Homework 1 Solution

Math 461: Probability Theory, Spring 2022
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1. For years, area zip codes in the United States and Canada consisted of a sequence of five digits. The first digit was an integer between 2 and 9, the second digit was either 0 or 1, and the third to fifth digits were any integer from 1 to 9.

   (a) How many area codes were possible?

   (b) How many area codes starting with a 4 were possible?

   (c) How many area codes were possible with all distinct digits?

Solution: (a) Number of possible area codes: $8 \cdot 2 \cdot 9^3 = 11664$ (Generalized counting principle: There are eight choices for the first digit, two for the second digit, and nine for the last third digits.)
(b) Number of area codes starting with a 4 is $1 \cdot 2 \cdot 9^3 = 1458$.
(c) There are eight choices for the first digit. If the second digit is 1, number of codes is $8 \cdot 1 \cdot 7 \cdot 6 \cdot 5 = 1680$. If the second digit is 0, number of codes is $8 \cdot 1 \cdot 8 \cdot 7 \cdot 6 = 2688$. This the answer is $2688 + 4032 = 4368$.

2. Five separate awards (best scholarship, best leadership qualities, and so on) are to be presented to selected students from a class of 25. How many different outcomes are possible if

   (a) a student can receive any number of awards?

   (b) a student can receive at most 1 award?

   (c) a student can receive at most 4 awards?

Solution: (a) For each of the award, we can choose any one of the students out of total 25 students in 25 ways. Thus the total number of outcomes possible os 25^5.
(b) Since each student can receive at most 1 award, for the first award we can choose a student in 25 ways, then for the second award we can choose a student from the remaining students in 24 ways and so on, for the fifth award we can choose a student in 21 ways. Thus total number of possible outcomes is $25 \cdot 24 \cdot 23 \cdot 22 \cdot 21$.
(c) $25^5 - 25$.

3. A person has 15 friends, of whom 6 will be invited to a party.

   (a) How many choices are there in total?

   (b) How many choices are there if 2 of the friends are feuding and will not attend together?

   (c) How many choices if 3 of the friends will only attend together?

Solution: (a) $\binom{15}{6} = 5005$.
(b) Suppose that A and B are feuding and will not attend together. Total number of ways to invite a group of 6 friends out of 15 is $\binom{15}{6} = 5005$. Out of these, A and B will be together in $\binom{13}{4} = 715$ cases, as the remaining 4 invited friends will be from the 13 remaining friends. Thus number of choices in which A and B will not attend together is $\binom{15}{6} - \binom{13}{4} = 4290$.

(c) If A, B and C will only attend together, number of choices where they will attend is $\binom{12}{3} = 220$ and number of choices where that will not attend is $\binom{12}{6} = 924$. Thus the total number of choices is $\binom{12}{3} + \binom{12}{6} = 1144$.

4. How many different letter arrangements can be made from the letters

   (a) BLOCK (b) CLASS (c) ALLOWANCE (d) MISSISSIPPI

Solution:

(a) $5! = 120$ (five distinct letters)

(b) $\frac{5!}{2!} = 60$ (five letters, two As)

(c) $\frac{9!}{2!2!} = 90720$ (nine letters, including two As and two Ls)

(d) $\frac{11!}{4!\cdot 4!\cdot 2!} = 34,650$ (eleven letters, including four Is, four Ss, and two Ps)

5. From a group of 10 women and 8 men, a committee consisting of 4 men and 4 women is to be formed. How many different committees are possible if

   (a) 2 of the men refuse to serve together?

   (b) 2 of the women refuse to serve together?

   (c) 1 man and 1 woman refuse to serve together?

Solution:

(a) Let's find the number of bad choices first. If the two finicky men serve together, there is room for 2 more men on the committee, and there are 6 men left to take this spot, i.e., there are $\binom{10}{4} \cdot \binom{6}{2}$ bad choices, so that there are $\binom{10}{4} \cdot \left(\binom{8}{4} - \binom{6}{2}\right) = 11550$ good choices.

