Section 1.1 : Systems of Linear Equations

Chapter 1 : Linear Equations

Math 1554 Linear Algebra

Section 1.1 Systems of Linear Equations

Topics

We will cover these topics in this section.

1. Systems of Linear Equations
2. Matrix Notation
3. Elementary Row Operations
4. Questions of Existence and Uniqueness of Solutions

Objectives

For the topics covered in this section, students are expected to be able to do the following.

1. Characterize a linear system in terms of the number of solutions, and whether the system is consistent or inconsistent.
2. Apply elementary row operations to solve linear systems of equations.
3. Express a set of linear equations as an augmented matrix.

A Single Linear Equation

A linear equation has the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

$a_1, \ldots, a_n$ and b are the **coefficients**, $x_1, \ldots, x_n$ are the **variables** or **unknowns**, and n is the **dimension**, or number of variables.

For example,

- $2x_1 + 4x_2 = 4$ is a line in two dimensions
- $3x_1 + 2x_2 + x_3 = 6$ is a plane in three dimensions

Systems of Linear Equations

When we have more than one linear equation, we have a **linear system** of equations. For example, a linear system with two equations is

$$\begin{cases} x_1 + 1.5x_2 + \pi x_3 = 4 \\ 5x_1 + 0 \cdot x_2 + 7x_3 = 5 \end{cases}$$

dimension = 3.

2 Eqn.

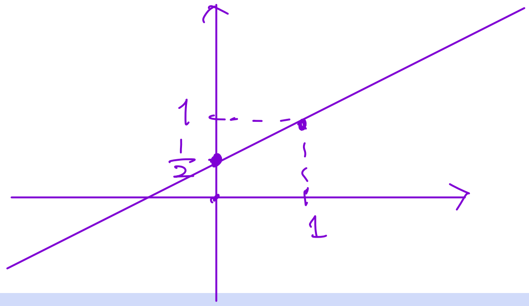
Definition: Solution to a Linear System

The set of all possible values of $x_1, x_2, \ldots x_n$ that satisfy all equations is the **solution** to the system.

A system can have a unique solution, no solution, or an infinite number of solutions.

Theorem, later.

$$\begin{cases} \boxed{x_1 - 2x_2 = -1} & \text{— ①} \longrightarrow (x_1, x_2) \text{ satisfies ①} \\ -x_1 + 3x_2 = 3 & \text{— ②} \longrightarrow \text{'' ②} \end{cases}$$



$x_1 = 1 : \quad 1 - 2 \cdot x_2 = -1, \quad x_2 = 1$

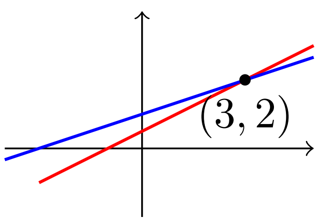
$x_1 = 0 : \quad 0 - 2 \cdot x_2 = -1, \quad x_2 = \frac{1}{2}$

$\vdots$

Two Variables

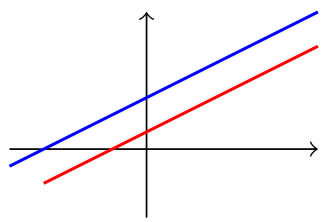
Consider the following systems. How are they different from each other?

$$\begin{cases} x_1 - 2x_2 = -1 \\ -x_1 + 3x_2 = 3 \end{cases}$$



non-parallel lines

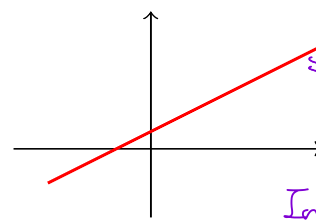
$$\begin{aligned} x_1 - 2x_2 &= -1 \\ -x_1 + 2x_2 &= 3 \end{aligned}$$



parallel lines

Empty Solution Set

$$\begin{aligned} x_1 - 2x_2 &= -1 \\ -x_1 + 2x_2 &= 1 \end{aligned}$$



= Solution

identical lines

Infinitely many solutions.

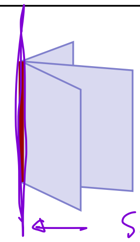
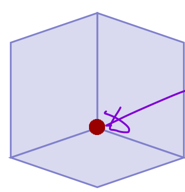
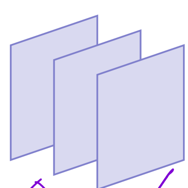
Solution of the system

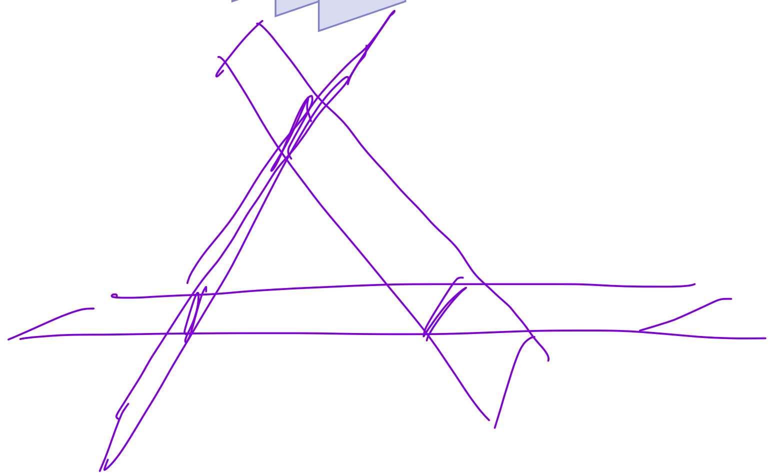
= Intersection of two lines.

Three Equns → Three planes

Three-Dimensional Case

An equation $a_1x_1 + a_2x_2 + a_3x_3 = b$ defines a plane in $\mathbb{R}^3$. The **solution** to a system of **three equations** is the set of intersections of the planes.

solution set	sketch	number of solutions
Intersection of planes		
line		Infinity
point		1
empty		0



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Row Reduction by Elementary Row Operations

How can we find the solution set to a set of linear equations?
We can manipulate equations in a linear system using **row operations**.

1. (Replacement/Addition) Add a multiple of one row to another.
2. (Interchange) Interchange two rows.
3. (Scaling) Multiply a row by a non-zero scalar.

Let's use these operations to solve a system of equations.

Example 1

Identify the solution to the linear system.

$$\begin{array}{rrrl} x_1 & -2x_2 & +x_3 & = 0 \quad \text{——} \ 1 \\ & 2x_2 & -8x_3 & = 8 \quad \text{——} \ 2 \\ 5x_1 & & -5x_3 & = 10 \quad \text{——} \ 3 \end{array}$$

0

3 − 5 × 1 ← Replace

$: (5x_1 - 5 \cdot x_1) + 10x_2 + (-5x_3 - 5x_3) = 10$

$\Rightarrow \quad 0 + 10x_2 - 10x_3 = 10$ — 4

4 × $\frac{1}{10}$ ← Scaling

$x_2 - x_3 = 1$

$\Rightarrow$ Reduce # of variables.

$a_1 x_1 + a_2 \cdot x_2 + \cdots + a_n \cdot x_n = b$ Linear Eqn.

System of Linear Eqn $\left\{ \begin{array}{c} \vdots \\ \vdots \end{array} \right.$

\# of variables = dimension.

① Replace
② Swap
③ Scaling.

