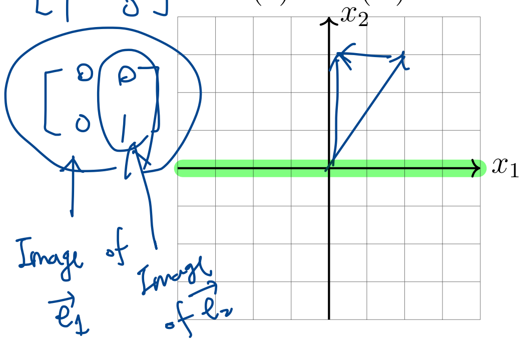
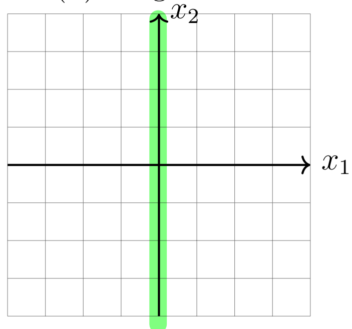
Midterm 2 Lecture Review Activity, Math 1554

1. (3 points) T_A is the linear transform $x \to Ax$, $A \in \mathbb{R}^{2\times 2}$, that projects points in $\mathbb{R}^2$ onto the x_2-axis. Sketch the nullspace of A, the range of the transform, and the column space of A. How are the range and column space related to each other?

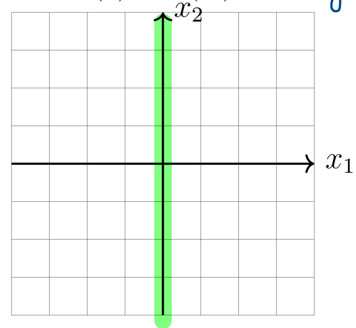
(a) Null(A)



(b) range of T_A



(c) Col(A) = Range(T_A)



2. Indicate **true** if the statement is true, otherwise, indicate **false**.

		true	false
a)	$S = \{\vec{x} \in \mathbb{R}^3 \,|\, x_1 = a, x_2 = 4a, x_3 = x_1 x_2\}$ is a subspace for any $a \in \mathbb{R}$.	○	○
b)	If A is square and non-zero, and $A\vec{x} = A\vec{y}$ for some $\vec{x} \neq \vec{y}$, then $\det(A) \neq 0$.	○	○

3. If possible, write down an example of a matrix or quantity with the given properties. If it is not possible to do so, write *not possible*.

(a) A is 2×2, Col$A$ is spanned by the vector $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and $\dim(\text{Null}(A)) = 1$. $A = \begin{pmatrix} & \\ & \end{pmatrix}$

(b) A is 2×2, Col$A$ is spanned by the vector $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and $\dim(\text{Null}(A)) = 0$. $A = \begin{pmatrix} & \\ & \end{pmatrix}$

(c) A is in RREF and $T_A: \mathbb{R}^3 \to \mathbb{R}^3$. The vectors u and v are a basis for the range of T.

$$u = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, v = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, A = \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}$$

4. Indicate whether the situations are possible or impossible by filling in the appropriate circle.

		possible	impossible
4.i)	Vectors $\vec{u}$ and $\vec{v}$ are eigenvectors of square matrix A, and $\vec{w} = \vec{u}+\vec{v}$ is also an eigenvector of A.	○	○
4.ii)	$T_A = A\vec{x}$ is one-to-one, $\dim(\text{Col}(A)) = 4$, and $T_A : \mathbb{R}^3 \to \mathbb{R}^4$.	○	○

5. (2 points) Fill in the blanks.

   (a) If A is a 6×4 matrix in RREF and $\text{rank}(A) = 4$, what is the rank of A^T? ☐

   (b) $T_A = A\vec{x}$, where $A \in \mathbb{R}^{2\times 2}$, is a linear transform that first rotates vectors in $\mathbb{R}^2$ clockwise by π radians about the origin, then scales their x-component by a factor of 3, then projects them onto the x_1-axis. What is the value of $\det(A)$? ☐

6. (3 points) A virus is spreading in a lake. Every week,

   - 20% of the healthy fish get sick with the virus, while the other healthy fish remain healthy but could get sick at a later time.
   - 10% of the sick fish recover and can no longer get sick from the virus, 80% of the sick fish remain sick, and 10% of the sick fish die.

