MATH 461 LECTURE NOTE
WEEK 10

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1. Exponential Random Variables (Sec 5.4-7)

Definition: Exponential random variable
A random variable X is exponential with parameter $\lambda > 0$ if its density is given by

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0, \\ 0, & x < 0. \end{cases}$$

We denote by $X \sim \exp(\lambda)$.

Proposition 1. *Let $X \sim \exp(\lambda)$ for $\lambda > 0$.*

(i) *The cumulative distribution function $F(x) = 1 - e^{-\lambda x}$.*
(ii) $\mathbb{E}[X] = \frac{1}{\lambda}$.
(iii) $\mathrm{Var}(X) = \frac{1}{\lambda^2}$.

Proposition 2 (Memoryless property)**.** *Let $s, t > 0$ and $X \sim \exp(\lambda)$ for $\lambda > 0$, then*

$$\mathbb{P}(X > s + t | X > t) = \mathbb{P}(X > s).$$

Example 3. Suppose that the number of miles that a car can run before its battery wears out is exponentially distributed with an average value of 10,000 miles. If a person desires to take a 5000-mile trip, what is the probability that he or she will be able to complete the trip without having to replace the car battery? What can be said when the distribution is not exponential?

2. Other Continuous RVs (Sec 5.4-7)

Gamma random variables
A random variable X is a Gamma random variable with parameter $\lambda > 0$ and $\alpha > 0$ if its density is given by

$$f(x) = \begin{cases} \frac{\lambda e^{-\lambda x} (\lambda x)^{\alpha - 1}}{\Gamma(\alpha)}, & x > 0 \\ 0, & x \leq 0 \end{cases}$$

where $\Gamma(\alpha) = \int_0^\infty e^{-y} y^{\alpha-1}\, dy$ is the gamma function. We denote by $X \sim \Gamma(\alpha, \lambda)$.

Remark 4. Note that $\Gamma(1, \lambda) \sim \exp(\lambda)$.

Remark 5. By integration by parts, one can show that $\Gamma(\alpha) = (\alpha - 1)\Gamma(\alpha - 1)$. In particular, $\Gamma(1) = 1$ and $\Gamma(n) = (n - 1)!$. Thus, the gamma function is a generalization of the factorial.

Proposition 6. *Let $X \sim \Gamma(\alpha, \lambda)$, then $\mathbb{E}[X] = \frac{\alpha}{\lambda}$ and $\mathrm{Var}(X) = \frac{\alpha}{\lambda^2}$.*

Remark 7. Suppose that the number of events occur in the time interval $[0, t]$ is a Poisson random variable $\mathrm{Poisson}(\lambda t)$ for some $\lambda > 0$. (See [SR, p.144] for this approximation.) If T_n is the time at which the n-th event occurs, then T_n is a gamma random variable with n and λ.

Weibull random variables
A random variable X is a Weibull random variable with parameter $\nu, \alpha > 0$ if its cdf is given by

$$F(x) = \begin{cases} 0, & x < \nu \\ 1 - e^{-\left(\frac{x-\nu}{\alpha}\right)^{\beta}}, & x > \nu. \end{cases}$$

Weibull is often used to model lifetime of some object, device, individual, etc.

Beta random variables
A random variable X is a beta random variable with parameter $a, b > 0$ if its density is given by

$$f(x) = \begin{cases} \frac{x^{a-1}(1-x)^{b-1}}{B(a,b)}, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

where $B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}\,dx$ is the beta function.

The beta distribution can be used to model a random phenomenon whose set of possible values is some finite interval.

3. Joint Distribution Functions (Sec 6.1)

In this section, we consider a collection of random variables $X_1, X_2, \cdots, X_n$ defined on a sample space S. In particular, we are interested in modeling relationships between them. For example, collections of random variables are involved in the following:

(i) X_1 is the price of a product today, X_2 the price tomorrow, and so on.
(ii) X_1 is rainfall in IL, X_2 rainfall in IN, and so on.
(iii) Statistical topics such as time series, multivariate analysis, multiple linear regression, factor models.
(iv) Probability topics such as Markov chains, stochastic processes.

First, we focus on two random variable X and Y on a sample space S. The probability of X and Y can be realized by their joint cumulative distribution function.

Definition: Joint cumulative distribution functions
For random variables X and Y, the joint cumulative distribution function $F(a,b)$ on $\mathbb{R}^2$ is defined by

$$F(a,b) = \mathbb{P}(X \leq a, Y \leq b).$$

The distributions of each X and Y can be obtained from the joint distribution. Indeed, we have

$$F_X(a) = \mathbb{P}(X \leq a) = \lim_{b\to\infty} \mathbb{P}(X \leq a, Y \leq b) = \lim_{b\to\infty} F(a,b),$$
$$F_Y(b) = \mathbb{P}(Y \leq b) = \lim_{a\to\infty} \mathbb{P}(X \leq a, Y \leq b) = \lim_{a\to\infty} F(a,b).$$

Example 8. Describe $\mathbb{P}(X > a, Y > b)$ and $\mathbb{P}(a_1 < X \leq a_2, b_1 < Y \leq b_2)$ for $a, a_1, a_2, b, b_1, b_2 \in \mathbb{R}$ in terms of distribution functions.