Example 1

Identify the solution to the linear system.

$$\begin{array}{rrrl} x_1 & -2x_2 & +x_3 & = 0 \quad \text{—— ①} \\ & 2x_2 & -8x_3 & = 8 \quad \text{—— ②} \\ 5x_1 & & -5x_3 & = 10 \quad \text{—— ③} \end{array}$$

Replacement

③ + (−5)①

$(-5x_1 + 10x_2 - 5x_3 = 0)$

$$\left\{ \begin{array}{rl} x_1 - 2x_2 + x_3 = 0 & \text{— ①} \\ 2x_2 - 8x_3 = 8 & \text{— ②} \\ 10x_2 - 10x_3 = 10 & \text{— ③'} \end{array} \right.$$

② → $\frac{1}{2}$ × ② , ③' → $\frac{1}{10}$ × ③ ← Scaling

$$\left\{ \begin{array}{rl} x_1 - 2x_2 + x_3 = 0 & \\ x_2 - 4x_3 = 4 & \text{— ⓐ} \\ x_2 - x_3 = 1 & \text{— ⓑ} \end{array} \right.$$

2 variables

ⓐ + (−1)·ⓑ :

$$\begin{array}{rl} & x_2 - 4x_3 = 4 \\ + & -x_2 + x_3 = -1 \\ \hline & \boxed{0 - 3x_3 = 3} \end{array} \Rightarrow x_3 = -1$$

$x_2 - (-1) = 1$ $x_2 = 0$

$x_1 - 2 \cdot 0 + (-1) = 0$ $\therefore x_1 = 1$

Augmented Matrices

It is redundant to write x_1, x_2, x_3 again and again, so we rewrite systems using matrices. For example,

$$\begin{array}{rrrl} 1 \cdot x_1 & -2x_2 & +1x_3 & = 0 \\ 0 \cdot x_1 & +2x_2 & -8x_3 & = 8 \\ 5x_1 & +0 \cdot x_2 & -5x_3 & = 10 \end{array}$$

can be written as the **augmented matrix**,

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 5 & 0 & -5 & 10 \end{array}\right]$$

The vertical line reminds us that the first three columns are the coefficients to our variables x_1, x_2, and x_3.

Systems $\longleftrightarrow$ Augmented Matrices

Row operations $\to$ { Replace, Swap, Scaling }

Simpler form

Row operations

another matrices

row Equivalent

Consistent Systems and Row Equivalence

Definition (Consistent)

A linear system is **consistent** if it has at least one solution.

Definition (Row Equivalence)

Two matrices are **row equivalent** if a sequence of row operations transforms one matrix into the other.

Note: if the augmented matrices of two linear systems are row equivalent, then they have the same solution set.

Fundamental Questions

Two questions that we will revisit many times throughout our course.

1. Does a given linear system have a solution? In other words, is it consistent?
2. If it is consistent, is the solution unique?

Section 1.2 : Row Reduction and Echelon Forms

Chapter 1 : Linear Equations

Math 1554 Linear Algebra

Section 1.2 : Row Reductions and Echelon Forms

Topics
We will cover these topics in this section.

1. Row reduction algorithm
2. Pivots, and basic and free variables
3. Echelon forms, existence and uniqueness

Objectives
For the topics covered in this section, students are expected to be able to do the following.

1. Characterize a linear system in terms of the number of leading entries, free variables, pivots, pivot columns, pivot positions.
2. Apply the row reduction algorithm to reduce a linear system to echelon form, or reduced echelon form.
3. Apply the row reduction algorithm to compute the coefficients of a polynomial.

Definition: Echelon Form and RREF

A rectangular matrix is in **echelon form** if

1. All zero rows (if any are present) are at the bottom.
2. The first non-zero entry (or **leading entry**) of a row is to the right of any leading entries in the row above it (if any).
3. All elements below a leading entry (if any) are zero.

A matrix in echelon form is in **reduced row echelon form** (RREF) if

1. All leading entries, if any, are equal to 1.
2. Leading entries are the only nonzero entry in their respective column.

$$\begin{matrix} 0 \\ \vdots \\ 0 \\ \boxed{} = 1 \\ 0 \\ 0 \\ 0 \end{matrix}$$

Example of a Matrix in Echelon Form

$\blacksquare$ = non-zero number, $\quad * =$ any number

$$\begin{bmatrix} 0 & \blacksquare & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

zero row

↓

bottom

Example 1

$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

Which of the following are in RREF?

$a) \quad \begin{bmatrix} 1 & 0 \\ 0 & \cancel{2} \end{bmatrix}$ ← Identity

1

$d) \quad \begin{bmatrix} 0 & 6 & 3 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & 1 & \frac{1}{2} & 0 \end{bmatrix}$

$b) \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$e) \quad \begin{bmatrix} 1 & 17 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$c) \quad \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

Definition: Pivot Position, Pivot Column

A **pivot position** in a matrix A is a location in A that corresponds to a leading 1 in the reduced echelon form of A.

A **pivot column** is a column of A that contains a pivot position.

Example 2: Express the matrix in reduced row echelon form and identify the pivot columns.

pivot position

$$A = \begin{bmatrix} 0 & -3 & -6 & 4 \\ -1 & -2 & -1 & 3 \\ -2 & -3 & 0 & 3 \end{bmatrix} \longrightarrow$$

pivot columns

RREF

$$\begin{bmatrix} 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Row Reduction Algorithm

The algorithm we used in the previous example produces a matrix in RREF. Its steps can be stated as follows.

Step 1a Swap the 1st row with a lower one so the leftmost nonzero entry is in the 1st row

Step 1b Scale the 1st row so that its leading entry is equal to 1

Step 1c Use row replacement so all entries below this 1 are 0

Step 2a Swap the 2nd row with a lower one so that the leftmost nonzero entry below 1st row is in the 2nd row

etc. ...

Now the matrix is in echelon form, with leading entries equal to 1.

Last step Use row replacement so all entries above each leading entry are 0, starting from the right.

$$\begin{cases} x_1 + 3x_2 + 0\cdot x_3 + 7x_4 + 0x_5 = 4 \\ x_3 + 4x_4 = 5 \\ x_5 = 6 \end{cases}$$

Basic And Free Variables

Consider the augmented matrix

$$\left[A \mid \vec{b}\right] = \begin{array}{c} \begin{array}{cccccc} x_1 & & x_3 & & x_5 & \end{array} \\ \left[\begin{array}{ccccc|c} 1 & 3 & 0 & 7 & 0 & 4 \\ 0 & 0 & 1 & 4 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 6 \end{array}\right] \\ \begin{array}{cccccc} & x_2 & & x_4 & & \end{array} \end{array}$$

$\rightarrow$ basic variables

$\rightarrow$ free variables

The leading one's are in first, third, and fifth columns. So:

- the pivot variables of the system $A\vec{x} = \vec{b}$ are x_1, x_3, and x_5.
- The free variables are x_2 and x_4. **Any choice** of the free variables leads to a solution of the system.

Note that A does not have basic variables or free variables. Systems have variables.

Existence and Uniqueness

Theorem

A linear system is consistent if and only if (exactly when) the last column of the **augmented** matrix does not have a pivot. This is the same as saying that the RREF of the augmented matrix does **not** have a row of the form

$$\begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & | & 1 \end{pmatrix}$$

Moreover, if a linear system is consistent, then it has

1. a unique solution if and only if there are no free variables.
2. infinitely many solutions that are parameterized by free variables.

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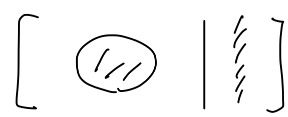
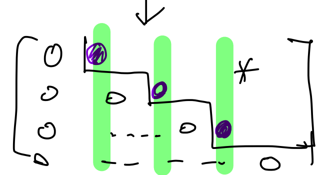
Recall

Linear Systems $\rightarrow$ Augmented matrix



row operations { Replace, Swap, Scaling.



pivot columns $\rightarrow$ basic variables

non pivot " $\rightarrow$ free variables

Consistent $\Leftrightarrow$ X [0 0 ... 0 | 1]

If consistent,

free $\rightarrow$ Infinitely many Sol.

no free $\rightarrow$ 1 solution.

Section 1.3 : Vector Equations

Chapter 1 : Linear Equations

Math 1554 Linear Algebra

1.3: Vector Equations

Topics

We will cover these topics in this section.

1. Vectors in $\mathbb{R}^n$, and their basic properties
2. Linear combinations of vectors

Objectives

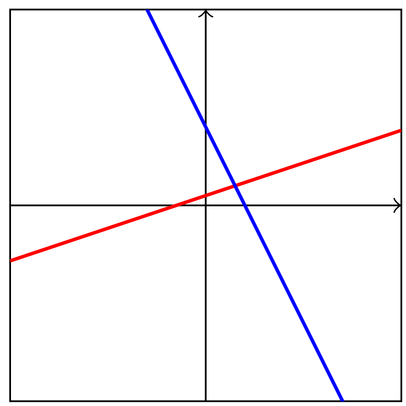
For the topics covered in this section, students are expected to be able to do the following.

