W subspace in $\mathbb{R}^n$

$\dim(W) + \dim(W^{\perp}) = n$

Midterm 3 Lecture Review Activity, Math 1554

1. Indicate **true** if the statement is true, otherwise, indicate **false**.

	true	false
a) If S is a two-dimensional subspace of $\mathbb{R}^{50}$, then the dimension of $S^{\perp}$ is 48.	X	○
b) An eigenspace $= \text{Null}(A - \lambda I)$ is a subspace spanned by a single eigenvector.	○	X
c) The $n \times n$ zero matrix can be diagonalized.	X	○
d) A least-squares line that best fits the data points $(0, y_1), (1, y_2), (2, y_3)$ is unique for any values y_1, y_2, y_3.	X	○

eigenvectors corresponding to a single eigenvalue

2. If possible, give an example of the following.

2.1) A matrix, A, that is in echelon form, and $\dim\left((\text{Row}A)^{\perp}\right) = 2$, $\dim\left((\text{Col}\, A)^{\perp}\right) = 1$

$\lambda = 0$

2.2) A singular 2×2 matrix whose eigenspace corresponding to eigenvalue $\lambda = 2$ is the line $x_1 = 2x_2$. The other eigenspace of the matrix is the x_2 axis.

$$A = P \cdot D P^{-1} = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}^{-1}$$

2.3) A subspace S, of $\mathbb{R}^4$, that satisfies $\dim(S) = \dim(S^{\perp}) = 3$.

$\dim(S) + \dim(S^{\perp}) = 4$

Not Possible

2.4) A 2×3 matrix, A, that is in RREF. $(\text{Row}\, A)^{\perp}$ is spanned by $\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$.

$= \text{Null}(A)$

$$\begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -3 \end{bmatrix}$$

$$\begin{cases} (0, y_1) \quad (1, y_2) \quad (2, y_3) \\ y = \beta_0 + \beta_1 x \end{cases}$$

$$\Rightarrow \begin{cases} y_1 = \beta_0 + \beta_1 \cdot 0 \\ y_2 = \beta_0 + \beta_1 \cdot 1 \\ y_3 = \beta_0 + \beta_1 \cdot 2 \end{cases}$$

$$\underline{\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix}$$

$$\mathrm{Row}(A)^{\perp} = \mathrm{Col}(A^T)^{\perp} = \mathrm{Nul}((A^T)^T) = \mathrm{Nul}(A)$$

How to find $\dim(\mathrm{Row}(A))$

$$A \longrightarrow \cdots \longrightarrow \begin{bmatrix} 1 & \text{---} \\ 0 & - - - \\ & \end{bmatrix}$$

$\underline{\dim(\mathrm{Row}(A)^{\perp})} = \underline{\underline{2}}$

$\dim(\mathrm{Row}(A)) = 1 = \dim(\mathrm{Col}(A))$

$$\begin{bmatrix} 1 & * & * \\ 0 & 0 & 0 \end{bmatrix}$$

$\dim(\mathrm{Col}(A)^{\perp}) = 1$

$\mathrm{Col}(A)$ in $\mathbb{R}^2$

3. Circle **possible** if the set of conditions are create a situation that is possible, otherwise, circle **impossible**. For the situations that are possible give an example.

3.1) A is $n \times n$, $A\vec{x} = A\vec{y}$ for a particular $\vec{x} \neq \vec{y}$, $\vec{x}$ and $\vec{y}$ are in $\mathbb{R}^n$, and $\boxed{\dim((\text{Row } A)^{\perp})} \neq 0$.

possible **impossible**

$\text{Row}(A)^{\perp} = \text{Col}(A^T)^{\perp}$

$= \text{Nul}(A)$

$z \neq 0$
$z \in \text{Nul}(A)$

$z \neq 0$
$Az = 0$

$(x-y) \neq 0$
$A(x-y) = 0$

3.2) A is $n \times n$, $\lambda \in \mathbb{R}$ is an eigenvalue of A, and $\dim((\text{Col}(A - \lambda I))^{\perp}) = 0$.

possible **impossible**

$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

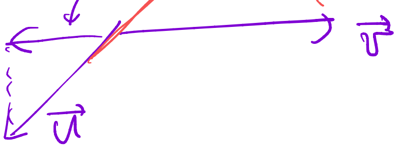
Nul(A)

$x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$y = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

3.3) $\text{proj}_{\vec{v}}\vec{u} = \text{proj}_{\vec{u}}\vec{v}$, $\vec{v} \neq \vec{u}$, and $\vec{u} \neq \vec{0}$, $\vec{v} \neq \vec{0}$.

possible **impossible**



$\vec{u} \perp \vec{v} \rightarrow \text{proj}_v u = \text{proj}_u v = 0$

4. Consider the matrix A.

$$A = \begin{pmatrix} 1 & -3 & 0 & 2 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$x_2 \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 0 \\ 3 \\ 1 \end{bmatrix}$

Construct a basis for the following subspaces and state the dimension of each space.

