E}[Y] = 0 \cdot \frac{1}{4} + 1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{4} = 1.$

$\mathbb{E}[X \cdot Y] = \sum_x \sum_y x \cdot y \; p(x,y)$

$= 0 \cdot 0 \cdot p(0,0) + 0 \cdot 1 \cdot p(0,1) + 1 \cdot 1 \cdot p(1,1)$
$+ 1 \cdot 2 \; p(1,2)$

$= \frac{3}{4}$

(a) $$\underbrace{E[X \mid Y=1]}_{\text{a new RV w/}} = 0 \cdot \frac{1}{2} + 1 \cdot \frac{1}{2} = \frac{1}{2}.$$

$$P_{X|Y}(x|1) = P(X=x \mid Y=1)$$

$$= \frac{P(X=x, Y=1)}{P(Y=1)} = \frac{\overset{\frac{1}{4}}{P(x,1)}}{P_Y(1)} \quad P_Y(1) = \frac{1}{2}$$

$$= \frac{1}{2} \quad \text{for } x=0,1 \qquad \text{for } x=0,1$$

(c) Are they Indep?

- $\boxed{P(x,y) \neq P_X(x) \cdot P_Y(y)}$

$$P(X \in A, Y \in B) = P(X \in A) \cdot P(Y \in B).$$

$$P(0,0) = \frac{1}{4}, \quad P_X(0) = \frac{1}{2}, \quad P_Y(0) = \frac{1}{4}$$

Dependent.

- $$\frac{P(x,y)}{P_Y(y)} = P_X(x)$$

$$P_{X|Y}(x|y) = P_X(x)$$

- Note If X, Y are independent, then

$$E[XY] = E[X] \cdot E[Y]$$

$$\left(\Longleftrightarrow \operatorname{Cov}(X,Y) = 0 \right)$$

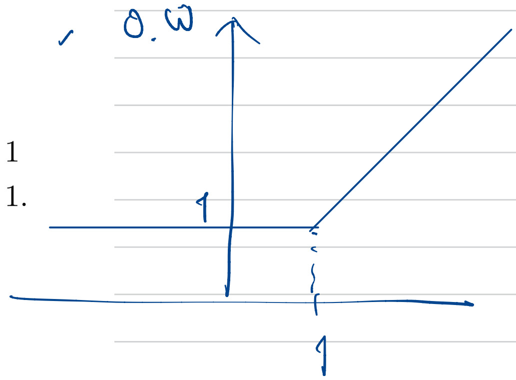
5. Let $X \sim \text{Exp}(2)$.

$$f(x) = \begin{cases} 2e^{-2x}, & x \geq 0 \\ 0, & \text{o.w.} \end{cases}$$

~~(a) For $0 < p < 1$, find the $(100p)$-th percentile of X.~~

(b) Let $h(x)$ be a function defined by

$$h(x) = \begin{cases} 1, & x \leq 1 \\ x, & x > 1. \end{cases}$$



Compute the expectation $\mathbb{E}[h(X)]$.

(c) Compute the expectation $\mathbb{E}[\max\{X, X^2\}]$.

(b) $$\mathbb{E}[h(X)] = \int_{-\infty}^{\infty} h(x) \cdot f(x)\,dx$$

$$= \int_0^{\infty} h(x)\, 2e^{-2x}\,dx$$

$$= \int_0^1 1 \cdot 2e^{-2x}\,dx + \int_1^{\infty} x \cdot 2e^{-2x}\,dx$$

$$= \left[-e^{-2x}\right]_0^1 + \left[-xe^{-2x} - \frac{1}{2}e^{-2x}\right]_1^{\infty}$$