1. Apply geometric and algebraic properties of vectors in $\mathbb{R}^n$ to compute vector additions and scalar multiplications.
2. Characterize a set of vectors in terms of **linear combinations**, their **span**, and how they are related to each other geometrically.

Motivation

We want to think about the **algebra** in linear algebra (systems of equations and their solution sets) in terms of **geometry** (points, lines, planes, etc).

$$x - 3y = -3$$
$$2x + y = 8$$



- This will give us better insight into the properties of systems of equations and their solution sets.
- To do this, we need to introduce n-dimensional space $\mathbb{R}^n$, and **vectors** inside it.

$\mathbb{R}^n$

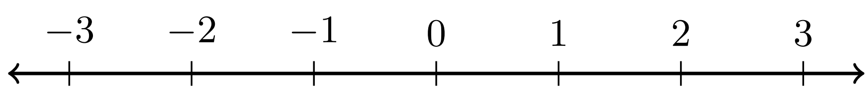
$= \{1, 2, \ldots, -1, \ldots, \pi, \ldots\}$

Recall that $\mathbb{R}$ denotes the collection of all real numbers.

Let n be a positive whole number. We define

$$\mathbb{R}^n = \text{all ordered } n\text{-tuples of real numbers } (x_1, x_2, x_3, \ldots, x_n).$$

When $n = 1$, we get $\mathbb{R}$ back: $\mathbb{R}^1 = \mathbb{R}$. Geometrically, this is the **number line**.

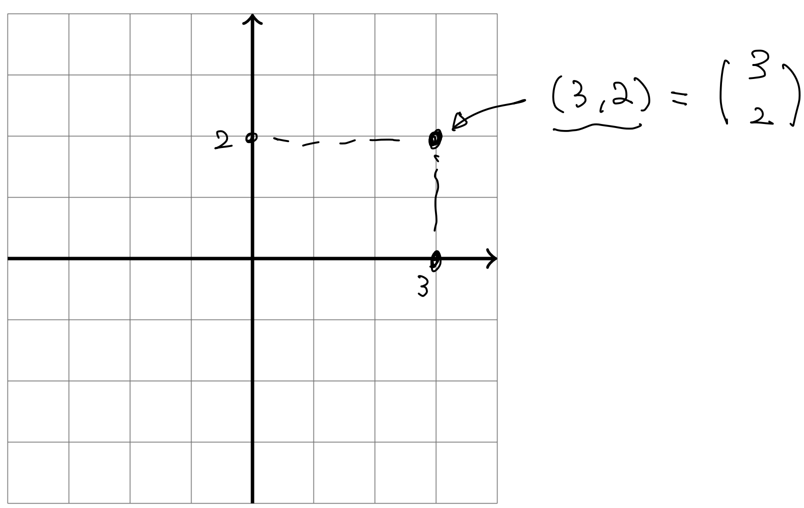


$\mathbb{R}^2$

Note that:

- when $n = 2$, we can think of $\mathbb{R}^2$ as a **plane**
- every point in this plane can be represented by an ordered pair of real numbers, its x- and y-coordinates

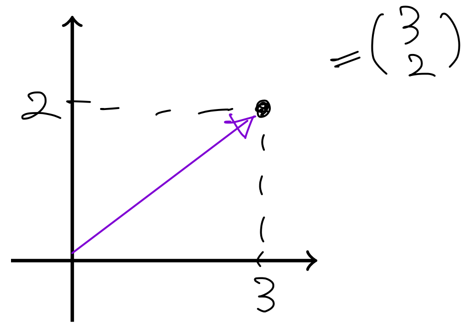
Example: Sketch the point $(3, 2)$ and the vector $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$.



Vectors

In the previous slides, we were thinking of elements of $\mathbb{R}^n$ as **points**: in the line, plane, space, etc.

We can also think of them as **vectors**: arrows with a given length and direction.



For example, the vector $\begin{pmatrix}3\\2\end{pmatrix}$ points **horizontally** in the amount of its x-coordinate, and **vertically** in the amount of its y-coordinate.

$$\mathbb{R}^n = \{ (x_1, x_2, \dots, x_n) : x_1, \dots, x_n \in \mathbb{R} \}$$

= { All points in n-dim. space }

= { Arrows from the origin }

$$= \left\{ \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \right\}$$

↖ vectors.

addition & multiplication

Vector Algebra

When we think of an element of $\mathbb{R}^n$ as a vector, we write it as a matrix with n rows and one column:

($\mathbb{R}^n$ vector = $n \times 1$ matrix)

$$\vec{v} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

Suppose

$$\vec{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad \vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$

Vectors have the following properties.

1. **Scalar Multiple**:

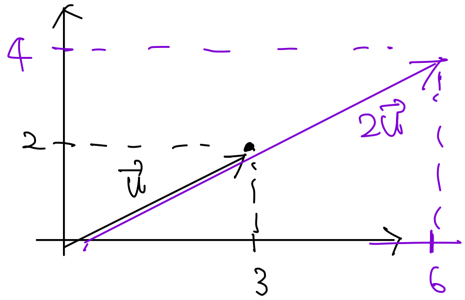
$c \in \mathbb{R}$

$$c\vec{u} = \begin{pmatrix} c \cdot u_1 \\ c\ u_2 \end{pmatrix}$$

2. **Vector Addition**:

$$\vec{u} + \vec{v} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} u_1 + v_1 \\ u_2 + v_2 \end{pmatrix}$$

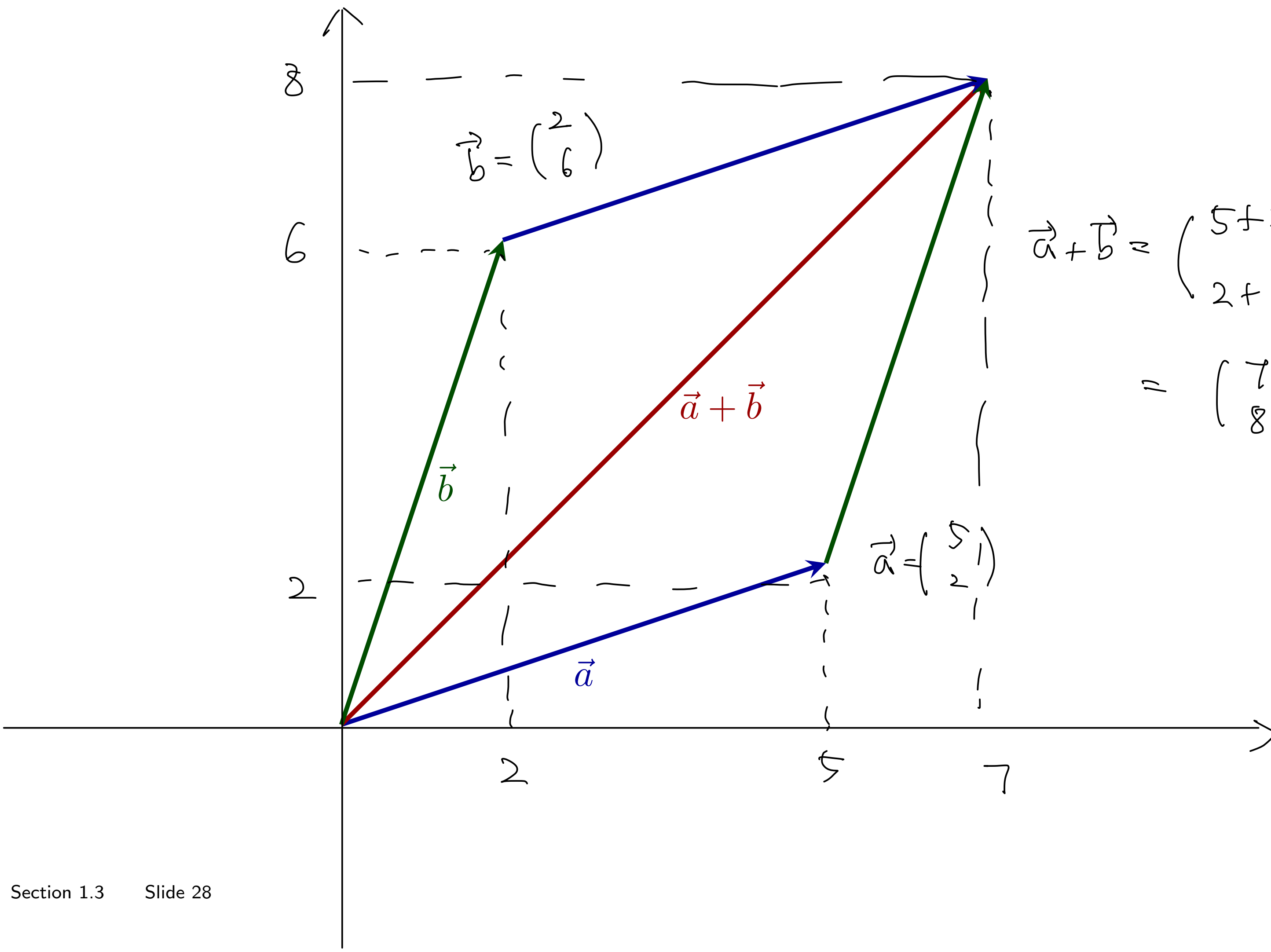
Note that vectors in higher dimensions have the same properties.



