Math 461: Probability Theory

Midterm 1 Solution, Spring 2022

1. (10 points) Circle True or False. Do not justify your answer.

   (a) **TRUE** False The number of different orderings of the letters *PROBABILITY* is $\binom{11}{2}\binom{9}{2}7!$.

   > **Solution:** Among 11 letters, there are two B's and two I's. Thus, the number of orderings is $11!/(2!2!)$.

   (b) **TRUE** False If $\mathbb{P}(E) = 0.2$ and $\mathbb{P}(F) = 0.3$, then $\mathbb{P}(E \cup F)$ cannot be greater than 0.5.

   > **Solution:** Since $\mathbb{P}(E) + \mathbb{P}(F) - \mathbb{P}(E \cap F) = \mathbb{P}(E \cup F)$ and $\mathbb{P}(E \cap F) \geqslant 0$, we have $\mathbb{P}(E \cup F) \leqslant \mathbb{P}(E) + \mathbb{P}(F) = 0.5$.

   (c) True **FALSE** If $\mathbb{E}[X] = 2$ and $\mathbb{E}[X^2] = 5$, then $\mathrm{Var}(X) = 3$.

   > **Solution:** $\mathrm{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = 5 - 2^2 = 1$.

   (d) True **FALSE** A urn has 8 blue balls and 7 red balls. We draw balls from the urn at random (equally likely) with replacement until 3 blue balls are drawn. If X is the number of balls drawn, then X is a binomial random variable with parameters 3 and $8/(8+7)$.

   > **Solution:** Since we draw a ball with replacement, each experiment is independent and identical, with success probability (drawing a blue ball) $8/15$. Since X is the waiting time until 3 successes, it is a negative binomial with 3 and $8/15$.

   (e) True **FALSE** Let A, B, C be events satisfying $\mathbb{P}(A) \geqslant \mathbb{P}(B)$ and $\mathbb{P}(C) > 0$. Then, we have $\mathbb{P}(A|C) \geqslant \mathbb{P}(B|C)$.

   > **Solution:** We toss 2 fair coins. Let A be the event that at least one is Heads, and $B = C$ the event that the both are Tails. Then, $\mathbb{P}(A) = 3/4 \geqslant \mathbb{P}(B) = \mathbb{P}(C) = \mathbb{P}(B \cap C) = 1/4$ and $\mathbb{P}(A \cap C) = 0$. Thus,
   >
   > $$0 = \mathbb{P}(A|C) = \frac{\mathbb{P}(A \cap C)}{\mathbb{P}(C)} < 1 = \frac{\mathbb{P}(B \cap C)}{\mathbb{P}(C)} = \mathbb{P}(B|C).$$

2. (10 points) Give definitions of the following.

   (a) Conditional probability of E given F.

Solution: If $\mathbb{P}(F) > 0$, then conditional probability of E given F is defined by

$$\mathbb{P}(E|F) = \frac{\mathbb{P}(E \cap F)}{\mathbb{P}(F)}.$$

(b) Three events E, F, G are independent.

Solution: Three events E, F, G are independent if

$$\mathbb{P}(E \cap F) = \mathbb{P}(E)\mathbb{P}(F), \quad \mathbb{P}(E \cap G) = \mathbb{P}(E)\mathbb{P}(G), \quad \mathbb{P}(F \cap G) = \mathbb{P}(F)\mathbb{P}(G),$$
$$\mathbb{P}(E \cap F \cap G) = \mathbb{P}(E)\mathbb{P}(F)\mathbb{P}(G).$$

(c) Cumulative distribution function.

Solution: Let X be a random variable. The cumulative distribution function is a function from $\mathbb{R}$ to $[0, 1]$ defined by $F(x) = \mathbb{P}(X \leqslant x)$, $x \in \mathbb{R}$.

(d) Expectation of a discrete random variable X.

Solution: Suppose a discrete random variable X takes its values $x_1, x_2, \cdots$. Then, the expectation of X is defined by

$$\mathbb{E}[X] = \sum_{k=1}^{\infty} x_k \mathbb{P}(X = x_k).$$

(e) Inclusion-Exclusion principle for three events E, F, G.

Solution:

$$\mathbb{P}(E \cup F \cup G) = \mathbb{P}(E) + \mathbb{P}(F) + \mathbb{P}(G) - \mathbb{P}(E \cap F) - \mathbb{P}(F \cap G) - \mathbb{P}(E \cap G) + \mathbb{P}(E \cap F \cap G).$$

3. (10 points) Let X be a discrete random variable with PMF

$$\mathbb{P}(X = k) = \begin{cases} ck^2, & k = 1, 2, 3, 4, \\ 0, & \text{otherwise} \end{cases}$$

for some constant $c > 0$. Find the constant c and the expectation $\mathbb{E}[X]$.

Solution: Since $1 = \mathbb{P}(X = 1) + \mathbb{P}(X = 2) + \mathbb{P}(X = 3) + \mathbb{P}(X = 4) = c(1 + 4 + 9 + 16)$, $c = 1/30$. Then,

$$\mathbb{E}[X] = \mathbb{P}(X = 1) + 2\mathbb{P}(X = 2) + 3\mathbb{P}(X = 3) + 4\mathbb{P}(X = 4) = \frac{1}{30}(1 + 8 + 27 + 64) = 10/3.$$

4. An airline overbooks a flight, selling more tickets for the flight than there are seats on the plane (figuring that it is likely that some people won't show up). The plane has 100 seats, and 110 people have booked the flight. Each person will show up for the flight with probability 0.9, independently.

   (a) (5 points) Let X be the number of people showing up. Is X a binomial random variable? Justify your answer. If so, find the parameters.

   (b) (5 points) Find the approximate probability that exactly 100 passengers show up.

Solution:

(a) Since each person will show up independently with the same success probability 0.9, it is a binomial random variable with 110 (since there are 110 people) and 0.9 (success probability).

(b) Let Y be the number of people not showing up, then $Y = 110 - X$ and it is a binomial with $n = 110$ and $p = 0.1$. Since $\mathrm{Bin}(n, p) \approx \mathrm{Pois}(\lambda)$ for large n and small p where $\lambda = np$, Y can be approximately considered as a Poisson random variable with $110 \times 0.1 = 11$. Thus,

$$\mathbb{P}(X = 100) = \mathbb{P}(Y = 10) \approx e^{-11}\frac{(11)^{10}}{10!}.$$

5. (10 points) A hat contains 10 coins, where 9 are fair but one is double-headed (always landing Heads). A coin is chosen equally likely and the chosen coin is flipped 4 times. Find the probability that the chosen coin is double-headed given that it lands Heads all 4 times.

Solution: Let E be the event that the double-headed coin is chosen, and F the event that the chosen coin lands Heads all 4 times. By Bayes formula,

$$\begin{aligned}\mathbb{P}(E|F) &= \frac{\mathbb{P}(F|E)\mathbb{P}(E)}{\mathbb{P}(F|E)\mathbb{P}(E) + \mathbb{P}(F|E^c)\mathbb{P}(E^c)} \\ &= \frac{1/10}{1/10 + (1/2)^4 \cdot 9/10} = \frac{16}{25}\end{aligned}$$

6. (10 points) Let X be a Poisson random variable with parameter λ. Show that $\mathbb{E}[X] = \lambda$.

Solution: It follows that

$$\mathbb{E}[X] = \sum_{k=0}^{\infty} k e^{-\lambda}\frac{\lambda^k}{k!} = \lambda \sum_{k=1}^{\infty} e^{-\lambda}\frac{\lambda^{k-1}}{(k-1)!} = \lambda.$$