# Math 3215: Intro to Probability and Statistics

## Exam 1 Solution

1. Let $A$ and $B$ be two events such that $\mathbb{P}(A) = 1/2$, $\mathbb{P}(B) = 1/4$, and $\mathbb{P}(A \cup B) = 7/12$.
   (a) (5 points) Find $\mathbb{P}(A \cap B)$.
   ANS: $\mathbb{P}(A \cap B) = 1/6$.
   (b) (5 points) Find $\mathbb{P}(A^c|B)$.
   ANS: $\mathbb{P}(A^c|B) = 1/3$.
   (c) (5 points) Is $A$ and $B$ independent? Justify your answer.
   ANS: No, because $\mathbb{P}(A \cap B) = 1/6 \neq \mathbb{P}(A)\mathbb{P}(B)$.

2. Consider an urn containing 12 balls, of which 5 are green and 7 are red. A sample of size 9 is to be drawn at random. Let $A$ be the event that first 3 balls are green, and $B$ the event that exactly 4 green balls are drawn.
   (a) (8 points) If 9 balls are drawn **with replacement**, what is $\mathbb{P}(A|B)$?
   ANS: $\mathbb{P}(A|B) = \frac{1}{21}$.
   (b) (7 points) If 9 balls are drawn **without replacement**, what is $\mathbb{P}(A|B)$? (Hint: You may use Bayes' formula.)
   ANS: $\mathbb{P}(A|B) = \frac{1}{21}$.

3. (10 points) A hat contains 10 coins, where 9 are fair but one is double-headed (always landing Heads). A coin is chosen uniformly at random. The chosen coin is flipped 4 times. Find the probability that the chosen coin is double-headed given that it lands Heads all 4 times.
   ANS: Let $A$ be the event that the chosen coin is double-headed and $B$ the event that it lands Heads all 4 times. Then $\mathbb{P}(A|B) = \frac{16}{25}$.

4. Suppose $X$ is a random variable taking values in $S = \{0, 1, 2, 3, \ldots\}$ with PMF

$$f(k) = \begin{cases} c, & k = 0, \\ \frac{1}{3^k k!}, & k = 1, 2, \cdots. \end{cases}$$

   (a) (5 points) Find the value of $c$ that would make this a valid probability model.
   ANS: $c = 2 - e^{1/3}$
   (b) (5 points) Find $\mathbb{E}[X]$.
   ANS: $\mathbb{E}[X] = \frac{1}{3}e^{1/3}$.
   (c) (5 points) Find $\mathbb{E}[3^X]$.
   ANS: $\mathbb{E}[3^X] = 1 - e^{1/3} + e$.

5. Let $X$ be a random variable with moment generating function given by $M(t) = c(1-2t)^{-5}$ for $t < \frac{1}{2}$ for some constant $c$.
   (a) (5 points) Find the constant $c$.
   ANS: $c = 1$.
   (b) (5 points) Find the expectation of $X$.
   ANS: $\mathbb{E}[X] = 10$.
   (c) (5 points) Find the variance of $X$.
   ANS: $\mathrm{Var}(X) = 20$.

6. An urn contains 4 white and 5 black balls. We randomly choose 4 balls. If 2 of them are white and 2 are black, we stop. If not, we replace the balls in the urn and again randomly select 4 balls. This continues until exactly 2 of the 4 chosen are white.

(a) (7 points) What is the probability that we shall make exactly 10 selections?

ANS: $(11/21)^9(10/21)$.

(b) (8 points) What is the probability that we select 4 balls more than 5 times?

ANS: $(11/21)^5$.

7. In a weekly lottery you have probability .05 of winning a prize with a single ticket. Suppose you buy 1 ticket per week for 20 weeks and let $X$ be the number of winning tickets.

(a) (5 points) Find $\mathbb{E}[X]$.

ANS: $\mathbb{E}[X] = 1$.

(b) (5 points) What is the probability that among the twenty tickets you buy more than 3 winning tickets? Use the corresponding tables to find an approximate value for this probability.

ANS: $\mathbb{P}(X > 3) = 0.0159$.

(c) (5 points) Using a Poisson approximation, write down an expression for the probability that among the twenty tickets you buy more than 3 winning tickets. Use the corresponding tables to find an approximate value for this probability.

ANS: $\mathbb{P}(X > 3) = 0.019$.