   Initially there are exactly 1000 fish in the lake.

   a) What is the stochastic matrix, P, for this situation? Is P regular?

   b) Write down any steady-state vector for the corresponding Markov-chain.

Midterm 2. Your initials: ______________

You do not need to justify your reasoning for questions on this page.

1. (a) (8 points) Suppose A is an $m \times n$ matrix and $\vec{b} \in \mathbb{R}^m$ unless otherwise stated. Select **true** if the statement is true for all choices of A and $\vec{b}$. Otherwise, select **false**.

true	false	
(X)	( )	If A, B and C are $n \times n$ matrices, B is invertible and $AC = B$, then C is invertible.
(X)	( )	If $A = LU$ is an LU-factorization of a square matrix A, then $\det(A) = \det(U)$
(X)	( )	If $\vec{x}$ is a vector in $\mathbb{R}^3$ and B is a basis for $\mathbb{R}^3$, then $[\vec{x}]_B$ has 3 entries.
(X)	( )	If A, B and C are $n \times n$ matrices, A is invertible and $AB = AC$, then $B = C$.
( )	(X)	If $A \in \mathbb{R}^{m\times n}$ and $\vec{b} \in \mathbb{R}^m$, then the set of solutions $\vec{x}$ to the system $A\vec{x} = \vec{b}$ is a subspace of $\mathbb{R}^n$.
( )	(X)	The set of all probability vectors in $\mathbb{R}^n$ is a subspace of $\mathbb{R}^n$.
( )	( )	If two matrices A, B share an eigenvector $\vec{v}$, with eigenvalue λ for matrix A and eigenvalue μ for the matrix B, then $\vec{v}$ is an eigenvector of the matrix $(A + 2B)$ with eigenvalue $\lambda + 2\mu$.
( )	( )	For any 2×2 real matrix A, we have $\det(-A) = -\det(A)$.

(b) (4 points) Indicate whether the following situations are possible or impossible.

possible	impossible	
( )	( )	A matrix $A \in \mathbb{R}^{n\times n}$ such that A is invertible and A^T is singular.
( )	( )	A 3×3 matrix A with $\dim(\text{Null}(A)) = 0$ such that the system $A\vec{x} = \begin{pmatrix}1\\0\\0\end{pmatrix}$ has no solution.
( )	( )	$T : \mathbb{R}^3 \to \mathbb{R}^3$ that is onto and its standard matrix has determinant equal to -1.
( )	( )	Two square matrices A, B with $\det(A)$ and $\det(B)$ both non-zero, and the matrix AB is singular.

Midterm 2. Your initials: ______________

You do not need to justify your reasoning for questions on this page.

2. (2 points) Suppose A and B are invertible $n \times n$ matrices. Find the inverse of the partitioned matrix

$$\begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}^{-1} = \begin{pmatrix} \_\_\_\_ & \_\_\_\_ \\ \_\_\_\_ & \_\_\_\_ \end{pmatrix}.$$

3. (2 points) Suppose A is a $m \times n$ matrix and B is $m \times 5$ matrix. Find the dimensions of the matrix C in the block matrix

partitioned

$$\begin{pmatrix} A & B \\ I_n & C \end{pmatrix}.$$

C has [n] rows and [5] columns.

7. (5 points) **Show all work for problems on this page.**
Given that 4 is an eigenvalue of the matrix

$$A = \begin{pmatrix} 6 & -2 & 2 \\ 2 & 2 & -2 \\ 1 & -1 & 4 \end{pmatrix},$$

find an eigenvector $\vec{v}$ of A such that $A\vec{v} = 4\vec{v}$.

