$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$

$\lambda^2 - \lambda - 1 = 0$

In-Class Final Exam Review Set A, Math 1554, Fall 2019

1. Indicate whether the statements are true or false.

$Ax = b$ $\quad m \times n$ $\quad n > m$

true	false	
⊗	○	If a linear system has more unknowns than equations, then the system has either no solutions or infinitely many solutions.
○	⊗	A $n \times n$ matrix A and its echelon form E will always have the same eigenvalues.
⊗	○	$x^2 - 2xy + 4y^2 \geq 0$ for all real values of x and y. $= Q(x,y)$ $\quad A = \begin{bmatrix} 1 & -1 \\ -1 & 4 \end{bmatrix}$ $\quad \lambda^2 - 5\lambda + 3 = 0$
⊗	○	If matrix A has linearly dependent columns, then $\dim((\text{Row}A)^{\perp}) > 0$.
⊗	○	If λ is an eigenvalue of A, then $\dim(\text{Null}(A - \lambda I)) > 0$.
⊗	○	If A has QR decomposition $A = QR$, then $\text{Col}A = \text{Col}Q$.
⊗	○	If A has LU decomposition $A = LU$, then $\text{rank}(A) = \text{rank}(U)$.
⊗	○	If A has LU decomposition $A = LU$, then $\dim(\text{Null}\,A) = \dim(\text{Null}\,U)$.

2. Give an example of the following.

   i) A 4×3 lower triangular matrix, A. such that $\text{Col}(A)^{\perp}$ is spanned by the vector $\vec{v} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 1 \end{pmatrix}$. $\quad A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{pmatrix}$

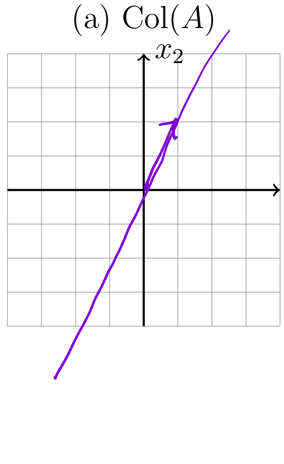
   $\text{Nul}(A^T)$ $\quad \begin{array}{c} \\ 3 \end{array} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -3 \end{bmatrix} = A^T$ (4)

   ii) A 3×4 matrix A, that is in RREF, and satisfies $\dim\left((\text{Row}\,A)^{\perp}\right) = 2$ and $\dim\left((\text{Col}\,A)^{\perp}\right) = 2$. $A = \begin{pmatrix} \text{NP.} \end{pmatrix}$

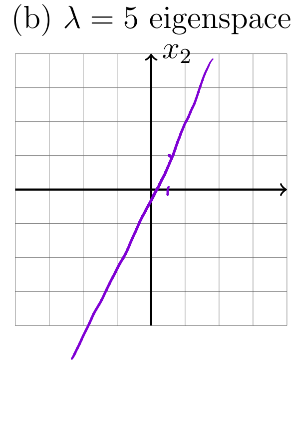
   $\text{Nul}(A)$ $\quad \dim(\text{Col}(A)) = 1$ $\quad \text{Col}(A) \subseteq \mathbb{R}^3$

3. (3 points) Suppose $A = \begin{pmatrix} 3 & 1 \\ 6 & 2 \end{pmatrix}$. On the grid below, sketch a) $\text{Col}(A)$, and b) the eigenspace corresponding to eigenvalue $\lambda = 5$.

(a) Col(A)



(b) $\lambda = 5$ eigenspace $= \text{Nul}(A - 5I)$



$= \text{Nul}\begin{bmatrix} -2 & 1 \\ 6 & -3 \end{bmatrix}$

$= \text{Nul}\begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 0 \end{bmatrix}$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}x_2 \\ x_2 \end{bmatrix}$

$= x_2 \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$

Eqn Var

4. Fill in the blanks.
   (a) If $A \in \mathbb{R}^{M\times N}$, $M < N$, and $A\vec{x} = 0$ does not have a non-trivial solution, ~~how many pivot columns does A have?~~ $\boxed{}$
   (b) Consider the following linear transformation.