$$= 1 - e^{-2} + \frac{1}{2}e^{-2} = 1 + \frac{1}{2}e^{-2}.$$

$$+ e^{-2}$$

(c) $$\mathbb{E}[\max\{X, X^2\}]$$

$$= \mathbb{E}[h(X)]$$

$$h(x) = \begin{cases} x^2 & x > 1 \\ x & 0 < x \leq 1 \end{cases}$$

$$= \int_0^1 x\, 2e^{-2x}\,dx + \int_1^{\infty} x^2\, 2e^{-2x}\,dx$$

$= \cdots$

$h(x) = \max\{x, x^2\}$

$x > x^2 \leftarrow$

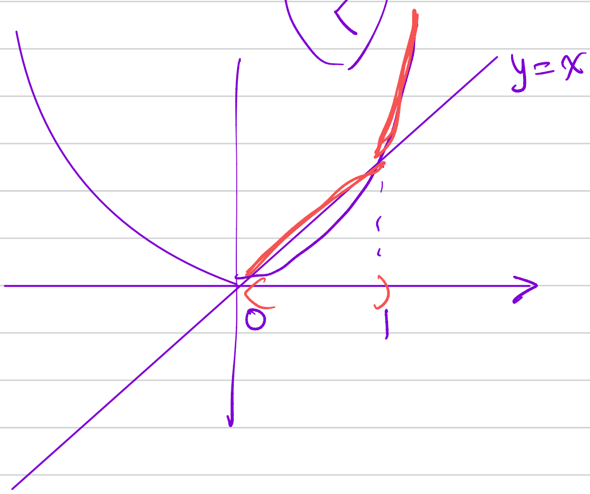
$<$

$x = x^2$

$x^2 - x = 0$

$x(x-1) = 0$

$x = 0, 1$



19. Let $X \sim N(0,4)$ and $W = X^2$. Find the PDF of W.

a function of X : a New RV

$h(x) = \begin{cases} x^2 \\ e^x \end{cases}$

$W = X^2$

$Z = e^X$

Start with the CDF of New one!

$$F_W(t) = P(W \le t) = P(X^2 \le t)$$

$$= \begin{cases} P(-\sqrt{t} \le X \le \sqrt{t}) & \text{if } t \ge 0 \\ 0 & \text{if } t < 0 \end{cases}$$

$$P(-\sqrt{t} \le X \le \sqrt{t}) = \int_{-\sqrt{t}}^{\sqrt{t}} \frac{1}{\sqrt{2\pi}\cdot 2} e^{-\frac{x^2}{8}} dx$$

$$F_W'(t) = f_W(t) = \begin{cases} (\qquad)' & t \ge 0 \\ 0, & t < 0 \end{cases}$$

$X \sim N(0,4)$ $\quad \mu = 0, \ \sigma^2 = 4$

$$\frac{1}{\sqrt{2\pi}\cdot\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

Q: $\frac{d}{dt}\left(\int_{-\sqrt{t}}^{\sqrt{t}} f(x)\, dx\right) = \frac{d}{dt}\left[F(\sqrt{t}) - F(-\sqrt{t})\right]$

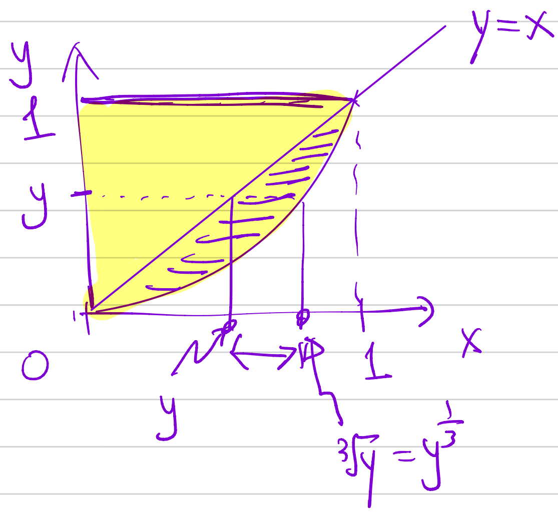
$$= f(\sqrt{t})\cdot(\sqrt{t})' - f(-\sqrt{t})\cdot(-\sqrt{t})'$$

26. Let X and Y have the joint PDF

$$f(x,y) = \frac{4}{3}, \quad \boxed{0 < x < 1, \quad x^3 < y < 1}$$

and 0 otherwise. Find $\mathbb{P}(X > Y)$.

