Chapter 1. Probability

Math 3215 Summer 2023

Georgia Institute of Technology

Section 1.
Properties of Probability

Why Probability and Statistics?

Two main reasons are uncertainty and complexity.

Uncertainty is all around us and is usually modeled as randomness: it appears in call centers, electronic circuits, quantum mechanics, medical treatment, epidemics, financial investments, insurance, games (both sports and gambling), online search engines, for starters.

Probability is a good way of quantifying and discussing what we know about uncertain things, and then making decisions or estimating outcomes.

Why Probability and Statistics?

Some things are too complex to be analyzed exactly (like weather, the brain, social science), and probability is a useful way of reducing the complexity and providing approximations.

Definition: Experiments, Sample spaces, Events

We consider experiments for which the outcome cannot be predicted with certainty.

Such experiments are called random experiments. = Toss a coin

The collection of all possible outcomes is denoted by S $= \{H, T\}$ and is called the sample space.

Given a sample space S, let A be a $\underbrace{\text{part of the collection}}_{\text{subset of } S}$ of outcomes in S. A is called an event.

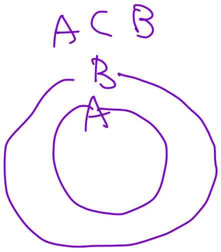
$\phi \subset S$

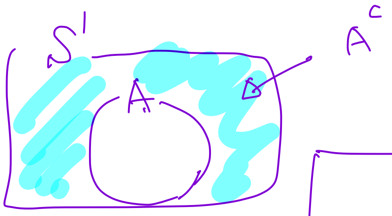
$\{H\} \subset S$

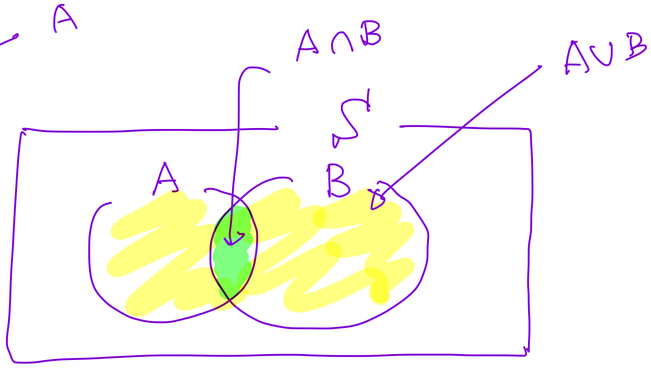
$\{T\} \subset S$

$\{H, T\} \subseteq S$

H is not an event.







Algebra of sets

Empty set (Null set): ϕ

A is a subset of B: $\underset{\text{set}}{A} \subset \underset{\text{set}}{B}$ (If $\overset{\text{element}}{x} \in A$ then $x \in B$)

x belongs to A

The union of A and B = $A \cup B = \{ x : x \in A$ OR $x \in B \}$

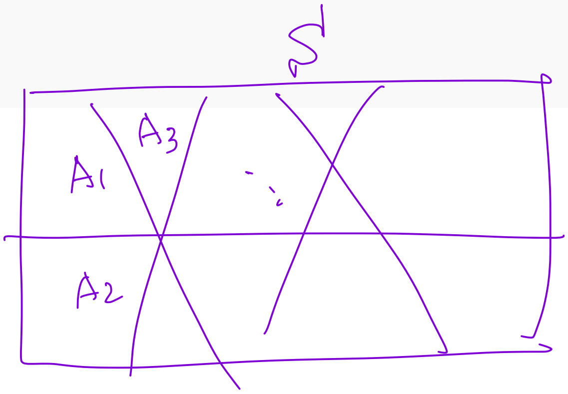
The intersection of A and B = $A \cap B = \{ x : x \in A$ AND $x \in B \}$

The complements of A = $A' = A^c = \{ x : x \notin A \}$

$A_1, A_2, \cdots, A_k$ are mutually exclusive events: for $i \neq j$ $A_i \cap A_j = \phi$

$A_1, A_2, \cdots, A_k$ are exhaustive events: $A_1 \cup A_2 \cup \cdots \cup A_k = S$

$A_1, A_2, \cdots, A_k$ are mutually exclusive and exhaustive events: = Partition



Algebra of sets

Commutative Laws Order doesn't matter

$A \cup B = B \cup A$ and $A \cap B = B \cap A$

Associative Laws Can $\cup / \cap$ with more than 2 sets.

$(A \cup B) \cup C = A \cup (B \cup C)$ and $(A \cap B) \cap C = A \cap (B \cap C)$ $= A \cap B \cap C$

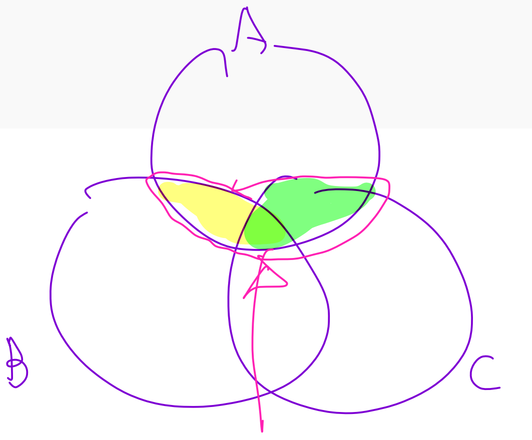
$= A \cup B \cup C$

Distributive Laws mixed $\cup / \cap$

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ and $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

De Morgan's Laws mixed $\cup / \cap$ and complement.

$(A \cup B)^c = A^c \cap B^c$ and $(A \cap B)^c = A^c \cup B^c$



$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

$$(A\cup B)^c = \{x : \underbrace{x \notin A\cup B}_{\sim(\underline{x \in A\cup B})}\} = A^c \cap B^c$$

$$x \in A \text{ or } x \in B$$

$$= \{x : x \notin A \ \underline{\text{and}}\ x \notin B\}$$

$$= \{x : x \notin A\}^{=A^c} \cap \{x : x \notin B\}^{=B^c}$$

Definition of Probability

Consider repeating the experiment a number of times-say, n times. We call these repetitions trials.

Count the number of times that event A actually occurred throughout these n trials; this number is called the frequency of event A and is denoted by $N(A)$.

The ratio $N(A)/n$ is called the relative frequency of event A in these n repetitions of the experiment.

As n increase, one can expect that the relative frequency tends to stabilize, close to some number p.

This p is called the probability of A.

Intuitive Definition

Exp = Tossing a coin

Repeat n times

$A = \{H\}$

$N(A)$ = # of event A occur.

