Homework 2

Math 461: Probability Theory, Spring 2022
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Due date: Feb 4, 2022

Instruction

1. Each problem is worth 10 points and only five randomly chosen problems will be graded.
2. Convert a photocopy of your solutions to **one single pdf file** and upload it on Moodle.
3. Please indicate whom you worked with, it will not affect your grade in any way.

1. (a) How many vectors $(x_1, x_2, \dots, x_n)$ are there for which each x_i is either 0 or 1 and
$$x_1 + x_2 + \cdots + x_n = k.$$
(b) How many vectors $(x_1, x_2, \dots, x_n)$ are there for which each x_i is either 0 or 1 and
$$x_1 + x_2 + \cdots + x_n \leqslant k.$$

2. (a) How many vectors $(x_1, x_2, \dots, x_n)$ are there for which each $x_i \geqslant 0$ is a non-negative integer and
$$x_1 + x_2 + \cdots + x_n = k.$$
(b) Suppose $k \geqslant n$. How many vectors $(x_1, x_2, \dots, x_n)$ are there for which each $x_i \geqslant 1$ is a positive integer and
$$x_1 + x_2 + \cdots + x_n = k.$$
(c) How many vectors $(x_1, x_2, \dots, x_n)$ are there for which each $x_i \geqslant 0$ is a non-negative integer and
$$x_1 + x_2 + \cdots + x_n \leqslant k.$$

3. Consider the set S of numbers $\{1, 2, \dots, n\}$. One can see that the number of subsets of S size k is $\binom{n}{k}$. Count the same number in a different way depending on how many subsets of size k have i as their highest numbered member, to give a proof of the following identity known as Fermat's combinatorial identity: For all integers $n \geqslant k$
$$\binom{n}{k} = \sum_{i=k}^{n} \binom{i-1}{k-1}.$$

4. (a) In how many ways can n identical balls be distributed into r bins such that each bin contains at least two balls. Assume that $n \geqslant 2r$.
(b) Do the same problem as in (a), but now each bin contains at least three balls and $n \geqslant 3r$.

5. A group of individuals containing b boys and g girls is lined up in random order; that is, each of the $(b+g)!$ permutations is assumed to be equally likely. What is the probability that the person in the i-th position, $1 \leqslant i \leqslant b+g$, is a girl?

6. Two cards are randomly selected from an ordinary playing deck. What is the probability that they form a blackjack? That is, what is the probability that one of the cards is an ace and the other one is either a ten, a jack, a queen, or a king?

7. A card player is dealt a 13 card hand from a well-shuffled, standard deck of cards. What is the probability that the hand is void in at least one suit ("void in a suit" means having no cards of that suit)?
Hint: Let E_i be the event that the hand is void in the suit i for $i = 1, 2, 3, 4$ (*clubs, hearts, diamonds* and *spades*).

8. For a group of 10 people, find the probability that all 4 seasons (winter, spring, summer, fall) occur at least once each among their birthdays, assuming that all seasons are equally likely.
Hint: Let E_i be the event that there are no birthdays in the i-th season.

9. A system is comprised of 5 components, each of which is either working or failed. Consider an experiment that consists of observing the status of each component, and let the outcome of the experiment be given by the vector $(x_1, x_2, x_3, x_4, x_5)$, where x_i is equal to 1 if component i is working and is equal to 0 if component i is failed.

   (a) How many outcomes are in the sample space of this experiment?

   (b) Suppose that the system will work if components 1 and 2 are both working, or if components 3 and 4 are both working, or if components 1, 3, and 5 are all working. Let W be the event that the system will work. Specify all the outcomes in W.

   (c) Let A be the event that components 4 and 5 are both failed. How many outcomes are contained in the event A?

   (d) Write out all the outcomes in the event AW.

10. Consider an experiment that consists of determining the type of job – either blue-collar or white-collar – and the political affiliation – Republican, Democratic, or Independent – of the 20 members of an adult soccer team. How many outcomes are

   (a) in the sample space?

   (b) in the event that at least one of the team members is a blue-collar worker?

   (c) in the event that none of the team members considers himself or herself an Independent?