$\vec{u} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

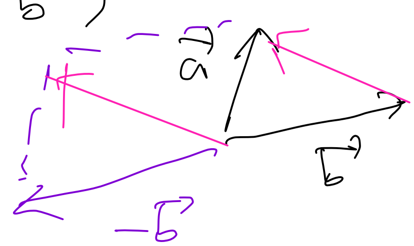
$2 \cdot \vec{u} = \begin{pmatrix} 2\cdot 3 \\ 2 \cdot 2 \end{pmatrix} = \begin{pmatrix} 6 \\ 4 \end{pmatrix}$

Parallelogram Rule for Vector Addition



$$\vec{a} + \vec{b} = \begin{pmatrix} 5+2 \\ 2+6 \end{pmatrix} = \begin{pmatrix} 7 \\ 8 \end{pmatrix}$$

$$\vec{a} - \vec{b} = \vec{a} + (-\vec{b})$$



$\vec{u} + 2\vec{v}$, $\vec{u} - \vec{v}$, $\vec{u}$, $\vec{v}$, $100 \cdot \vec{u}$, ...

Example $\vec{u} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} 1 \\ -4 \end{pmatrix}$

$\vec{u} + \vec{v} = \begin{pmatrix} 3+1 \\ 2-4 \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$: linear combination of $\vec{u}$, $\vec{v}$

Linear Combinations and Span

Definition

1. Given vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p \in \mathbb{R}^n$, and scalars $c_1, c_2, \ldots, c_p$, the vector below

$$\vec{y} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_p\vec{v}_p$$

is called a **linear combination of** $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$ **with weights** $c_1, c_2, \ldots, c_p$.

2. The set of all linear combinations of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$ is called the **Span** of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$.

Span$(\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\}) = \{$ All linear combinations $\}$

Example

$$\text{Span}\left(\left\{ \underset{\vec{u}}{\begin{pmatrix} 3 \\ 2 \end{pmatrix}}, \underset{\vec{v}}{\begin{pmatrix} 1 \\ -4 \end{pmatrix}} \right\}\right)$$

$= \{\vec{0}, \vec{u}, \vec{v}, \vec{u}+\vec{v}, \vec{u}-\vec{v}, \ldots\}$

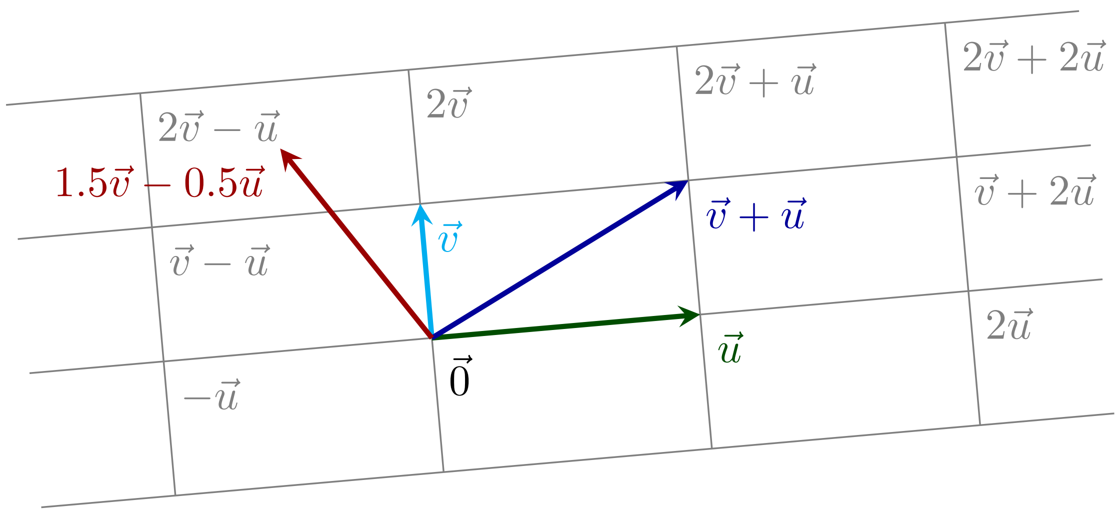
$= \{ a \cdot \vec{u} + b \cdot \vec{v} : a, b \in \mathbb{R} \}$

$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \vec{0} = 0 \cdot \vec{u} + 0 \cdot \vec{v}$

$\vec{u} = 1 \cdot \vec{u} + 0 \cdot \vec{v}$

Geometric Interpretation of Linear Combinations

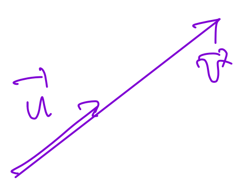
Note that any two vectors in $\mathbb{R}^2$ that are not scalar multiples of each other, span $\mathbb{R}^2$. In other words, any vector in $\mathbb{R}^2$ can be represented as a linear combination of two vectors that are not multiples of each other.



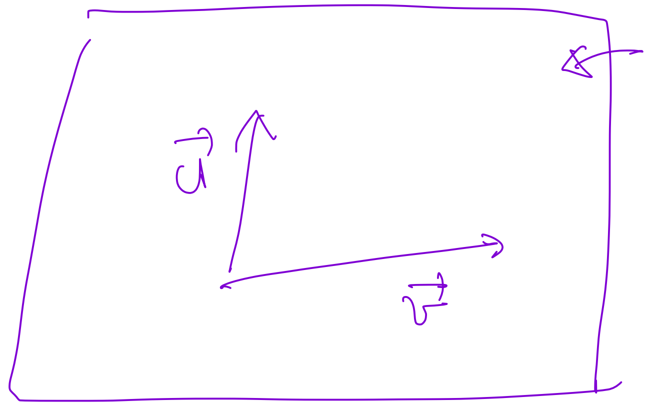
Span$\{\vec{u}, \vec{v}\}$ = Plane containing $\vec{u}$, $\vec{v}$

if $\vec{u}$, $\vec{v}$ are not on the same line.