Definition of Probability

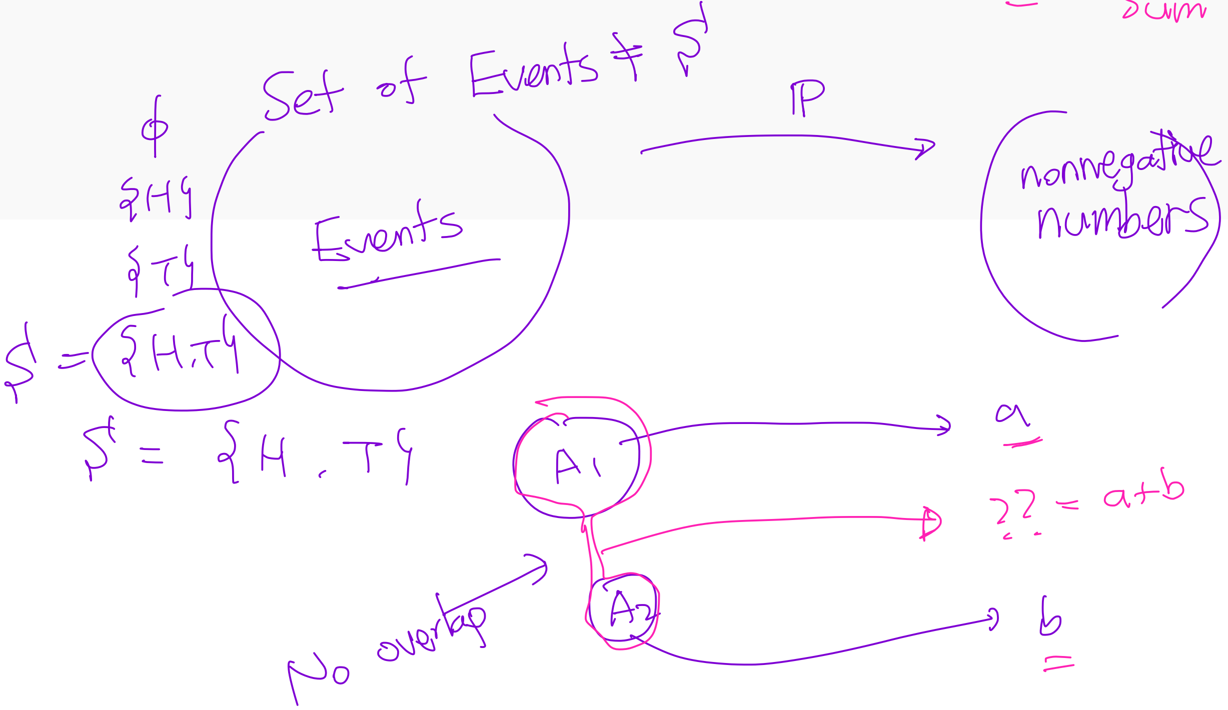
Definition

Probability is a real-valued set function $\mathbb{P}$ that assigns, to each event A in the sample space S, a number $\mathbb{P}(A)$, called the probability of the event A, such that the following properties are satisfied:

1. $\underline{\mathbb{P}(A) \geq 0}$ for all events A
2. $\underline{\mathbb{P}(S) = 1}$
3. For mutually exclusive events $\underline{A_1, A_2, \cdots,}$
   $\mathbb{P}(A_1 \cup A_2 \cup \cdots) = \mathbb{P}(A_1) + \mathbb{P}(A_2) + \cdots$

finite or infinite (countable)

: Prob. of Unions = Sum of Prob.



$$\left\{\begin{array}{l} \mathbb{P}(A) \geq 0 \\ \mathbb{P}(S) = 1 \\ A \cap B = \phi \end{array}\right. \Rightarrow \mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)$$

Definition of Probability

Theorem

Let A, B be events.

1. $\mathbb{P}(A) = 1 - \mathbb{P}(A^c)$
2. $\mathbb{P}(\varnothing) = 0$
3. If $A \subset B$, then $\mathbb{P}(A) \leq \mathbb{P}(B)$.
4. $\mathbb{P}(A) \leq 1$ for all events A.

B=S for ③

Exercise

Proof ① $A \cap A^c = \phi$, $A \cup A^c = S$ A, A^c are partition

$$\textcircled{1} = \mathbb{P}(\underbrace{A \cup A^c}_{= S}) = \mathbb{P}(A) + \mathbb{P}(A^c)$$

$$\mathbb{P}(A) = 1 - \mathbb{P}(A^c).$$

② $A = S$ using ①

Definition of Probability

H T H T T 5 times

H T H H 4 times

8

Example

A fair coin is flipped successively until the same face is observed on successive flips.

What is the probability that it will take three or more flips of the coin to observe the same face on two consecutive flips?

$A = \{ 3$ or more flips $\}$ $\underline{\mathbb{P}(A)}$

$A^c = \{ 2$ ~~or less~~ flips $\}$

$= \underbrace{\{ HH , TT \}} \subseteq \{ HH, TT, HT, TH \}$

$\mathbb{P}(A^c) = \frac{2}{4} = \frac{1}{2}$

$\mathbb{P}(A) = 1 - \mathbb{P}(A^c) = 1 - \frac{1}{2} = \frac{1}{2}$.

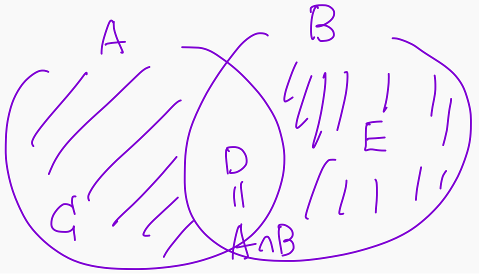
Definition of Probability

Theorem

For events A, B, C,

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$$

$$\mathbb{P}(A \cup \underbrace{B \cup C}_{B'}) = \boxed{\mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(C)} - \underline{\mathbb{P}(A \cap B)} - \underline{\mathbb{P}(B \cap C)} - \underline{\mathbb{P}(C \cap A)} + \underline{\mathbb{P}(A \cap B \cap C)}$$



$A = C \cup D \quad , \quad C \cap D = \phi$

$B = D \cup E \quad , \quad D \cap E = \phi$

$\mathbb{P}(A) = \mathbb{P}(C) + \mathbb{P}(D)$

$+ \ \underline{\mathbb{P}(B) = \mathbb{P}(D) + \mathbb{P}(E)}$

$\mathbb{P}(A) + \mathbb{P}(B) = \underbrace{\mathbb{P}(C) + \mathbb{P}(D) + \mathbb{P}(E)}$

$+ \underbrace{\mathbb{P}(D)}$

$= \mathbb{P}(A \cup B) + \mathbb{P}(A \cap B)$

$A \cup B = \underbrace{C \cup D \cup E}_{\text{mutually exclusive}}$

$\mathbb{P}(A \cup B) = \mathbb{P}(C) + \mathbb{P}(D) + \mathbb{P}(E)$

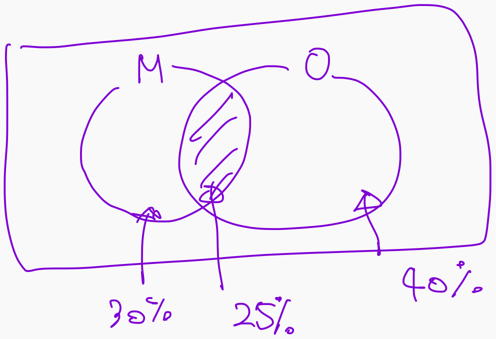
Definition of Probability

Example

Among a certain population of men, 30% are smokers, 40% are obese, and 25% are both smokers and obese.