4.1) $(\text{Row}A)^{\perp}$ $= \text{Nul}(A) = \left\{ \begin{bmatrix} \\ \end{bmatrix}, \begin{bmatrix} \\ \end{bmatrix} \right\}$ 2

4.2) Col A $= \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

4.3) $(\text{Col}A)^{\perp}$

$= \text{Nul}(A^T)$

$= \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

$\text{proj}_u v = v$

$\text{proj}_v u = u$

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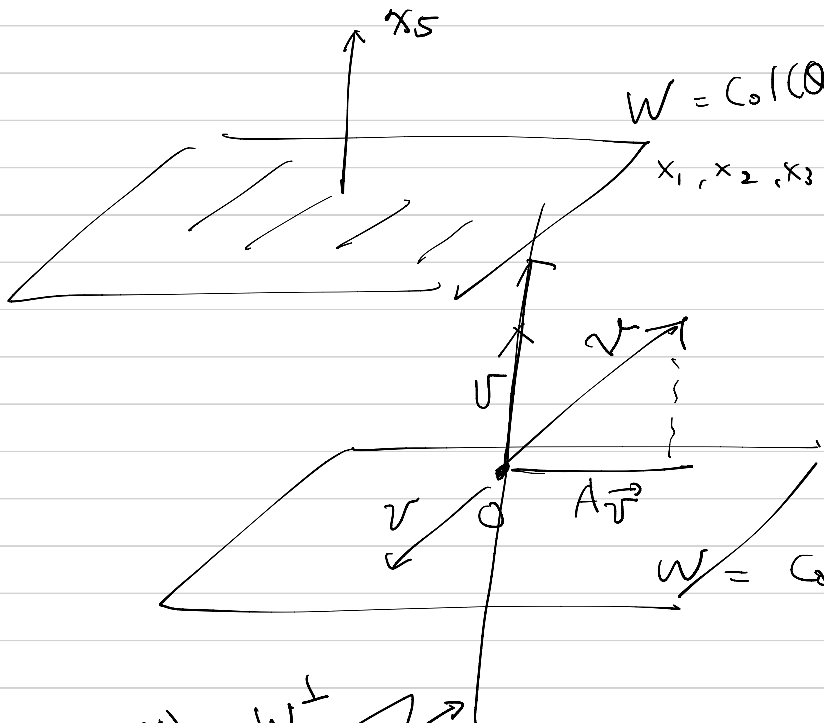
$\underline{W}$: $\underline{4\text{-dim}}$ in $\mathbb{R}^5$

$$A = Q\cdot Q^T = \begin{bmatrix} 1 & & & & \\ & 1 & & 0 & \\ & & 1 & & \\ & 0 & & 1 & \\ & & & & 0 \end{bmatrix}$$

$W = \mathrm{Col}(Q)$

x_1, x_2, x_3, x_4



$\underline{Av} = \underline{v}$ for all ~~$v \in \mathbb{R}^5$~~ $v \in W$

$W = \mathrm{Col}(\underset{\overset{\|}{A}}{Q\cdot Q^T})$

$\mathrm{Nul}(A) = W^\perp$

$W^\perp = \mathrm{Col}(A)^\perp = \mathrm{Nul}(A^T)$
$= \mathrm{Nul}(A)$

$A^T = (Q\cdot Q^T)^T = (Q^T)^T \cdot Q^T = Q\cdot Q^T = A$

λ : eigenvalue for A

$\Rightarrow \quad \det(A - \lambda I) = 0 = \det((A - \lambda I)^T)$
$= \det(A^T - \lambda I)$

$\Rightarrow \quad \lambda$: eigenvalue for A^T.

$\Rightarrow \quad \dim(\mathrm{Null}(A^T - \lambda I)) \neq 0$

$\quad\quad \dim(\mathrm{Col}(A - \lambda I)^\perp)$

$\underline{\text{Impossible}}$

$$\underset{10}{\overset{\shortparallel}{}}$$

$$\dim(\mathrm{Col}(A)^{\perp}) + \dim(\mathrm{Col}(A)) = 15$$

$$\underbrace{\dim(\mathrm{Row}(A)^{\perp})}_{\dim(\mathrm{Nul}(A))} + \underbrace{\dim(\mathrm{Row}(A))}_{\dim(\mathrm{Col}(A))} = 17$$

$$W^{\perp} = \mathrm{Nul}\left(\begin{bmatrix} 1 & 1 & 2 & -1 \end{bmatrix}\right)^{\perp}$$

$$= \mathrm{Col}\left(\left[\begin{pmatrix} 1 \\ 1 \\ 2 \\ -1 \end{pmatrix}\right]\right)$$

$$\frac{1}{\sqrt{7}}\begin{bmatrix} 1 \\ 1 \\ 2 \\ -1 \end{bmatrix}$$