Ex Span$\left\{ \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ -4 \end{pmatrix} \right\} = \mathbb{R}^2$



Ex $\mathbb{R}^3$



$\mathrm{Span}\{\vec{u}, \vec{v}\} \neq \mathbb{R}^3$

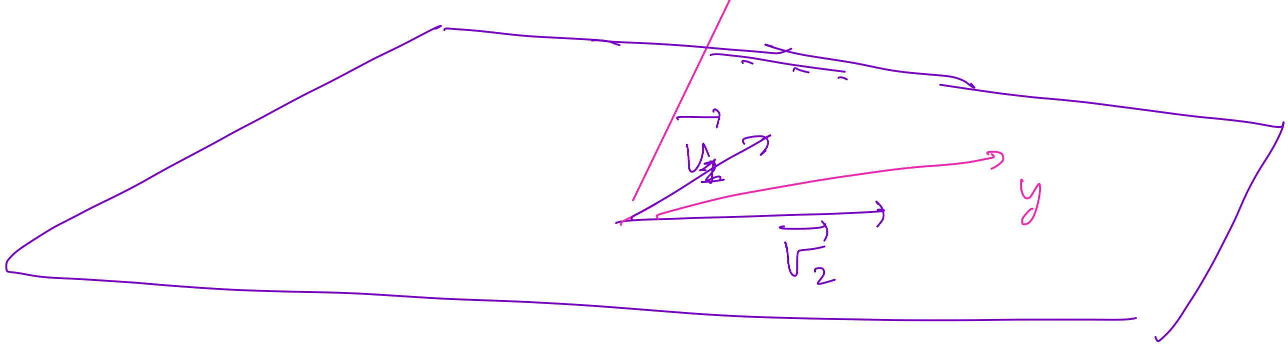
Example

Is $\vec{y}$ in the span of vectors $\vec{v}_1$ and $\vec{v}_2$?

$$\vec{v}_1 = \begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix}, \ \vec{v}_2 = \begin{pmatrix} 2 \\ 5 \\ 6 \end{pmatrix}, \text{ and } \vec{y} = \begin{pmatrix} 7 \\ 4 \\ 15 \end{pmatrix}.$$

y ?

Span



$\vec{y}$ belongs to the plane

$\iff \vec{y} \in \mathrm{Span}\{\vec{v}_1, \vec{v}_2\}$

$\iff \vec{y}$ is a linear combination of $\vec{v}_1, \vec{v}_2$

$\iff \vec{y} = x_1 \cdot \vec{v}_1 + x_2 \cdot \vec{v}_2$ for some $x_1, x_2 \in \mathbb{R}$

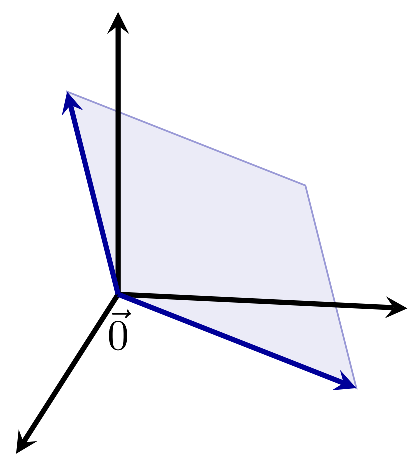
$$\iff \begin{pmatrix} 7 \\ 4 \\ 15 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix} + x_2 \begin{pmatrix} 2 \\ 5 \\ 6 \end{pmatrix} = \begin{pmatrix} x_1 + 2x_2 \\ -2x_1 + 5x_2 \\ -3x_1 + 6x_2 \end{pmatrix}$$

$$\begin{cases} x_1 + 2x_2 = 7 \\ -2x_1 + 5x_2 = 4 \\ -3x_1 + 6x_2 = 15 \end{cases} \longrightarrow \left[\begin{array}{cc|c} 1 & 2 & 7 \\ -2 & 5 & 4 \\ -3 & 6 & 15 \end{array}\right]$$

The Span of Two Vectors in $\mathbb{R}^3$

In the previous example, did we find that $\vec{y}$ is in the span of $\vec{v}_1$ and $\vec{v}_2$?

In general: Any two non-parallel vectors in $\mathbb{R}^3$ span a plane that passes through the origin. Any vector in that plane is also in the span of the two vectors.



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$\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_p \in \mathbb{R}^n$, $c_1, c_2, \cdots, c_p \in \mathbb{R}$

$c_1 \cdot \vec{v}_1 + c_2\vec{v}_2 + \cdots + c_p \cdot \vec{v}_p$: a linear combination of $\vec{v}_1, \cdots, \vec{v}_p$

$\text{Span}(\{\vec{v}_1, \cdots, \vec{v}_p\}) = \{\text{All linear combinations}\}$

$= \{c_1 \cdot \vec{v}_1 + c_2\vec{v}_2 + \cdots + c_p \cdot \vec{v}_p : c_1, \cdots, c_p \in \mathbb{R}\}$

Section 1.4 : The Matrix Equation

Chapter 1 : Linear Equations

Math 1554 Linear Algebra

"Mathematics is the art of giving the same name to different things."
- H. Poincaré

In this section we introduce another way of expressing a linear system that we will use throughout this course.

1.4 : Matrix Equation $A\vec{x} = \vec{b}$

Topics

We will cover these topics in this section.

1. Matrix notation for systems of equations.
2. The matrix product $A\vec{x}$.

Objectives

For the topics covered in this section, students are expected to be able to do the following.

1. Compute matrix-vector products.
2. Express linear systems as vector equations and matrix equations.
3. Characterize linear systems and sets of vectors using the concepts of span, linear combinations, and pivots.

Notation

symbol	meaning
Elements $\in$ Set	belongs to
$\mathbb{R}^n$	the set of vectors with n real-valued elements
$\mathbb{R}^{m\times n}$	the set of real-valued matrices with m rows and n columns

Example: the notation $\vec{x} \in \mathbb{R}^5$ means that $\vec{x}$ is a vector with five real-valued elements.

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}$$

$$A \in \mathbb{R}^{2\times 3} \qquad A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$$

$n \times 1$ matrices $\quad\quad$ $m \times 1$ mat.

$$(m \times n \text{ Matrices}) \cdot (\mathbb{R}^n \text{ vectors}) = (\mathbb{R}^m \text{ vectors})$$

Linear Combinations

$\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n \in \mathbb{R}^m$

$A = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \end{bmatrix} \quad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

Definition

m rows, n columns

A is a $m \times n$ matrix with columns $\vec{a}_1, \dots, \vec{a}_n$ and $x \in \mathbb{R}^n$, then the **matrix vector product** $A\vec{x}$ is a linear combination of the columns of A:

$$A\vec{x} = \begin{bmatrix} | & | & \cdots & | \\ \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \\ | & | & \cdots & | \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1\underbrace{\vec{a}_1}_{\mathbb{R}^m} + x_2\underbrace{\vec{a}_2}_{\mathbb{R}^m} + \cdots + x_n\underbrace{\vec{a}_n}_{\mathbb{R}^m} \in \mathbb{R}^m$$

Note that $A\vec{x}$ is in the span of the columns of A.

Example

The following product can be written as a linear combination of vectors:

$$\underbrace{\begin{bmatrix} 1 & 0 & -1 \\ 0 & -3 & 3 \end{bmatrix}}_{2 \times 3} \underbrace{\begin{bmatrix} 4 \\ 3 \\ 7 \end{bmatrix}}_{3 \times 1 \text{ or } \mathbb{R}^3} = \underbrace{4 \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{2 \times 1 \text{ or } \mathbb{R}^2} + 3 \cdot \begin{bmatrix} 0 \\ -3 \end{bmatrix} + 7 \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

Solution Sets

$$\underset{m \times n}{\begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \end{bmatrix}} \cdot \underset{\substack{n \times 1 \\ \mathbb{R}^n}}{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}} = \underset{\substack{m \times 1 \\ \mathbb{R}^m}}{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}}$$

Theorem

If A is a $m \times n$ matrix with columns $\vec{a}_1, \ldots, \vec{a}_n$, and $x \in \mathbb{R}^n$ and $\vec{b} \in \mathbb{R}^m$, then the solutions to

$$A\vec{x} = \vec{b}$$ ← A Matrix Equation.

has the same set of solutions as the vector equation

$$x_1\vec{a}_1 + \cdots + x_n\vec{a}_n = \vec{b}$$ ⇒ $\vec{b}$ is a linear combi. of $\vec{a}_1, \ldots, \vec{a}_n$

which as the same set of solutions as the set of linear equations with the augmented matrix

$$\left[\begin{array}{cccc|c} \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n & \vec{b} \end{array}\right]$$

$\left\{ \begin{array}{ll} m & \text{Equs} \\ n & \text{variables} \end{array} \right.$

$$\vec{a}_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} \qquad \vec{a}_2 = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} \qquad \cdots \qquad \vec{a}_n = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

$$x_1 \cdot \vec{a}_1 + \cdots + x_n \cdot \vec{a}_n = \vec{b}$$

$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$$\begin{bmatrix} a_{11} \cdot x_1 + a_{12} x_2 + \cdots + a_{1n} x_n \\ a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n \\ \vdots \\ a_{m1} x_1 + a_{m2} x_2 + \cdots + a_{mn} x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

Existence of Solutions

$A = [\vec{a_1}, \cdots, \vec{a_n}]$

Theorem

linear system is consistent

The equation $A\vec{x} = \vec{b}$ has a solution if and only if $\vec{b}$ is a linear combination of the columns of A.