Suppose we select a man at random from this population.

What is the probability that the selected man is either a smoker or obese?



$P(M) = 0.3$

$P(O) = 0.4$

$P(M \cap O) = 0.25$

$P($ A random person is smoker $\underline{\text{or}}$ obese $)$

$$= P(M \cup O) = P(M) + P(O) - P(M \cap O)$$

$$= 0.3 + 0.4 - 0.25 = 0.45.$$

$$P(A \cup B) = 1 - P(\underbrace{(A \cup B)^c}_{= A^c \cap B^c}) = 1 - \underline{P(A^c \cap B^c)}$$

Probability with Equally likely outcomes

S finite.

equally likely outcomes.

Let $S = (e_1, e_2, ..., e_m)$.

If each of these outcomes has the same probability of occurring, we say that the m outcomes are equally likely. $\Rightarrow \mathbb{P}(\{e_1\}) = \mathbb{P}(\{e_2\}) = \cdots$

$$= \frac{1}{\#\text{ of outcomes}}$$

In this case, $\mathbb{P}(A)$ is equal to

$$\mathbb{P}(\underset{\sim}{A}) = \mathbb{P}(\{e_1\}) + \mathbb{P}(\{e_2\}) + \mathbb{P}(\{e_3\}) = \frac{\#\text{ of outcomes in } A}{\#\text{ of total outcomes}}$$

$$A = \{e_1, e_2, e_3\}$$
$$= \{e_1\} \cup \{e_2\} \cup \{e_3\}$$

Probability with Equally likely outcomes

Example

Let a card be drawn at random from an ordinary deck of 52 playing cards.

What is the probability that a king is drawn?

Section 2.

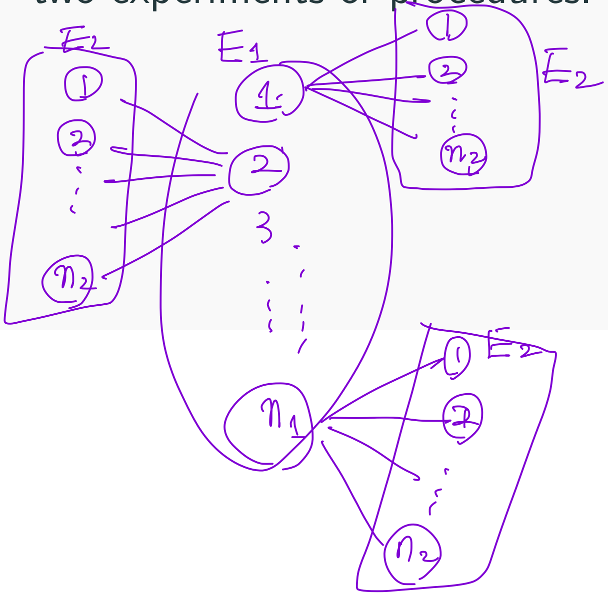
Methods of Enumeration

Multiplication Principle

Suppose that an experiment E_1 has n_1 outcomes and, for each of these possible outcomes, an experiment E_2 has n_2 possible outcomes.

Then the composite experiment $E_1 E_2$ that consists of performing first E_1 and then E_2 has $n_1 n_2$ possible outcomes.

The multiplication principle can be extended to a sequence of more than two experiments or procedures.



$\Rightarrow$ Total outcomes.

$= n_1 \cdot n_2$.

Multiplication Principle

Example

A cafe lets you order a deli sandwich your way.

There are: E_1, six choices for bread; E_2, four choices for meat; E_3, four choices for cheese; and E_4, 12 different garnishes (condiments).

What is the number of different sandwich possibilities, if you may choose one bread, 0 or 1 meat, 0 or 1 cheese, and from 0 to 12 condiments?

$C_1 \quad C_2 \quad \cdots \quad C_{12}$

$2 \quad 2 \quad \cdots \quad 2$

$6 \cdot 5 \cdot 5 \cdot \underline{13}$

?

$\boxed{12!} = 12 \cdot 11 \cdot 10 \cdots 1$

$\underline{2^{12}}$

$\underline{12! + 1}$

Note: $n! = n \cdot (n-1) \cdot \cdots 2 \cdot 1$.

Permutation

4 spots

$4 \times 3 \times 2 \times 1 = 4! = 24.$

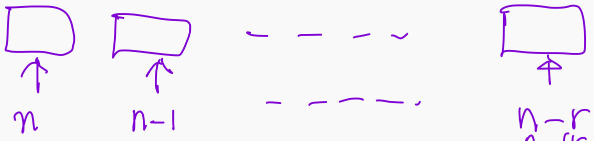
Example

What is the number of the arrangements of four letters a, b, c, d? $4!$

Definition

Each of the arrangements (in a row) of n different objects is called a permutation of the n objects. $= n!$

Permutation



r spots.

$n-(r-1) = n-r+1$ ✓

If only r positions are to be filled with objects selected from n different objects, $r \leq n$, then the number of possible ordered arrangements is

Definition

Each of the ${}_nP_r$, arrangements is called a permutation of n diff. objects taken r at a time.

$$= n\cdot(n-1)\cdots(n-r+1)$$

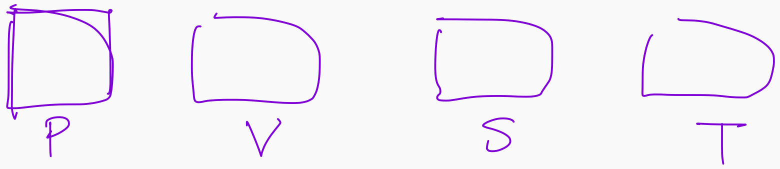
$$= \frac{n\cdot(n-1)\cdots(n-r+1)\cdot(n-r)(n-r-1)\cdots 2\cdot 1}{(n-r)(n-r-1)\cdots 2\cdot 1}$$

$$= \frac{n!}{(n-r)!} = {}_nP_r.$$

Permutation

Example

What is the number of ways of selecting a president, a vice president, a secretary, and a treasurer in a club consisting of ten persons?



$$10 \cdot 9 \cdot 8 \cdot 7 = {}_{10}P_4 = \frac{10!}{(10-4)!}$$

Sampling

Suppose that a set contains n objects. Consider the problem of selecting r objects from this set.

- If r objects are selected from a set of n objects, and if the order of selection is noted, then the selected set of r objects is called an ordered sample of size r.
- Sampling with replacement occurs when an object is selected and then replaced before the next object is selected.
- Sampling without replacement occurs when an object is not replaced after it has been selected.