(b) Same reasoning as in (a): There are $\binom{8}{2} \cdot \binom{8}{4}$ bad choices, so that there are $\left(\binom{10}{4} - \binom{8}{2}\right)\binom{8}{4} = 12740$ good choices.

(c) Here, there are $\binom{9}{3} \cdot \binom{7}{3}$ bad choices, so that there are $\binom{10}{4} \cdot \binom{8}{4} - \binom{9}{3} \cdot \binom{7}{3} = 11760$ good choices.

6. In how many ways can 2 chemistry, 3 mathematics books, and 4 biology book be arranged on a bookshelf if

   (a) the books can be arranged in any order?

   (b) the mathematics books must be together and the biology books must be together?

   (c) the biology books must be together, but the other books can be arranged in any order?

Solution: (a) Arrange all 9 books in 9! ways.
(b) 2 chemistry books, one math subject and 1 biology subject can be arranged in 4! ways. The math books themselves can be arranged in 3! ways. Similar for biology books. So the answer is 4!3!4!.
(c) Similar to (b). 6!4!

7. (a) How many ways are there to split a dozen people into 3 teams, where one team has 2 people, and the other two teams have 5 people each? The teams are unordered.

   (b) How many ways are there to split a dozen people into 3 teams, where each team has 4 people? The teams are unordered.

Solution: (a) The number of ways to divide 12 people into three groups with first group having 2 people, second group having 5 people and the last group having 5 people is $\binom{12}{2,5,5} = \frac{12!}{2!5!^2}$. However, since the groups are unordered the two groups of size 5 each can be permutated in 2! ways. Thus the number is $\frac{12!}{2!^2 5!^2}$.
(b) The number of ways to split 12 people into three teams each having 4 people is $\binom{12}{4,4,4}\frac{1}{3!} = \frac{12!}{4!^3 3!}$. The last $\frac{1}{3!}$ factor is coming because the three teams are indistinguishable.

8. (a) If 14 people are to be divided into 5 distinct committees of respective sizes 2, 2, 2, 3 and 5, how many divisions are possible?
(b) What if the committees are not distinct, i.e., we want 3 committees of size 2 each, 1 of size 3 and 1 of size 5.

Solution: (a) $\binom{14}{2,2,2,3,5} = \frac{14!}{2^3 3! 5!}$.
(b) $\frac{1}{3!}\binom{14}{2,2,2,3,5}$

9. (a) How many paths are there from the point $(0,0)$ to the point $(10,10)$ in the plane such that each step either consists of going one unit up or one unit to the right?
   (b) How many paths are there from $(0,0)$ to $(20,20)$, where each step consists of going one unit up or one unit to the right, and the path has to go through $(10,10)$?

Solution: (a) Since the paths can go only up and right, to go from $(0,0)$ to $(10,10)$ one needs to take 10 up steps and 10 right steps in any order. Total number of steps will be $10+10=20$. Thus out of 20 steps we need to choose 10 steps in which one should go right. Thus the number of paths is $\binom{20}{10}$.
(b) Any path that goes from $(0,0)$ to $(20,20)$ passing through $(10,10)$ can be broken into two paths, one from $(0,0)$ to $(10,10)$ and the other from $(10,10)$ to $(20,20)$ using only up and right directions. The first segment can be chosen in $\binom{20}{10}$ ways using part (a). For the second segment the number is = number of ways to choose $20-10=10$ right steps out of total $20+20-10-10=20$ steps, which is $\binom{20}{10}$. Thus the total number of paths is $\binom{20}{10}^2$.

10. Expand (a) $(2y+z^2)^5$ and (b) $(x+2y+z)^3$.

Solution: (a) Binomial theorem:

$$\begin{aligned}(2y+z^2)^5 &= \sum_{k=0}^{5}\binom{5}{k}(2y)^k(z^2)^{5-k}\\ &= z^{10} + 10yz^8 + 40y^2z^6 + 80y^3z^4 + 80y^4z^2 + 32y^5.\end{aligned}$$

(b) Multinomial theorem

$$(x+2y+z)^3 = 6x^2y + 3x^2z + x^3 + 12xy^2 + 12xyz + 3xz^2 + 12y^2z + 8y^3 + 6yz^2 + z^3.$$