Linear system is Consistent $\Leftrightarrow$ $b \in \text{Span}\{\vec{a_1}, \cdots, \vec{a_n}\}$

$\Updownarrow$

Aug. Mat.

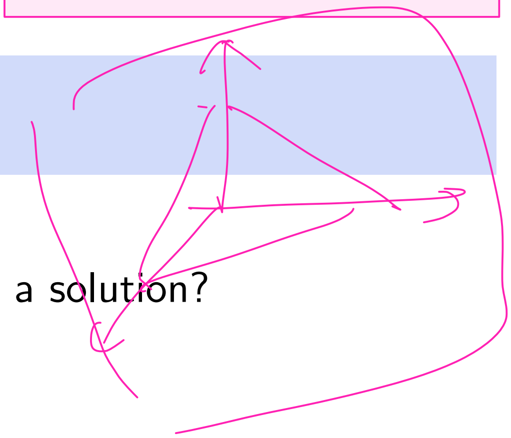
Last columns $\neq$ pivot

$[0 \; \cdots \; 0 \mid 1]$

$A\vec{x} = \vec{b}$ has a solution

$$\iff \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \vec{b} \in \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 8 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ 4 \\ -2 \end{pmatrix} \right\}$$

$$\iff \quad b_3 - \frac{b_2 - 2b_1}{2} = 0, \quad \boxed{2b_1 - b_2 + 2b_3 = 0}$$



Example

For what vectors $\vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ does the equation have a solution?

$$\overbrace{\begin{pmatrix} 1 & 3 & 4 \\ 2 & 8 & 4 \\ 0 & 1 & -2 \end{pmatrix}}^{=A} \vec{x} = \vec{b}$$

Linear System :

$$\begin{cases} x_1 + 3x_2 + 4x_3 = b_1 \\ 2x_1 + 8x_2 + 4x_3 = b_2 \\ \qquad\; x_2 - 2x_3 = b_3 \end{cases}$$

Augmented Mat. :

$$\left[\begin{array}{ccc|c} 1 & 3 & 4 & b_1 \\ 2 & 8 & 4 & b_2 \\ 0 & 1 & -2 & b_3 \end{array}\right]$$

$$\xrightarrow[② \to ② - 2\times①]{} \left[\begin{array}{ccc|c} 1 & 3 & 4 & b_1 \\ 0 & 2 & -4 & b_2 - 2\cdot b_1 \\ 0 & 1 & -2 & b_3 \end{array}\right]$$

$$\xrightarrow[② \to \frac{1}{2}\times ②]{} \left[\begin{array}{ccc|c} 1 & 3 \to 0 & 4 & b_1 \\ 0 & 1 & -2 & \frac{b_2 - 2\cdot b_1}{2} \\ 0 & 1 \to 0 & -2 & b_3 \end{array}\right] \xrightarrow[③ \to ③ - ②]{}$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 4 & b_1 \\ 0 & 1 & -2 & \frac{b_2 - 2b_1}{2} \\ 0 & 0 & 0 & b_3 - \frac{b_2-2b_1}{2} \end{array}\right]$$

$b_3 - \frac{b_2-2b_1}{2}$: 0 → consistent; Not zero → make it 1 → last column is pivot

If $b_3 - \frac{b_2-2b_1}{2} = 0$

$A\vec{x} = \vec{b}$ has a solution

Example

For what vectors $\vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ does the equation have a solution?

$$\begin{pmatrix} 1 & 3 & 4 \\ 2 & 8 & 4 \\ 0 & 1 & -2 \end{pmatrix} \vec{x} = \vec{b}$$

The Row Vector Rule for Computing $A\vec{x}$

$$\begin{bmatrix} 1 & 0 & 2 & 0 & 3 \\ 0 & 1 & 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \quad & \quad & \quad \\ \quad & \quad & \quad \end{bmatrix}$$

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Summary

We now have four **equivalent** ways of expressing linear systems.

1. A system of equations:

$$\begin{aligned} 2x_1 + 3x_2 &= 7 \\ x_1 - x_2 &= 5 \end{aligned}$$

2. An augmented matrix:

$$\left[\begin{array}{cc|c} 2 & 3 & 7 \\ 1 & -1 & 5 \end{array}\right] \xrightarrow{\text{row operations}} \text{RREF.}$$

3. A vector equation:

linear combination

$$\underbrace{x_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} 3 \\ -1 \end{pmatrix}}_{\text{linear combination}} = \begin{pmatrix} 7 \\ 5 \end{pmatrix}$$

4. As a matrix equation:

$$\begin{pmatrix} 2 & 3 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \end{pmatrix}$$

Each representation gives us a different way to think about linear systems.

Section 1.5 : Solution Sets of Linear Systems

Chapter 1 : Linear Equations

Math 1554 Linear Algebra

1.5 : Solution Sets of Linear Systems

Topics
We will cover these topics in this section.

1. Homogeneous systems
2. Parametric **vector** forms of solutions to linear systems

Objectives
For the topics covered in this section, students are expected to be able to do the following.

1. Express the solution set of a linear system in parametric vector form.
2. Provide a geometric interpretation to the solution set of a linear system.
3. Characterize homogeneous linear systems using the concepts of free variables, span, pivots, linear combinations, and echelon forms.

Linear system : $A \cdot \vec{x} = \vec{b}$, $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

Homogeneous Systems

Definition

Linear systems of the form $A \cdot \vec{x} = \vec{0} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ are **homogeneous**.

Linear systems of the form $A \cdot \vec{x} = \vec{b}$, $\vec{b} \neq \vec{0}$ are **inhomogeneous**.

always consistent.

Because homogeneous systems always have the **trivial solution**, $\vec{x} = \vec{0}$, the interesting question is whether they have nontrivial solution (nonzero) solutions.

$A\vec{0} = \vec{0}$ $\Downarrow$ $\vec{0}$ is a solution to $A\vec{x} = \vec{0}$

Observation

$A\vec{x} = \vec{0}$ has a nontrivial solution $\iff$ $A\vec{x} = \vec{0}$ has infinitely many solution

$\iff$ there is a free variable

$\iff$ A has a column with no pivot.

$[\,A \mid \vec{0}\,]$, enough to consider A $\xrightarrow{\text{row operations}}$ RREF

nonpivot $\longleftrightarrow$ free variables $\updownarrow$ Infinitely many solution

x_3 : free.

$$\begin{pmatrix}2x_3\\-x_3\\x_3\end{pmatrix} = x_3\begin{pmatrix}2\\-3\\1\end{pmatrix}$$

$$\begin{cases} x_1 - 2x_3 = 0 \\ x_2 + x_3 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = 2x_3 \\ x_2 = -x_3 \end{cases}$$

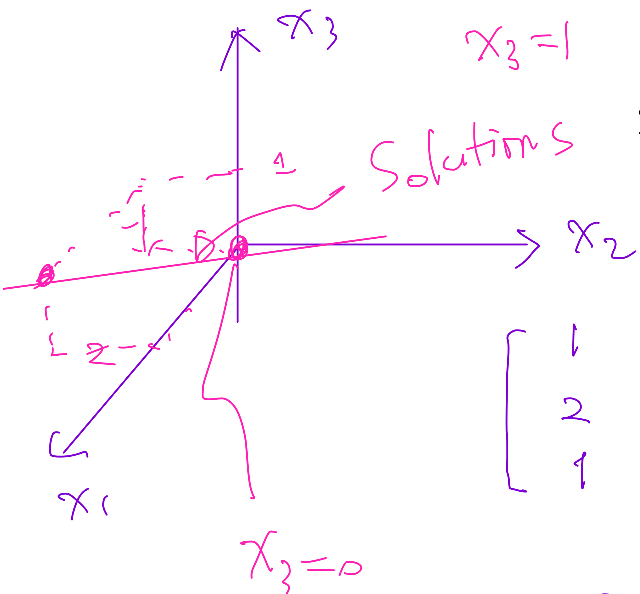
Solution Set $= \{(2x_3, -x_3, x_3) : x_3 \in \mathbb{R}\}$

Example: a Homogeneous System

Identify the free variables, and the solution set, of the system.

$$\begin{aligned} x_1 + 3x_2 + x_3 &= 0 \\ 2x_1 - x_2 - 5x_3 &= 0 \\ x_1 - 2x_3 &= 0 \end{aligned}$$

$\rightarrow$ Homogeneous.