Sampling

Example

What are the number of ordered samples of five cards that can be drawn with/without replacement?

from ordinary deck

has 52 cards

w/o replacement

$$= 52 \times 51 \times 50 \times 49 \times 48 = {}_{52}P_5 = \frac{52!}{(52-5)!}$$

with replacement

$$= 52 \times 52 \times 52 \times 52 \times 52 = 52^5 .$$

Def. of Probability $\begin{cases} P(A) \geq 0 \\ P(S) = 1 \\ P(\underline{A_1 \cup A_2 \cup \cdots \cup A_k}) = P(A_1) + \cdots + P(A_k) \end{cases}$

partition

$S = \{e_1, \cdots, e_n\}$ equally likely $\Rightarrow$ $P(A) = \dfrac{\text{\# of outcomes in } A}{\text{\# of outcomes}}$ ← $\underline{\underline{\text{count}}}$

- n object # arrangement $= n! = n \cdot (n-1) \cdots 2 \cdot 1$
- n object " of r chosen $= n(n-1) \cdots (n-r+1) = {}_nP_r$
- n object choose r without order $= \dfrac{{}_nP_r}{r!} = {}_nC_r = \binom{n}{r} = \dfrac{n!}{r!(n-r)!}$

Combination

Definition

Each of the unordered subsets of $\{1, 2, \cdots, n\}$ is called a combination of n objects taken r at a time. $= {}_nC_r = \binom{n}{r}$

diff.

$\underline{r \text{ chosen}} \Rightarrow$ give order

$$\binom{n}{r} \times \underbrace{r \times (r-1) \times \cdots \times 1}_{\overset{\shortparallel}{r!}} = {}_nP_r = \frac{n!}{(n-r)!}$$

$${}_nC_r = \binom{n}{r} = \frac{n!}{(n-r)!\,r!}$$

Combination

Example

The number of possible five-card hands (in five-card poker) drawn from a deck of 52 playing cards.

$n = 52$

↓ Choose 5 at random

no order

without replacement.

$$\# = \binom{52}{5} = \frac{52!}{5!\,(52-5)!} \quad //$$

$(a+b)^{②} = (a+b)(a+b) = a^2 + ab + ba + b^2$

$= a^2 + 2ab + b^2$

$\binom{n}{2} a^{n-2} \cdot b^2$

$(a+b)^n = (a+b)(a+b) \cdots (a+b) = 1 \cdot a^n + \binom{n}{1} a^{n-1} \cdot b + \cdots + \cdots \; b^n$

n spots

$\boxed{b} \quad \boxed{a} \quad \boxed{b} \quad \cdots \quad \boxed{a} = a^2 \cdot b^{n-2}$

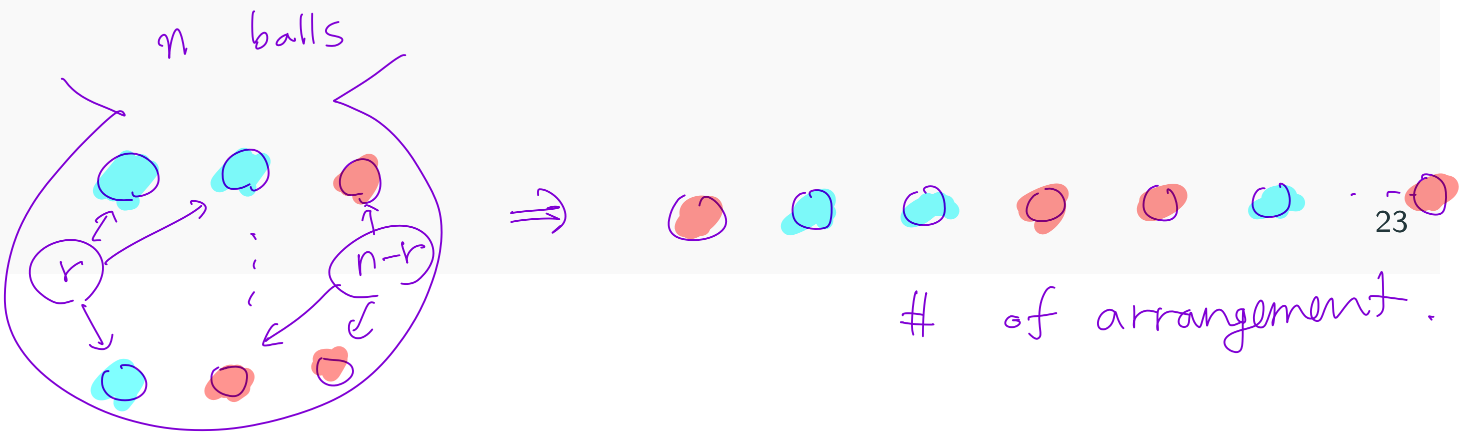
Binomial Theorem

Binomial Theorem

$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$ $= \binom{n}{n} a^n + \binom{n}{n-1} a^{n-1} \cdot b^1 + \binom{n}{n-2} a^{n-2} b^2 + \cdots$

Suppose that a set contains n objects of two types: r of one type and $n - r$ of the other type.

The number of distinguishable arrangements is



# of arrangement.

If balls are distinguishable ⇒ # = n!

But in this case: $\# = \binom{n}{r}$ ← total spots

← for ○

Binomial Theorem

Example

A coin is flipped ten times and the sequence of heads and tails is observed.

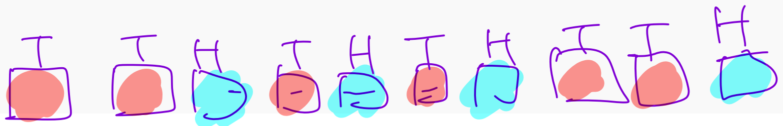
The number of possible 10-tuplets that result in four heads and six tails is

Outcomes are HHTTTHT HTTT

⋮

HHHH TTTTTT $= \binom{10}{4}$

How many total outcomes? $= 2^{10}$



$2 \times 2 \times \cdots \times 2$

$$\text{ANS} = \binom{10}{4} = \frac{10!}{6!\;4!} = \frac{10\cdot\cancel{9}^{3}\cdot\cancel{8}\cdot 7}{\cancel{4}\cdot\cancel{3}\cdot\cancel{2}\cdot 1} = 210.$$

$$\binom{10}{6} = \frac{10!}{6!\,(10-6)!} = \frac{10!}{6!\,4!} = \binom{10}{4}.$$

In general $\binom{n}{r} = \binom{n}{n-r}$

$$(a+b)^n = (a+b) \cdots (a+b)$$

$$\Downarrow$$

$$(a+b+c)^{10} = (a+b+c)(a+b+c)\cdots(a+b+c)$$

$$= \cdots + \boxed{?}\, a^3 b^4 c^3 + \cdots$$

Binomial Theorem

$$\square \quad \square \quad \square \quad \cdots \quad \square$$

① Choose 3 for a

② Among 7 remaining spots choose 4 for b

Multinomial coefficients

The coefficient of $a_1^{r_1} a_2^{r_2} \cdots a_s^{r_s}$ in the expansion of $(a_1 + \cdots + a_s)^n$ is

$$\binom{10}{3}\cdot\binom{7}{4} = \frac{10!}{3!\,\cancel{7!}}\cdot\frac{\cancel{7!}}{4!\cdot 3!} = \frac{10!}{3!\,4!\cdot 3!}$$

In general

$$(a_1 + a_2 + \cdots + a_s)^n = \sum_{n_1+n_2+\cdots+n_s=n} \binom{n}{n_1, n_2, \cdots, n_s} a_1^{n_1} \cdot a_2^{n_2} \cdots a_s^{n_s}$$

$$\binom{n}{n_1, n_2, \cdots, n_s} = \frac{n!}{n_1!\, n_2! \cdots n_s!}$$

Section 3.
Conditional Probability

Conditional Probability

Example

Suppose that we are given 20 tulip bulbs that are similar in appearance and told that eight will bloom early, 12 wil bloom late, 13 will be red, and seven will be yellow.