$\vec{v}$= [ ]

8. (6 points) Find the LU-factorization of

$$A = \begin{pmatrix} 1 & 2 & 5 \\ 1 & -1 & 8 \\ 2 & 8 & 6 \end{pmatrix}.$$

$L =$ [ ]

$U =$ [ ]

$$P = \underset{B}{\begin{bmatrix} 1 & 0 & 0 \\ 1 & -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 \end{bmatrix}} \underset{A^{-1}}{\begin{bmatrix} 1 & 0 & -1 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix}^{-1}}$$

$PA = B$

$PA \cdot A^{-1} = B \cdot A^{-1}$

$P = B \cdot A^{-1}$

$$P \cdot \begin{bmatrix} | & | & | \\ \vec v_1 & \vec v_2 & \vec v_3 \\ | & | & | \end{bmatrix} = \begin{bmatrix} | & | & 0 \\ 1\cdot \vec v_1 & \frac{1}{2}\vec v_2 & 0 \\ | & | & 0 \end{bmatrix}$$

$$P\vec v_1 = 1\cdot \vec v_1, \quad P\vec v_2 = \frac{1}{2}\cdot \vec v_2, \quad P\vec v_3 = 0\cdot \vec v_3 = 0$$

Midterm 2. Your initials: ______________

9. (6 points) **Show all work for problems on this page.**
Consider the Markov chain $\vec{x}_{k+1} = P\vec{x}_k,\ k = 0, 1, 2, \ldots$.
Suppose P has eigenvalues $\lambda_1 = 1$, $\lambda_2 = 1/2$ and $\lambda_3 = 0$. Let $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$ be eigenvectors corresponding to λ_1, λ_2, and λ_3, respectively:

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, \quad \vec{v}_3 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}.$$

Steady state vector = eigenvector corresponding to $\lambda = 1$.

Note: you may leave your answers as linear combinations of the vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$.

(i) If $\vec{x}_0 = \frac{1}{2}\vec{v}_1 + \frac{1}{2}\vec{v}_2$, then what is $\vec{x}_3$?

$$\vec x_1 = P x_0 = P\cdot\left(\tfrac{1}{2}v_1 + \tfrac{1}{2}v_2\right) = \tfrac{1}{2}\cdot P v_1 + \tfrac{1}{2}\cdot P\cdot v_2 = \tfrac{1}{2}v_1 + \tfrac{1}{2}\cdot\left(\tfrac{1}{2}\right)v_2$$

$$\vec x_2 = P\cdot \vec x_1 = P\cdot\left(\tfrac{1}{2}v_1 + \tfrac{1}{2}\cdot\left(\tfrac{1}{2}\right)\cdot v_2\right) = \tfrac{1}{2}v_1 + \left(\tfrac{1}{2}\right)^3 v_2$$

$$\vec x_3 = P\vec x_2 = \tfrac{1}{2}v_1 + \left(\tfrac{1}{2}\right)^4 v_2$$

$\vec{x}_3 =$ [ ]

(ii) If $\vec{x}_0 = \begin{pmatrix} 1/4 \\ 1/2 \\ 1/4 \end{pmatrix}$, then what is $\vec{x}_1$?
Hint: write $\vec{x}_0$ as a linear combination of $\vec{v}_1, \vec{v}_2, \vec{v}_3$.

$$\vec x_0 = a\cdot\vec v_1 + b\cdot\vec v_2 + c\vec v_3 \qquad \text{Find } a, b, c$$

$\vec{x}_1 =$ [ ]

$$\begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ \frac{1}{4} \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & \frac{1}{4} \\ 1 & -1 & 1 & \frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{4} \end{array}\right] \longrightarrow a = \tfrac{1}{2},\ b, c = \ldots$$

(iii) If $\vec{x}_0 = \begin{pmatrix} 1/4 \\ 1/2 \\ 1/4 \end{pmatrix}$, then what is $\vec{x}_k$ as $k \to \infty$?

$$x_0 = a\cdot v_1 + b\cdot v_2 + c\cdot v_3$$

$$x_k = a\, 1^k v_1 + b\cdot\left(\tfrac{1}{2}\right)^k v_2 + c\cdot 0^k v_3$$

($b\cdot(\frac{1}{2})^k v_2 \to 0$, $c\cdot 0^k v_3 \to 0$)

$$\longrightarrow a\cdot v_1 : \text{prob. vector.} \quad = a\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ a \\ 0 \end{bmatrix}$$

$\lim_{k\to\infty} \vec{x}_k =$ $\frac{1}{2}v_1$

$x_k = P^k x_0 = P^k (a v_1 + b v_2 + c v_3) = a \cdot 1^k v_1 + b \cdot (\frac{1}{2})^k \cdot v_2 + c \cdot 0^k v_3$

Fact $\vec{x}$: prob. vector. $\Rightarrow$ $P \cdot \vec{x}$: prob. vector

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