$A \cdot \vec{x} = \vec{0}$

$$\begin{bmatrix} 1 & 3 & 1 \\ 2 & -1 & -5 \\ 1 & 0 & -2 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & 1 \\ 2 & -1 & -5 \\ 1 & 0 & -2 \end{bmatrix}$$

$$\xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & 3 & 1 \\ 0 & -7 & -7 \\ 0 & -3 & -3 \end{bmatrix} \xrightarrow[R_3 \times (-\frac{1}{3})]{R_2 \times (-\frac{1}{7})} \begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\xrightarrow[R_3 \to R_3 - R_2]{R_1 \to R_1 - 3R_2} \underbrace{\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}}_{\text{RREF}}$$

Ex) If x_2, x_3 : free, $x_1 = x_2 + x_3$

$\Rightarrow$ Solution $= \{ \underline{(x_2+x_3, x_2, x_3)} : x_2, x_3 \in \mathbb{R} \}$

$= \text{Span}\left\{ \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix} \right\}$

$$\begin{pmatrix} x_2+x_3 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_2 \\ 0 \end{pmatrix} + \begin{pmatrix} x_3 \\ 0 \\ x_3 \end{pmatrix} = \underbrace{x_2 \overbrace{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}}^{=\vec{v}_2} + x_3 \overbrace{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}}^{=\vec{v}_3}}_{\text{lin. combi.}}$$

Parametric Forms, Homogeneous Case

In the example on the previous slide we expressed the solution to a system using a vector equation. This is a **parametric form** of the solution.

In general, suppose the free variables for $A\vec{x} = \vec{0}$ are $x_k, \ldots, x_n$. Then all solutions to $A\vec{x} = \vec{0}$ can be written as

$$\vec{x} = x_k\vec{v}_k + x_{k+1}\vec{v}_{k+1} + \cdots + x_n\vec{v}_n$$

for some $\vec{v}_k, \ldots, \vec{v}_n$. This is the **parametric form** of the solution.

$$\begin{cases} x_1 = 2x_3 \\ x_2 = -x_3 \end{cases} \Rightarrow \text{Solution} = \{ (2x_3, -x_3, x_3) : x_3 \in \mathbb{R} \}$$

$$= \{ (2t, -t, t) : t \in \mathbb{R} \}$$

$$\begin{cases} x_3 = t \\ x_1 = 2t \\ x_2 = -t \end{cases}$$

Example 2 (non-homogeneous system)

Write the parametric vector form of the solution, and give a geometric interpretation of the solution.

$$
\begin{aligned}
x_1 + 3x_2 + x_3 &= 9 \\
2x_1 - x_2 - 5x_3 &= 11 \\
x_1 - 2x_3 &= 6
\end{aligned}
$$

(Note that the left-hand side is the same as Example 1).

$$\left[\begin{array}{ccc|c} 1 & 3 & 1 & 9 \\ 2 & -1 & -5 & 11 \\ 1 & 0 & -2 & 6 \end{array}\right]$$

$$\xrightarrow[\substack{R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - R_1}]{} \left[\begin{array}{ccc|c} 1 & 3 & 1 & 9 \\ 0 & -7 & -7 & -7 \\ 0 & -3 & -3 & -3 \end{array}\right] \xrightarrow[\substack{R_2 \times (-\frac{1}{7}) \\ R_3 \times (-\frac{1}{3})}]{} \left[\begin{array}{ccc|c} 1 & 3 & 1 & 9 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{array}\right]$$

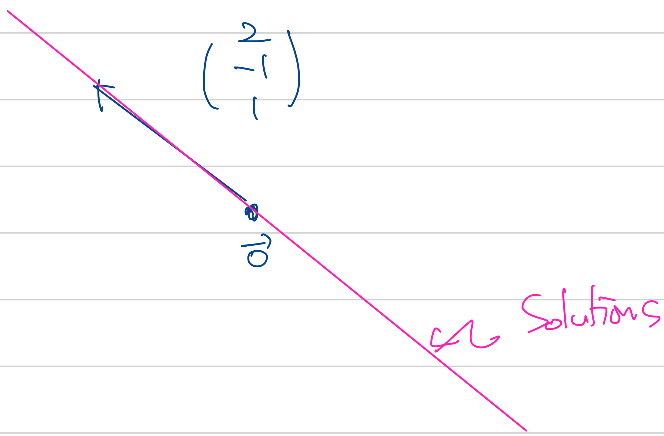
$$\xrightarrow[\substack{R_1 \to R_1 - 3R_2 \\ R_3 \to R_3 - R_2}]{} \underbrace{\left[\begin{array}{ccc|c} 1 & 0 & -2 & 6 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]}_{\text{RREF}} \Rightarrow \begin{cases} x_1 - 2x_3 = 6 \\ x_2 + x_3 = 1 \end{cases}$$

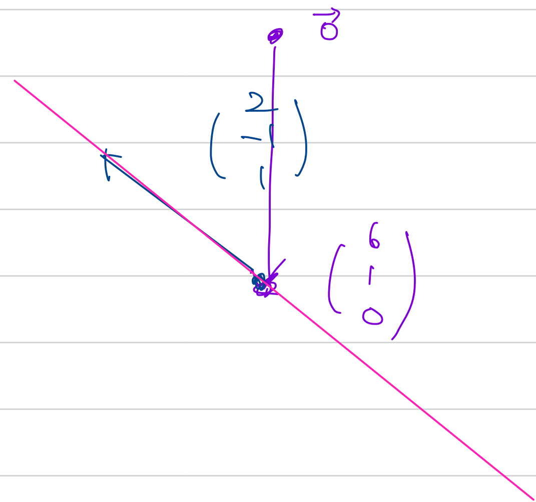
$$
\begin{aligned}
x_1 &= 2x_3 + 6 \\
x_2 &= -x_3 + 1 \\
x_3 &= x_3
\end{aligned}
$$

Solution $= \{ (2x_3+6, -x_3+1, x_3) : x_3 \in \mathbb{R} \}$

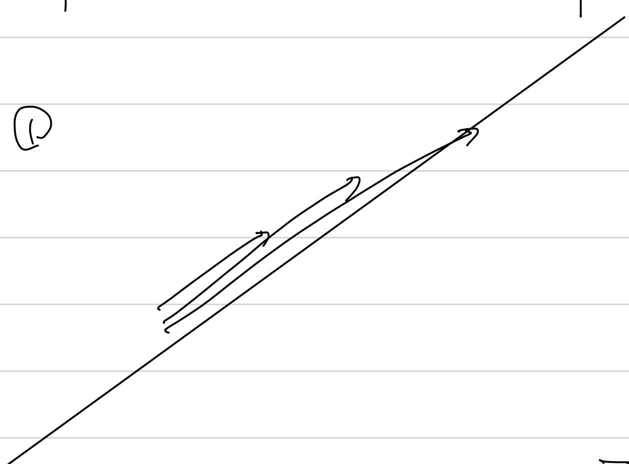
$$\begin{pmatrix} 2x_3+6 \\ -x_3+1 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2x_3 \\ -x_3 \\ x_3 \end{pmatrix} + \begin{pmatrix} 6 \\ 1 \\ 0 \end{pmatrix} = x_3 \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} + \begin{pmatrix} 6 \\ 1 \\ 0 \end{pmatrix}$$

Homogeneous | Non-homogeneous



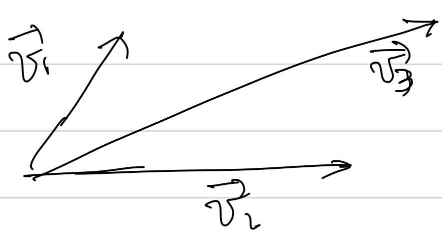


Span $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$

① 

On the same line

Span = line.