If one bulb is selected at random, the probability that it will produce a red tulip is $= \frac{13}{20}$

The probability that it will produce a red tulip given that it will bloom early is $=$

	Early	Late	
Red	3	10	13
Yellow	5	2	7
	8	12	

① $\frac{13}{8}$ pr. ≤ 1

② $\frac{13/20}{\boxed{8}}$

ANS= $\frac{3}{8}$

Need more information.

Conditional Probability

Definition

The conditional probability of an event A, given that event B has occurred, is defined by

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\underline{\mathbb{P}(B)}}$$

provided that $\mathbb{P}(B) > 0$.

$\neq 0$

New Sample space.

Example

$$\mathbb{P}(\text{Red} \mid \text{Early}) = \frac{\mathbb{P}(\text{Red} \cap \text{Early})}{\underline{\mathbb{P}(\text{Early})}}$$

$$= \frac{3/\cancel{20}}{8/\cancel{20}} = \frac{3}{8}$$

Conditional Probability

Example

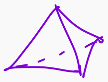
If $\mathbb{P}(A) = 0.4$, $\mathbb{P}(B) = 0.5$, and $\mathbb{P}(A \cap B) = 0.3$, then $\mathbb{P}(A|B) =$

$$\mathbb{P}(A|\underline{\underline{B}}) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} = \frac{0.3}{0.5}$$

$$= \frac{3}{5} = 0.6 .$$

Conditional Probability

$\boxed{1/3}$ ⋮



Example

Two fair four-sided dice are rolled and the sum is determined. Let A be the event that a sum of 3 is rolled, and let B be the event that a sum of 3 or a sum of 5 is rolled. The conditional probability that a sum of 3 is rolled, given that a sum of 3 or 5 is rolled, is

4 → (1, 1) ← 4

(1, 2)

(1, 3)

(1, 4)

⋮

$\Longrightarrow$

$\underline{\underline{A}}$

(1,2) , (2,1)

$\underline{B}$

(1, 2) , (2, 1)

(1, 4) (2, 3)

(3, 2) (4, 1)

$A \cap B = A$

$\boxed{\text{Total Outcomes}}$

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{\# \text{ of } \underline{\underline{A}}}{\# \text{ of } B}$$

$$= \frac{2}{6} = \frac{1}{3}.$$

Q: $\underline{P(A)} = \frac{2}{4 \cdot 4} = \frac{1}{8}.$

HW 1 Posted (Canvas & Gradescope)

Due 5/25 at 9:30 am

Properties of Conditional probabilities

Theorem

Suppose $\mathbb{P}(B) > 0$.

1. $\mathbb{P}(A|B) \geq 0$.
2. $\mathbb{P}(B|B) = 1$.
3. If $A_1, A_2, \cdots, A_k$ are mutually exclusive events, then

$$\mathbb{P}(A_1 \cup A_2 \cup \cdots \cup A_k|B) = \mathbb{P}(A_1|B) + \cdots + \mathbb{P}(A_k|B).$$

4. $\mathbb{P}(A^c|B) = 1 - \mathbb{P}(A|B)$.

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A\cap B)}{\mathbb{P}(B)} \Rightarrow \mathbb{P}(A\cap B) = \mathbb{P}(A|B)\cdot\mathbb{P}(B)$$

$$\mathbb{P}(B|A) = \frac{\mathbb{P}(A\cap B)}{\mathbb{P}(A)} \Rightarrow \quad = \mathbb{P}(B|A)\cdot\mathbb{P}(A).$$

The multiplication rule

The multiplication rule

The probability that two events, A and B, both occur is given by

$$\mathbb{P}(A\cap B) = \mathbb{P}(A|B)\mathbb{P}(B) = \mathbb{P}(B|A)\mathbb{P}(A).$$

$$\mathbb{P}(A\cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A\cap B)$$

$$\mathbb{P}(A\cap B) = \mathbb{P}(A|B)\cdot\mathbb{P}(B)$$

$$= \mathbb{P}(B|A)\cdot\mathbb{P}(A)$$

The multiplication rule

Example

At a county fair carnival game there are 25 balloons on a board, of which ten balloons are yellow, eight are red, and seven are green.

A plaver throws darts at the balloons to win a prize and randomly hits one of them.

Suppose the player throws darts twice.

What is the probability that the both baloons hit are yellow?

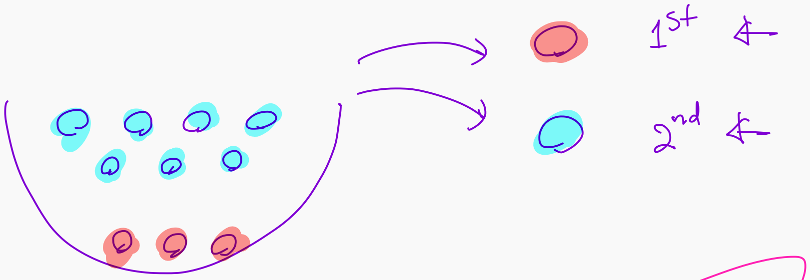
The multiplication rule

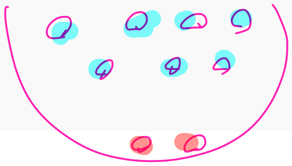
Example

A bowl contains seven blue chips and three red chips.

Two chips are to be drawn successively at random and without replacement.

Compute the probability that the first draw results in a red chip and the second draw results in a blue chip.





$$P(1^{st} = R) = \frac{3}{10}$$

$$P(2^{nd} = B \mid 1^{st} = R) = \boxed{\frac{7}{9}}$$

$$P(\underbrace{1^{st} = R}_{A} \overset{\cap}{\text{and}} \underbrace{2^{nd} = B}_{B}) = P(\overset{B}{2^{nd} = B} \overset{|}{\mid} \overset{A}{1^{st} = R}) \cdot \underline{\overset{A}{P(1^{st} = R)}}$$

$$= \frac{3}{10} \cdot \frac{7}{9} = \frac{7}{30}.$$

Q: $\underline{P(2^{nd} = B)} \overset{?}{=} P(1^{st} = R \overset{\cap}{,} 2^{nd} = B) + P(1^{st} = B \overset{\cap}{,} 2^{nd} = B)$

$$= \frac{7}{10} \cdot \frac{6}{9}$$

The multiplication rule

Example

A bowl contains seven blue chips and three red chips.

Two chips are to be drawn successively at random and without replacement.

