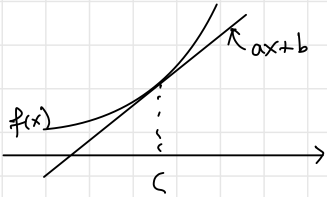
4/29/22

Jensen's : If f is convex on I & $P(X \in I) = 1$ then

$$E[f(X)] \geq f(E[X])$$

<u>Proof</u> Let $c := EX \in I$ (why?). Then $\exists\ a, b$ s.t. why?

$f(x) \geq ax + b$ & $f(c) = a \cdot c + b$.



$$Ef(X) \geq E[aX + b] = aEX + b$$
$$= f(c) = f(EX).$$

$\blacksquare$

<u>Example</u> $f(x) = x \log x$, $x > 0$ $X \geq 0$, $EX = 1$

$f'(x) = \log x + 1$

$f''(x) = \frac{1}{x} > 0$

$$E[X \log X] \geq (EX)\log(EX) = 0.$$

<u>Example</u> f: concave $\Rightarrow$ $-f$ is convex

$$E[-f(X)] \geq -f(EX) \Rightarrow E[f(X)] \leq f(EX).$$

Law of Large Numbers

$X_1, X_2, \cdots$ are Independent, Identically distributed.

$E[X_1] = \mu$, $Var(X_1) = \sigma^2$

<u>WLLN</u> : For $\varepsilon > 0$,

$$P\left(\left|\frac{X_1 + \cdots + X_n}{n} - \mu\right| \geq \varepsilon\right) \xrightarrow[n\to\infty]{} 0.$$

<u>Proof</u> Let $S_n = (X_1 + \cdots + X_n)$ then $E\frac{S_n}{n} = \mu$, $Var\left(\frac{S_n}{n}\right) = \frac{\sigma^2}{n}$

$$P\left(\left|\frac{S_n}{n} - \mu\right| \geq \varepsilon\right) \leq \frac{\sigma^2}{n\varepsilon^2} \longrightarrow 0 \text{ as } n \to \infty$$

$\blacksquare$