$$T(x_1, x_2) = (2x_1 - x_2,\ 4x_1 - 2x_2,\ x_2 - 2x_1)\,. = \overset{3\times 2}{A} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

The domain of T is $\boxed{}$. The image of $\vec{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ under $T(\vec{x})$ is $\begin{pmatrix} \ \end{pmatrix}$. The co-domain of T is $\boxed{}$. The range of T is:

$$\text{Col}(A) \qquad \underline{A} = \begin{bmatrix} T(e_1) & T(e_2) \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 4 & -2 \\ -2 & 1 \end{bmatrix} \rightarrow \cdots \begin{bmatrix} \ \end{bmatrix}$$

5. Four points in $\mathbb{R}^2$ with coordinates (t, y) are $(0, 1)$, $(\frac{1}{4}, \frac{1}{2})$, $(\frac{1}{2}, -\frac{1}{2})$, and $(\frac{3}{4}, -\frac{1}{2})$. Determine the values of c_1 and c_2 for the curve $y = c_1 \cos(2\pi t) + c_2 \sin(2\pi t)$ that best fits the points. Write the values you obtain for c_1 and c_2 in the boxes below.

$$c_1 = \boxed{} \qquad c_2 = \boxed{}$$

$$1 = c_1 \cdot \cos(0) + c_2 \sin(0) = 1\cdot c_1 + 0\cdot c_2$$

$$\frac{1}{2} = c_1 \cdot \cos\left(\frac{\pi}{2}\right) + c_2 \sin\left(\frac{\pi}{2}\right) = 0\, c_1 + 1\, c_2$$

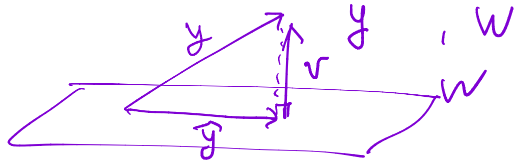
$$-\frac{1}{2} = c_1 \cos(\pi) + c_2 \sin(\pi) = -1\cdot c_1 + 0\cdot c_2$$

$$-\frac{1}{2} = c_1 \cos\left(\frac{3\pi}{2}\right) + c_2 \sin\left(\frac{3\pi}{2}\right) = 0\cdot c_1 - 1\cdot c_2$$

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$$

$$Ax = b$$
$$\Downarrow$$
$$A^TAx = A^Tb.$$

Normal Eqn.



$y = \underset{\underset{\mathrm{proj}_W(y)}{\parallel}}{\hat{y}} + \underline{w^\perp} \in W^\perp$

In-Class Final Exam Review Set B, Math 1554, Fall 2019

1. Indicate whether the statements are true or false.

true	false	
☒	◯	For any vector $\vec{y} \in \mathbb{R}^2$ and subspace W, the vector $\vec{v} = \vec{y} - \mathrm{proj}_W\vec{y}$ is orthogonal to W.
◯	☒	If A is $m \times n$ and has linearly dependent columns, then the columns of A cannot span $\mathbb{R}^m$.
◯	☒	If a matrix is invertible it is also diagonalizable. $[1 \quad 1]$ $\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -k \\ 0 & 1 \end{bmatrix}$
☒	◯	If E is an echelon form of A, then $\mathrm{Null}\,A = \mathrm{Null}\,E$. Null space doesn't change along row op.
◯	☒	If the SVD of $n \times n$ singular matrix A is $A = U\Sigma V^T$, then $\mathrm{Col}A \neq \underline{\mathrm{Col}U}$. $\mathbb{R}^n$
◯	◯	If the SVD of $n \times n$ matrix A is $A = U\Sigma V^T$, $r = \mathrm{rank}A$, then the first r columns of V give a basis for $\mathrm{Null}A$.