Compute the probability that the first draw results in a red chip and the second draw results in a blue chip.

$$\mathbb{P}((\overset{=D}{A\cap B})\cap C) = \mathbb{P}(C \mid D)\cdot \overset{= \underline{\mathbb{P}(A\cap B)}}{\underline{\mathbb{P}(D)}}$$

$$= \underline{\mathbb{P}(C \mid A\cap B)\cdot \mathbb{P}(A)\cdot \mathbb{P}(B\mid A)}$$

The multiplication rule

Multiplication rule for three events

$$\mathbb{P}(A\cap B\cap C) = \mathbb{P}(A)\mathbb{P}(B|A)\mathbb{P}(C|A\cap B).$$

The multiplication rule

52 $\longrightarrow$ 4 cards without replacement.

Example

Four cards are to be dealt successively at random and without replacement from an ordinary deck of playing cards. 52 cards

The probability of receiving, in order, a $\underbrace{\overset{13}{\text{spade}}}_{1^{st}}$, a $\underbrace{\overset{13}{\text{heart}}}_{2^{nd}}$, a $\underbrace{\overset{13}{\text{diamond}}}_{3^{rd}}$, and a $\underbrace{\overset{13}{\text{club}}}_{4^{th}}$ is

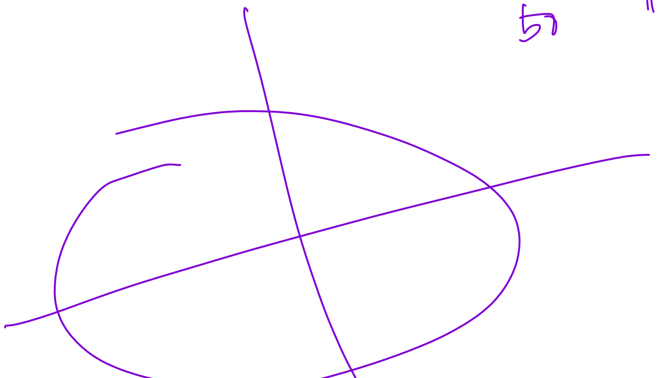
$$P(1^{st}=S \text{ and } 2^{nd}=H \text{ and } 3^{rd}=D \text{ and } 4^{th}=C)$$

$$= \underline{P(1^{st}=S)} \cdot P(2^{nd}=H \mid 1^{st}=S) \cdot P(3^{rd}=D \mid 1^{st}=S, 2^{nd}=H)$$

$$\cdot P(4^{th}=C \mid 1^{st}=S, 2^{nd}=H, 3^{rd}=D)$$

$$= \frac{13}{52} \cdot \frac{13}{51} \cdot \frac{13}{50} \cdot \frac{13}{49}.$$

$$\frac{13}{50} = P(3^{rd}=D \mid 2^{nd}=1^{st}=S)$$

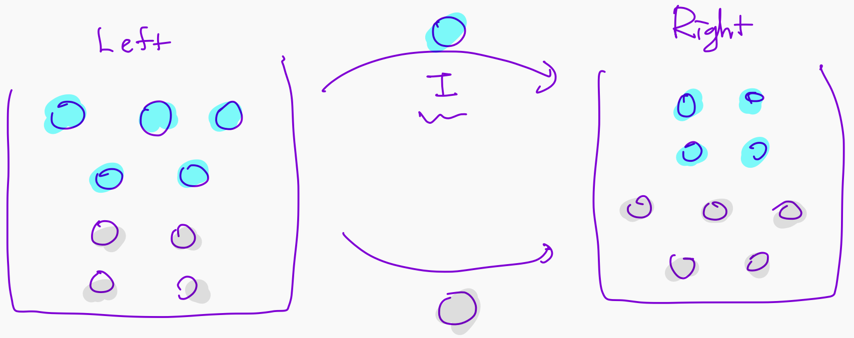


The multiplication rule

Example

A boy has five blue and four white marbles in his left pocket and four blue and five white marbles in his right pocket.

If he transfers one marble at random from his left to his right pocket, what is the probability of his then drawing a blue marble from his right pocket?



$\boxed{41/90}$.

II ○

$$P(\mathrm{II}=B) = P(\mathrm{I}=B,\ \mathrm{II}=B) + P(\mathrm{I}=W,\ \mathrm{II}=B)$$

$$= P(\mathrm{I}=B)\cdot P(\mathrm{II}=B \mid \mathrm{I}=B) + P(\mathrm{I}=W)\cdot P(\mathrm{II}=B \mid \mathrm{I}=W)$$

$$= \frac{5}{9}\cdot\frac{5}{10} + \frac{4}{9}\cdot\frac{4}{10} = \frac{5^2+4^2}{90}$$

Section 4.
Independent Events

Independent Events

For certain pairs of events, the occurrence of one of them may or may not change the probability of the occurrence of the other.

In the latter case, they are said to be independent events.

Independent Events

$\{\underset{\sim}{HH}, TT, \underset{\sim}{HT}, TH\} = S$

Example

Flip a coin twice. Let $\underset{\sim}{A}$ = heads on the first flip and B = tails on the second flip

$= \{HH, HT\}$ $= \{HT, TT\}$

Compute $\underline{\mathbb{P}(B|A)}$ and $\mathbb{P}(B)$.

$A \cap B = \{HT\}$

$$\mathbb{P}(B|A) = \frac{\#\text{ of } A \text{ and } B}{\#\text{ of } A} = \frac{1}{2}$$

$$\mathbb{P}(B) = \frac{2}{4} = \frac{1}{2}.$$

A , B indep.

$$\underset{\underset{\frac{\mathbb{P}(A\cap B)}{\boxed{\mathbb{P}(B)}}}{\shortparallel}}{\mathbb{P}(A|B)} = \mathbb{P}(A) \quad \Rightarrow \quad \underbrace{\mathbb{P}(A\cap B) = \mathbb{P}(A)\cdot\mathbb{P}(B)}$$

Independent Events

Definition: Independence

Events A and B are independent if and only $\overset{\text{if}}{\text{if}}$ $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$.

Otherwise, A and B are called dependent events.

Independent Events

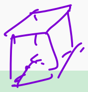
red → ← white

$S = \{ (1,1), (1,2), ---, (1,6)$

$(2,1), (2,2), ---, (2,6)$ 36 outcomes

$\vdots$

$(6,1), --- , (6,6) \}$



Example

A red die and a white die are rolled.

Let $A = \{\text{red is } 4\}$ and $B = \{\text{sum is odd}\}$. Are they independent?

$\{ (4,1), (4,2), ---, (4,6) \}$

6 outcomes

$B = \{ (1,2), (1,4), (1,6)$

$(2,1), (2,3), (2,5)$

$\vdots$

$(6,1), (6,3), (6,5) \}$ ← $6 \cdot 3 = 18$ outcomes

$\mathbb{P}(A) = 6/36 = \frac{1}{6}$

$\mathbb{P}(B) = 18/36 = \frac{1}{2}$

$\mathbb{P}(A \cap B) = 3/36 = \frac{1}{12}$

$\mathbb{P}(A) \cdot \mathbb{P}(B) = \frac{1}{6} \cdot \frac{1}{2} = \frac{1}{12}$.

