SYLLABUS FOR MATH 595
TOPIC IN FUNCTIONAL AND GEOMETRIC INEQUALITIES

DAESUNG KIM

General information.
- Instructor: Daesung Kim (daesungk@illinois.edu), 247B Illini Hall
- Time and place: MWF 10–10:50am, Henry Administration Bldg 137
- References
  - *Analysis* 2nd edition by Lieb and Loss (for rearrangement inequalities and Hardy-Littlewood-Sobolev inequalities)
  - *Topics in Optimal Transportation* by Cédric Villani (for optimal transport and log-Sobolev inequality)
  - further references will be given in lecture notes.
- Course webpage: https://daesungk.github.io/math595-s22-uiuc

Goal and Plan. Functional and geometric inequalities play an important role in several research areas such as calculus of variations, PDEs, harmonic analysis, probability, convex geometry, and so on. In particular, there have been of great interest in studying sharp forms and quantitative forms of such inequalities. In general, functional and geometric inequalities can be formulated as

$$\mathcal{G}(u) \geq c\mathcal{H}(u)$$

where $\mathcal{G}$ and $\mathcal{H}$ are nonnegative functional on a class of admissible functions (or sets) and c is a positive constant. For example, the Sobolev inequality is the case where $\mathcal{G}(u) = \|\nabla u\|_2$ and $\mathcal{H}(u) = \|u\|_q$, $q = 2d/(d-2)$ on $\mathbb{R}^d$, $d \geq 3$. We are interested in the following questions regarding functional and geometric inequalities:

(i) Is the inequality *sharp*? Namely, can we find a constant c satisfying $\mathcal{G}(u) \geq c\mathcal{H}(u)$ such that for all $\lambda > c$, there exists u_λ such that $\mathcal{G}(u_\lambda) < \lambda\mathcal{H}(u_\lambda)$? Such constant is called the sharp constant.

(ii) Once the sharp constant is obtained, can we chracterize the equality cases? In other words, the question is to find a collection of u, say X_0 such that $\mathcal{G}(u) = c\mathcal{H}(u)$ if and only if $u \in X_0$. Such u is called an optimizer.

(iii) After identifying the class of optimizers, a following question is whether a function u is close to the class of optimizers when u almost attains the equality. This question can be answered by a quantitative version of the inequality: a lower bound of the deficit $\delta(u) = \mathcal{G}(u) - \mathcal{H}(u)$ in terms of a distance between u and the class of the optimizers.

In this course, we discuss the above questions for the following inequalities: Sobolev inequality and Hardy–Littlewood–Sobolev inequality; Log-Sobolev inequality and related inequalities; Spectral inequalities (if time permits). We will see that these questions were answered by a new proof of inequalities. Thus, we will focus on exploring various proofs. Here is the temporary plan:

Part 1. Hardy-Littlewood-Sobolev inequality.
- Sobolev inequality
- Non-sharp proofs
- Rearrangement inequalities
- Existence of optimizers

- Conformal invariance
- Equality cases and sharp constants
- Stability results
- Duality

Part 2.Log Sobolev inequality.

- Two point process
- Semigroup proof
- Optimal transportation proof by Cordero-Erausquin
- Equality cases by Carlen
- Stability results

Presentation. Each student will give a 20 min presentation in class. Temporary schedule for presentations is March 7–11 and May 2–4. One can choose one of the following topics for presentations:

- the references given in "further reading" in the lecture note, or
- any papers related to course material, or
- material covered in class.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN
E-mail address: daesungk@illinois.edu