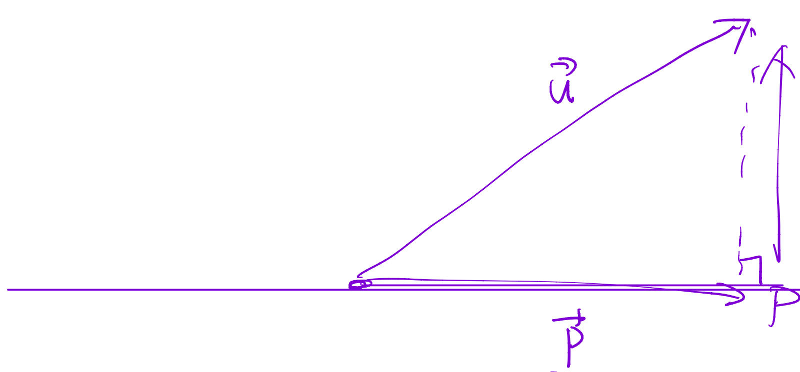
2. Give an example of:

a) a vector $\vec{u} \in \mathbb{R}^3$ such that $\mathrm{proj}_{\vec{p}}\vec{u} = \vec{p}$, where $\vec{u} \neq \vec{p}$, and $\vec{p} = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$: $\quad \vec{u} = \begin{pmatrix} a \\ 2 \\ c \end{pmatrix}$

b) an upper triangular 4×4 matrix A that is in RREF, 0 is its only eigenvalue, and its corresponding eigenspace is 1-dimensional. $\quad A = \begin{pmatrix} \quad \end{pmatrix}$

c) A 3×4 matrix, A, and $\mathrm{Col}(A)^\perp$ is spanned by $\begin{pmatrix} 1 \\ -3 \\ -4 \end{pmatrix}$.

d) A 2×2 matrix in RREF that is diagonalizable and not invertible.

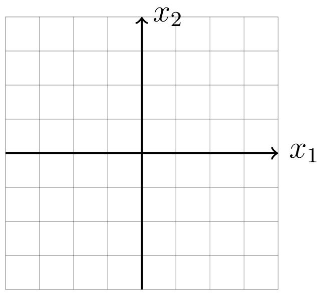


$\underline{w} \neq 0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

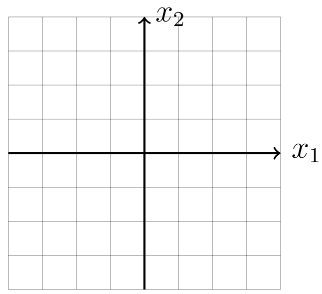
$w \cdot p = 0$

3. Suppose $A = \begin{pmatrix} 2 & -1 \\ 4 & -2 \end{pmatrix}$. On the grid below, sketch a) the range of $x \to Ax$, b) $(\text{Col}\, A)^{\perp}$, (c) set of solutions to $A\vec{x} = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$.

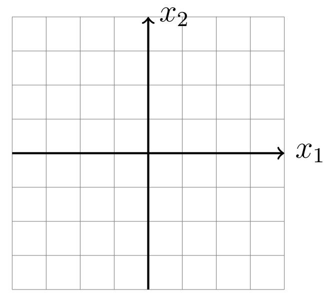
(a) range



(b) $(\text{Col}\, A)^{\perp}$



(c) solutions



4. Matrix A is a 2×2 matrix whose eigenvalues are $\lambda_1 = \frac{1}{2}$ and $\lambda_2 = 1$, and whose corresponding eigenvectors are $\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$. Calculate

   1. $A(\vec{v}_1 + 4\vec{v}_2)$
   2. A^{10}
   3. $\lim_{k \to \infty} A^k(\vec{v}_1 + 4\vec{v}_2)$

In-Class Final Exam Review Set C, Math 1554, Fall 2019

1. Indicate whether the statements are possible or impossible.

possible	impossible	
○	○	$Q(\vec{x}) = \vec{x}^T A\vec{x}$ is a positive definite quadratic form, and $Q(\vec{v}) = 0$, where $\vec{v}$ is an eigenvector of A.
○	○	The maximum value of $Q(\vec{x}) = ax_1^2 + bx_2^2 + cx_3^2$, where $a > b > c$, for $\vec{x} \in \mathbb{R}^3$, subject to $\|\|\vec{x}\|\| = 1$, is not unique.
○	○	The location of the maximum value of $Q(\vec{x}) = ax_1^2 + bx_2^2 + cx_3^2$, where $a > b > c$, for $\vec{x} \in \mathbb{R}^3$, subject to $\|\|\vec{x}\|\| = 1$, is not unique.
○	○	A is 2×2, the algebraic multiplicity of eigenvalue $\lambda = 0$ is 1, and $\dim(\mathrm{Col}(A)^\perp)$ is equal to 0.
○	○	Stochastic matrix P has zero entries and is regular.
○	○	A is a square matrix that is not diagonalizable, but A^2 is diagonalizable.
○	○	The map $T_A(\vec{x}) = A\vec{x}$ is one-to-one but not onto, A is $m \times n$, and $m < n$.