$\therefore$ Indep.

Independent Events

Example

A red die and a white die are rolled.

Let $A = \{\text{red is } 5\}$ and $B = \{\text{sum is } 11\}$. Are they independent?

$\{(5,1), (5,2), \text{----}, \underbrace{(5,6)}\}$

$B = \{(6,5), \underbrace{(5,6)}\}$ $\quad A \cap B = \{(5,6)\}$

$\mathbb{P}(A) = 6/36 = \frac{1}{6}.$ $\quad \mathbb{P}(B) = \frac{2}{36} = \frac{1}{18}$

$\mathbb{P}(A \cap B) = 1/36 \neq \mathbb{P}(A) \cdot \mathbb{P}(B) = \frac{1}{6} \cdot \frac{1}{18}$

dependent.

Independent Events

Theorem

If A and B are independent, then the following pairs are independent:

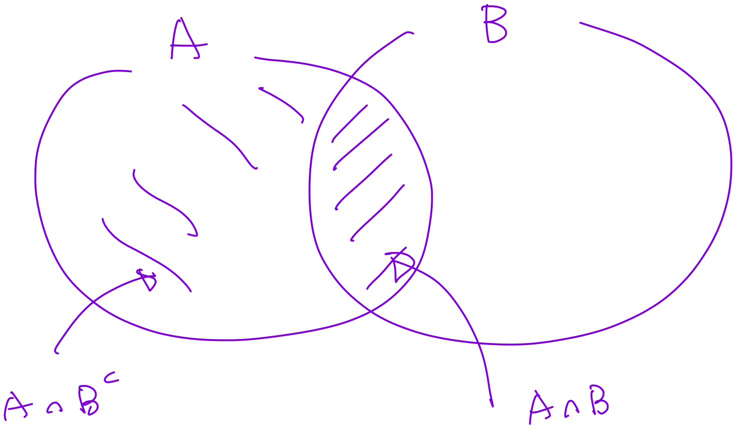
- A and B^c
- A^c and B
- A^c and B^c

$$\mathbb{P}(A) \cdot \underline{\mathbb{P}(B^c)} \overset{?}{=} \underline{\mathbb{P}(A \cap B^c)}$$

$$= \mathbb{P}(A) \cdot (1 - \mathbb{P}(B))$$

$$= \mathbb{P}(A) - \underbrace{\mathbb{P}(A) \cdot \mathbb{P}(B)}$$

$$= \underline{\mathbb{P}(A) - \mathbb{P}(A \cap B)} =$$



$$A = \underline{(A \cap B)} \cup \underline{(A \cap B^c)}$$

have no overlap.

$$\mathbb{P}(A) = \underbrace{\mathbb{P}(A \cap B)}_{R} + \mathbb{P}(A \cap B^c)$$

$$\mathbb{P}(A) - \mathbb{P}(A \cap B) = \mathbb{P}(A \cap B^c)$$

Independent Events

Definition: Mutually independence

Events A, B, and C are mutually independent if and only if A, B, and C are pairwise independent and

$$\mathbb{P}(A \cap B \cap C) = \mathbb{P}(A)\mathbb{P}(B)\mathbb{P}(C).$$

$$\begin{cases} \mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B) \\ \mathbb{P}(B \cap C) = \mathbb{P}(B) \cdot \mathbb{P}(C) \\ \mathbb{P}(C \cap A) = \mathbb{P}(C) \cdot \mathbb{P}(A) \end{cases}$$

Independent Events

Example

Three inspectors look at a critical component of a product.

Their probabilities of detecting a defect are different, namely, $\overset{A}{0.99}$, $\overset{B}{0.98}$, and $\underset{C}{0.96}$, respectively. Assume independence.

Compute the following probabilities: (a) that exactly $\underline{\text{two}}$ find the defect, and (b) that all three find the defect.

$$\mathbb{P}((a)) = \mathbb{P}(A \cap B \cap C^c) + \mathbb{P}(A \cap B^c \cap C)$$

$$+ \mathbb{P}(A^c \cap B \cap C)$$

$$= \mathbb{P}(A) \cdot \mathbb{P}(B) \cdot \underline{\underline{\mathbb{P}(C^c)}} + \mathbb{P}(A) \cdot \underline{\underline{\mathbb{P}(B^c)}} \cdot \mathbb{P}(C)$$

$$+ \mathbb{P}(A^c) \cdot \mathbb{P}(B) \cdot \mathbb{P}(C)$$

$$= 0.99 \cdot 0.98 \cdot (1 - 0.96) + 0.99 \cdot (1 - 0.98) \cdot 0.96$$

$$+ (1 - 0.99)\, 0.98 \cdot 0.96 .$$

Section 5.
Bayes' Theorem

Recall · Conditional Probability : $\mathbb{P}(A \mid B) = \dfrac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$

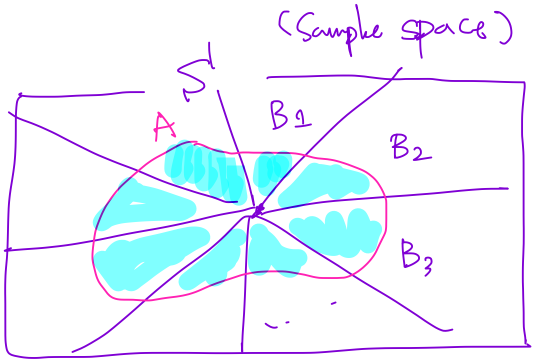
· Multiplication Rule : $\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B \mid A)$

$= \mathbb{P}(B) \cdot \mathbb{P}(A \mid B)$

· A and B are independent if $\mathbb{P}(A \cap B) = \underline{\mathbb{P}(A)} \cdot \underline{\mathbb{P}(B)}$

$\Rightarrow \mathbb{P}(A) = \mathbb{P}(A \mid B)$ if $\mathbb{P}(B) \neq 0$

· A, B, C mutually independent $\left\{ \begin{array}{ll} A, B & \text{ind.} \\ B, C & \text{"} \\ C, A & \text{"} \end{array} \right.$ & $\mathbb{P}(A \cap B \cap C) = \mathbb{P}(A) \cdot \mathbb{P}(B) \cdot \mathbb{P}(C)$



$$\mathbb{P}(A) = \mathbb{P}\big((A\cap B_1) \cup (A\cap B_2) \cup \cdots \cup (A\cap B_n)\big)$$

mutually exclusive

Partition

A collection of events

$B_1, B_2, \cdots, B_n$

$\left\{ \begin{array}{l} \text{mutually exclusive,} \\ \text{exhaustive.} \end{array} \right.$

$(B_1 \cup B_2 \cup \cdots \cup B_n = S)$

The law of total probabilities

$$\Rightarrow \quad \mathbb{P}(A) = \mathbb{P}(A\cap B_1) + \mathbb{P}(A\cap B_2) + \cdots + \mathbb{P}(A\cap B_n)$$
$$= \mathbb{P}(B_1)\cdot\mathbb{P}(A|B_1) + \mathbb{P}(B_2)\cdot\mathbb{P}(A|B_2) + \cdots + \mathbb{P}(B_n)\cdot\mathbb{P}(A|B_n)$$