$x^3 \le y$

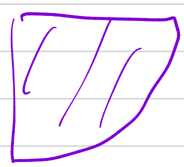
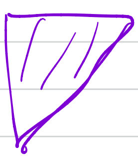


$$\mathbb{P}(X > Y) = \mathbb{P}((X,Y) \in A)$$

$$= \iint_A \frac{4}{3}\, dx\, dy \qquad A = \{x > y\}$$

$$\left(= \int_0^1 \int_y^{y^{1/3}} \frac{4}{3}\, dx\, dy \right)$$

$$= \frac{4}{3} \cdot \text{Area}(A)$$

$= \frac{4}{3}\Big($  $-$  $\Big) = 1 - \frac{2}{3} = \frac{1}{3}.$

$= \frac{3}{4}$, $= \frac{1}{2}$

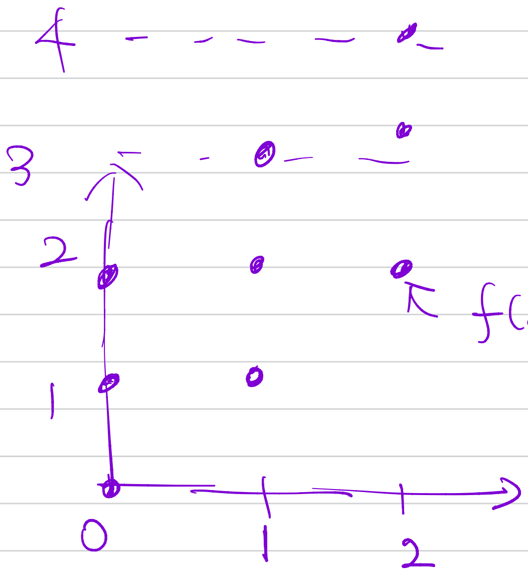
(b) Find the least square regression line.

24. Let X and Y be discrete random variables with joint PMF $f(x,y) = \frac{1}{9}$ for $(x,y) \in S$ where

$$S = \{(x,y) : 0 \le x \le 2, x \le y \le x+2, x, y \text{ are integers}\}.$$

Find $f_X(x)$, $f_{Y|X}(y|1)$, $\mathbb{E}[Y|X=1]$, and $f_Y(y)$.

9 pts



$$f_X(x) = \frac{1}{3} \quad \text{for} \quad x = 0, 1, 2$$

$$f_{Y|X}(y|1) = \frac{f(1,y)}{f_X(1)} = \frac{\frac{1}{9}}{\frac{1}{3}} = \frac{1}{3}$$

for $y = 1, 2, 3$

$$\mathbb{E}[Y|X=1] = 2$$

$$= \frac{1}{3} \cdot 1 + \frac{1}{3} \cdot 2 + \frac{1}{3} \cdot 3$$

$$f_Y(y) = \begin{cases} \frac{1}{9} & y = 0, 4 \\ \frac{2}{9} & y = 1, 3 \\ \frac{3}{9} & y = 2 \end{cases}$$

$T_1 \sim \text{Exp}(10)$ $\qquad$ $T_2 \sim \text{Exp}(20)$

$T = \min(T_1, T_2)$: a new RV

CDF of T

$$F_T(t) = \mathbb{P}(T \leq t)$$
$$= 1 - \mathbb{P}(\underline{T > t})$$
$$= 1 - \mathbb{P}(\underline{\min(T_1, T_2) > t})$$
$$= 1 - \mathbb{P}(T_1 > t, T_2 > t)$$

Indep.

$$= 1 - \mathbb{P}(\underline{T_1 > t}) \cdot \mathbb{P}(T_2 > t)$$
$$= 1 - e^{-10t} \cdot e^{-20 \cdot t}$$
$$= 1 - e^{-30t}$$

Execise what if $T = \max\{T_1, T_2\}$