2. Transform $T_A = A\vec{x}$ reflects points in $\mathbb{R}^2$ through the line $y = 2 + x$. Construct a standard matrix for the transform using homogeneous coordinates. Leave your answer as a product of three matrices.

3. Fill in the blanks.

   (a) $T_A = A\vec{x}$, where $A \in \mathbb{R}^{2\times 2}$, is a linear transform that first rotates vectors in $\mathbb{R}^2$ clockwise by $\pi/2$ radians about the origin, then reflects them through the line $x_1 = x_2$. What is the value of $\det(A)$? $\boxed{}$

   (b) B and C are square matrices with $\det(BC) = -5$ and $\det(C) = 2$. What is the value of $\det(B)\det(C^4)$? $\boxed{}$

   (c) A is a 6×4 matrix in RREF, and $\text{rank}(A) = 4$. How many different matrices can you construct that meet these criteria? $\boxed{}$

   (d) $T_A = A\vec{x}$, where $A \in \mathbb{R}^{2\times 2}$, projects points onto the line $x_1 = x_2$. What is an eigenvalue of A equal to? $\boxed{}$

   (e) If an eigenvalue of A is $\frac{1}{3}$, what is one eigenvalue of A^{-1} equal to? $\boxed{}$

   (f) If A is 30×12 and $A\vec{x} = \vec{b}$ has a unique least squares solution $\hat{x}$ for every $\vec{b}$ in $\mathbb{R}^{30}$, the dimension of Null$A$ is $\boxed{}$.

4. A is a 2×2 matrix whose nullspace is the line $x_1 = x_2$, and $C = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$. Sketch the nullspace of $Y = AC$.

5. Construct an SVD of $A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$. Use your SVD to calculate the condition number of A.

LU Decomposition

$$A \xrightarrow{E_1} \xrightarrow{E_2} \xrightarrow{\quad} \cdots \xrightarrow{E_k} \overset{\text{REF}}{U}$$

$$\underbrace{E_k \cdots E_2 E_1}\, A = U$$

upper triangular

$$\underset{m \times n}{A} = \underbrace{E_1^{-1} \cdots E_{k-1}^{-1} E_k^{-1}}_{= L} \cdot U$$

lower triangular

$$L = \begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ * & & & 1 \end{bmatrix}_{m \times m} \qquad U = \begin{bmatrix} & * \\ 0 & \end{bmatrix}_{m \times n}$$

$$Ax = b \longrightarrow L\underbrace{\boxed{Ux}}_{y} = b \longrightarrow \begin{cases} L \cdot y = b \\ Ux = y \end{cases}$$

L

$$\left[\begin{array}{cccc|c} 1 & & & 0 & \\ & 1 & & & b \\ & & \ddots & & \\ * & & & 1 & \end{array}\right]$$

$y_1 = b_1$

$y_2 = \cdots$

$\vdots$

$y_m = \cdots$

$$\left[\begin{array}{cc|c} & * & y_1 \\ 0 & & \vdots \\ & & y_m \end{array}\right]$$

$b_1 = \cdots$

$\vdots$

$b_m = \cdots$

Diagonalization

A

$\{v_1, \dots, v_n\}$ eigenvector & basis for $\mathbb{R}^n$

$$\Rightarrow \quad A = P \cdot D \cdot P^{-1}$$

$$P = [v_1, \dots, v_n] \qquad D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

$P^{-1} = P^T \iff P$ is an orthogonal matrix

$\iff \{v_1, \dots, v_n\}$ orthonormal

$\iff A$: symm.

Spectral Thm.

$\iff A = P \cdot D \cdot P^T$