The law of total probabilities

If $B_1, \cdots, B_n$ are mutually exclusive and exhaustive events (partition), then

$$\mathbb{P}(A) = \sum_{k=1}^{n} \mathbb{P}(A\cap B_k) = \sum_{k=1}^{n} \mathbb{P}(A|B_k)\mathbb{P}(B_k).$$

$$\mathbb{P}(B|A) = \frac{\mathbb{P}(A\cap B)}{\mathbb{P}(A)} = \frac{\mathbb{P}(A|B)\cdot\mathbb{P}(B)}{\mathbb{P}(A)}$$

(If $\mathbb{P}(A|B)$ is easy to compute)

$$\mathbb{P}(B_2 | A) = \frac{\mathbb{P}(A | B_1) \cdot \mathbb{P}(B_1)}{\mathbb{P}(A)}$$

$$= \frac{\mathbb{P}(A | B_2) \cdot \mathbb{P}(B_2)}{\mathbb{P}(A|B_1) \cdot \mathbb{P}(B_1) + \mathbb{P}(A|B_2) \cdot \mathbb{P}(B_2) + \dots + \mathbb{P}(A|B_n) \cdot \mathbb{P}(B_n)}$$

Bayes' Theorem

Bayes' Theorem

$$\mathbb{P}(B_k|A) = \frac{\mathbb{P}(A \cap B_k)}{\mathbb{P}(A)} = \frac{\mathbb{P}(B_k)\mathbb{P}(A|B_k)}{\mathbb{P}(A)} = \frac{\mathbb{P}(B_k)\mathbb{P}(A|B_k)}{\sum_{k=1}^{n} \mathbb{P}(A|B_k)\mathbb{P}(B_k)}.$$

Examples

Example

In a certain factory, machines I, II, and III are all producing springs of the same length.

Of their production, machines I, II, and III respectively produce 2%, 1%, and 3% defective springs.

Of the total production of springs in the factory, machine I produces 35%, machine II produces 25%, and machine III produces 40%.

If the selected spring is defective, what is the conditional probability that it was produced by machine III?

$D = \{ \text{defective one is chosen} \}$

$\mathrm{I} = \{ \text{from Machine I} \}$

$\mathrm{II} = \{ \quad " \quad \mathrm{II} \}$

$\mathrm{III} = \{ \quad " \quad \mathrm{III} \}$

$$\mathbb{P}(\mathrm{III} \mid D) = \frac{\underset{\frac{3}{100}}{\mathbb{P}(D \mid \mathrm{III})} \cdot \overset{\frac{4}{10}}{\mathbb{P}(\mathrm{III})}}{\mathbb{P}(D) = ?}$$

$$\mathbb{P}(D) = \mathbb{P}(D \cap \mathrm{I}) + \mathbb{P}(D \cap \mathrm{II}) + \mathbb{P}(D \cap \mathrm{III})$$

$$= \underline{\mathbb{P}(D|I)}\cdot \underline{\mathbb{P}(I)} + \underline{\mathbb{P}(D|II)}\cdot \mathbb{P}(II) + \mathbb{P}(D|III)\mathbb{P}(III)$$

$$= \frac{2}{100}\cdot\frac{35}{100} + \frac{1}{100}\cdot\frac{25}{100} + \frac{3}{100}\cdot\frac{40}{100}.$$

$$\mathbb{P}(III|D) = \frac{\frac{3}{\cancel{100}}\cdot\frac{\cancel{40}\,8}{\cancel{100}}}{\frac{2}{\cancel{100}}\cdot\frac{\cancel{35}\,7}{\cancel{100}} + \frac{1}{\cancel{100}}\cdot\frac{\cancel{25}\,5}{\cancel{100}} + \frac{3}{\cancel{100}}\cdot\frac{\cancel{40}\,8}{\cancel{100}}}$$

Examples

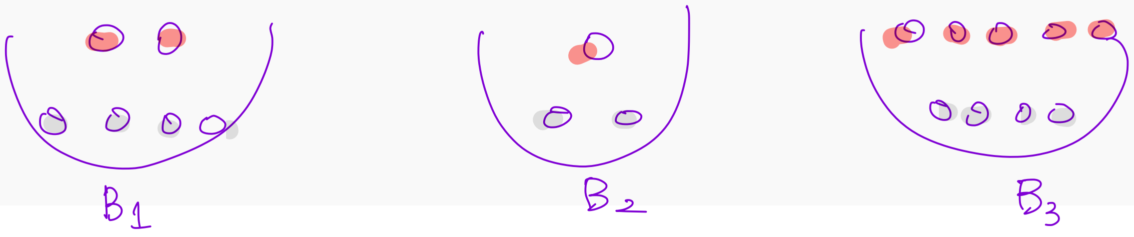
Example

Bowl B_1, contains two red and four white chips, bowl B_2 contains one red and two white chips, and bowl B_3 contains five red and four white chips.

Choose one of three bowls with $\mathbb{P}(B_1) = 1/3$, $\mathbb{P}(B_2) = 1/6$, and $\mathbb{P}(B_3) = 1/2$ and draw a chip from the chosen bowl.

Let R be the event that a red chip is chosen.

Compute $\mathbb{P}(R)$ and $\mathbb{P}(B_1|R)$.



$$\mathbb{P}(R) = \mathbb{P}(R\cap B_1) + \mathbb{P}(R\cap B_2) + \mathbb{P}(R\cap B_3)$$

$$= \underline{\mathbb{P}(R|B_1)}\cdot \underline{\mathbb{P}(B_1)} + \mathbb{P}(R|B_2)\cdot \underline{\mathbb{P}(B_2)} + \underline{\mathbb{P}(R|B_3)}\cdot \underline{\mathbb{P}(B_3)}$$

$$= \frac{2}{6}\cdot\frac{1}{3} + \frac{1}{3}\cdot\frac{1}{6} + \frac{5}{9}\cdot\frac{1}{2}$$

$$\mathbb{P}(B_1|R) = \frac{\mathbb{P}(R|B_1)\ \mathbb{P}(B_1)}{\mathbb{P}(R)} = \frac{\frac{2}{\cancel{6}}\cdot\frac{1}{\cancel{3}}}{\frac{2}{\cancel{6}}\cdot\frac{1}{\cancel{3}} + \frac{1}{\cancel{3}}\cdot\frac{1}{\cancel{6}} + \frac{5}{\cancel{9}}\cdot\frac{1}{\cancel{2}}} = \frac{2}{2+1+5}$$

Examples

Example

Bowl B_1, contains two red and four white chips, bowl B_2 contains one red and two white chips, and bowl B_3 contains five red and four white chips.

Choose one of three bowls with $\mathbb{P}(B_1) = 1/3$, $\mathbb{P}(B_2) = 1/6$, and $\mathbb{P}(B_3) = 1/2$ and draw a chip from the chosen bowl.

Let R be the event that a red chip is chosen.

Compute $\mathbb{P}(R)$ and $\mathbb{P}(B_1|R)$.