Definition: Joint probability mass function
For discrete random variables X and Y, the joint probability mass function is defined by

$$p(x,y) = \mathbb{P}(X = x, Y = y).$$

The probability mass functions for each X and Y are

$$p_X(x) = \mathbb{P}(X = x) = \sum_y \mathbb{P}(X = x, Y = y) = \sum_y p(x,y),$$
$$p_Y(y) = \mathbb{P}(Y = y) = \sum_x \mathbb{P}(X = x, Y = y) = \sum_x p(x,y).$$

Example 9. Two fair dice are rolled. Find the joint probability mass function of X and Y when X is the largest value obtained on any die and Y is the sum of the values.

Definition: Joint continuity
We say that X and Y are jointly continuous if there exists a nonnegative function $f(x,y)$ on $\mathbb{R}^2$ such that

$$\mathbb{P}((X,Y)\in C) = \iint_C f(x,y)\,dxdy$$

for every C in $\mathbb{R}^2$. The function $f(x,y)$ is called the joint probability density function of X and Y.

The probability mass functions for each X and Y are

$$p_X(x) = \mathbb{P}(X=x) = \sum_y \mathbb{P}(X=x, Y=y) = \sum_y p(x,y),$$
$$p_Y(y) = \mathbb{P}(Y=y) = \sum_x \mathbb{P}(X=x, Y=y) = \sum_x p(x,y).$$

Remark 10. Suppose X and Y are jointly continuous.

(i) For any sets $A, B \subseteq \mathbb{R}$,

$$\mathbb{P}(X\in A, Y\in B) = \int_B \int_A f(x,y)\,dxdy.$$

(ii) The cumulative distribution function $F(a,b)$ is

$$F(a,b) = \mathbb{P}(X\le a, Y\le b) = \int_{-\infty}^{b}\int_{-\infty}^{a} f(x,y)\,dxdy.$$

Differentiating with respect to a and b, we have

$$\frac{\partial^2}{\partial a \partial b}F(a,b) = f(a,b).$$

(iii) The density of X can be obtained from the joint density $f(x,y)$. For $A\subseteq\mathbb{R}$, one can see that

$$\int_A f_X(x)\,dx = \mathbb{P}(X\in A) = \mathbb{P}(X\in A, Y\in\mathbb{R}) = \int_{\mathbb{R}}\int_A f(x,y)\,dxdy = \int_A \left(\int_{\mathbb{R}} f(x,y)\,dy\right)\,dx$$

Let $A = (-\infty, x)$ and differentiate in x, then

$$f_X(x) = \int_{\mathbb{R}} f(x,y)\,dy.$$

Similarly, we have

$$f_Y(y) = \int_{\mathbb{R}} f(x,y)\,dx.$$

We call f_X and f_Y the marginal densities of X and Y.

Example 11. The joint density function of X and Y is given by

$$f(x,y) = \begin{cases} 2e^{-x}e^{-2y}, & 0<x<\infty, 0<y<\infty \\ 0, & \text{otherwise.}\end{cases}$$

Compute $\mathbb{P}(X<Y)$.

Example 12. The joint density function of X and Y is given by

$$f(x,y) = \begin{cases} c(y^2-x^2)e^{-y}, & -y\le x\le y, 0<y<\infty \\ 0, & \text{otherwise.}\end{cases}$$

(i) Find c.
(ii) Compute $\mathbb{P}(0<X<1, Y<1)$.
(iii) Find the marginal density of Y.
(iv) Compute $\mathbb{E}[Y]$.

More than two random Variables
For random variables $X_1, X_2, \cdots, X_n$, the joint cumulative distribution function is defined by

$$F(a_1, a_2, \cdots, a_n) = \mathbb{P}(X_1 \leq a_1, X_2 \leq a_2, \cdots, X_n \leq a_n).$$

The random variables $X_1, X_2, \cdots, X_n$ are jointly continuous if there exists a nonnegative function $f(x_1, x_2, \cdots, x_n)$ such that

$$\mathbb{P}((X_1, X_2, \cdots, X_n) \in C) = \iint \cdots \int_C f(x_1, x_2, \cdots, x_n)\, dx_1 dx_2 \cdots dx_n$$

for all $C \subseteq \mathbb{R}^n$.

References

[SR] Sheldon Ross, *A First Course in Probability*, 9th Edition, Pearson

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