② 

$\vec{v}_3$ belongs to the plane containing $\vec{v}_1, \vec{v}_2$

Span = plane

Q: $\mathrm{Span}(\{v_1, v_2, \dots, v_6\})$ = 6 - dimension

When?

Section 1.7 : Linear Independence

Chapter 1 : Linear Equations

Math 1554 Linear Algebra

1.7 : Linear Independence

Topics

We will cover these topics in this section.

- Linear independence
- Geometric interpretation of linearly independent vectors

Objectives

For the topics covered in this section, students are expected to be able to do the following.

1. Characterize a set of vectors and linear systems using the concept of linear independence.
2. Construct dependence relations between linearly dependent vectors.

Motivating Question

What is the smallest number of vectors needed in a parametric solution to a linear system?

No free variable.

$\Updownarrow$

Only trivial $\Leftrightarrow$ Unique solution.

Homogeneous System

$$\begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_k \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Linear Independence

A set of vectors $\{\vec{v}_1, \ldots, \vec{v}_k\}$ in $\mathbb{R}^n$ are **linearly independent** if

$$\underline{c_1}\vec{v}_1 + \underline{c_2}\vec{v}_2 + \cdots + \underline{c_k}\vec{v}_k = \vec{0} \leftarrow$$

vector equation for $\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{pmatrix}$

has only the **trivial** solution. It is **linearly dependent** otherwise.

$c_1 = 0,\ c_2 = 0,\ \ldots,\ c_k = 0$

In other words, $\{\vec{v}_1, \ldots, \vec{v}_k\}$ are linearly dependent if there are real numbers $c_1, c_2, \ldots, c_k$ **not all zero** so that

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k = \vec{0}$$

Consider the vectors:

$$\vec{v}_1, \vec{v}_2, \ldots \vec{v}_k$$

To determine whether the vectors are linearly independent, we can set the linear combination to the zero vector:

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_k \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = V\vec{c} \overset{??}{=} \vec{0}$$

Linear independence: There is NO non-zero solution $\vec{c}$

Linear dependence: There is a non-zero solution $\vec{c}$.

$$c_1 \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} = \vec{0}$$

Example 1

For what values of h are the vectors linearly independent?

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ h \end{bmatrix}, \underbrace{\begin{bmatrix} 1 \\ h \\ 1 \end{bmatrix}}_{=\vec{v}_2}, \underbrace{\begin{bmatrix} h \\ 1 \\ 1 \end{bmatrix}}_{=\vec{v}_3}$$

$\{v_1, v_2, v_3\}$ are linearly independent

$\overset{\text{definition}}{\Leftrightarrow}$ $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$ implies $c_1 = 0$, $c_2 = 0$, $c_3 = 0$

$\Leftrightarrow$ $\begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ has the only trivial solution.

$$= \begin{bmatrix} 1 & 1 & h \\ 1 & h & 1 \\ h & 1 & 1 \end{bmatrix}$$

$\Leftrightarrow$ $\begin{bmatrix} 1 & 1 & h \\ 1 & h & 1 \\ h & 1 & 1 \end{bmatrix} \xrightarrow[\text{operations}]{\text{row}}$ RREF has $\underline{\text{3 pivot}}$ columns

$$1^2 - h^2 = (1+h)(1-h)$$

$$\begin{bmatrix} 1 & 1 & h \\ 1 & h & 1 \\ h & 1 & 1 \end{bmatrix} \xrightarrow[\substack{R_2 \to R_2 - R_1 \\ R_3 \to R_3 - h\cdot R_1}]{} \begin{bmatrix} 1 & 1 & h \\ 0 & h-1 & 1-h \\ 0 & 1-h & 1-h^2 \end{bmatrix}$$

$1-h = -(h-1)$

$1-h^2 = (1-h)(1+h)$

① $h-1$ or $1-h$ $\neq 0$

Assume $h \neq 1$

$$\xrightarrow[\substack{\frac{1}{h-1} R_2 \\ \frac{1}{1-h} R_3}]{} \begin{bmatrix} 1 & 1 & h \\ 0 & 1 & -1 \\ 0 & 1 & 1+h \end{bmatrix}$$

$$\xrightarrow[R_3 \to R_3 - R_2]{} \begin{bmatrix} 1 & 1 & h \\ 0 & 1 & -1 \\ 0 & 0 & h+2 \end{bmatrix}$$

$h+2 \neq 0$

$h+2 \neq 0$

$h \neq -2$

$h \neq 1, -2 \Rightarrow 3$ pivots

$\Rightarrow$ Only trivial solution

$\Rightarrow \left\{ \begin{bmatrix} 1 \\ 1 \\ h \end{bmatrix}, \begin{bmatrix} 1 \\ h \\ 1 \end{bmatrix}, \begin{bmatrix} h \\ 1 \\ 1 \end{bmatrix} \right\}$ linearly indep.

Example 2 (One Vector)

Suppose $\vec{v} \in \mathbb{R}^n$. When is the set $\{\vec{v}\}$ linearly dependent?

$c_1 \cdot \vec{v} = \vec{0}$ implies $c_1 = 0$?

$\vec{v} = \vec{0}$

$\overset{c_1}{1} \cdot \vec{v} = \vec{0}$ $\quad c_1 \neq 0$

$\Rightarrow$ linearly dependent

$\vec{v} \neq \vec{0}$

$$c_1 \begin{bmatrix} v_1 \\ v_2 \neq 0 \\ \vdots \\ v_n \end{bmatrix} = \vec{v} = \begin{bmatrix} \vdots \\ \boxed{c_1 \cdot v_2} \\ \vdots \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\Rightarrow \frac{c_1 \cdot v_2^{\neq 0}}{v_2} = \frac{0}{v_2} \Rightarrow c_1 = 0$$

Example 3 (Two Vectors)

Suppose $\vec{v}_1, \vec{v}_2 \in \mathbb{R}^n$. When is the set $\{\vec{v}_1, \vec{v}_2\}$ linearly dependent? Provide a geometric interpretation.

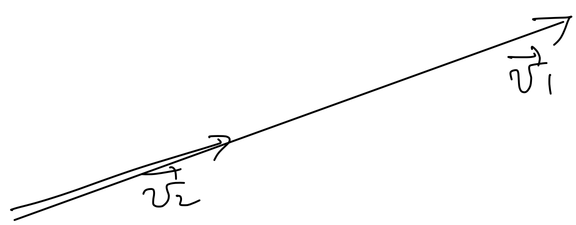
$c_1 \cdot \vec{v}_1 + c_2 \cdot \vec{v}_2 = \vec{0}$ and one of c_1, c_2 is nonzero.

If $c_1 \neq 0$,

$$\underbrace{c_1}_{\neq 0} \vec{v}_1 = -c_2 \cdot \vec{v}_2$$

$$\vec{v}_1 = \left(-\frac{c_2}{c_1}\right) \cdot \vec{v}_2$$

$= $ a scalar multiple of $\vec{v}_2$



$\Rightarrow$ $\vec{v}_1, \vec{v}_2$ are on the same line.

$\Rightarrow$ $\text{Span}(\{\vec{v}_1, \vec{v}_2\}) = $ a line.

or

$\{\vec{0}\}$

Two Theorems

Example $k = 6$, $n = 5$

Fact 1. Suppose $\vec{v}_1, \ldots, \vec{v}_k$ are vectors in $\mathbb{R}^n$. If $k > n$, then $\{\vec{v}_1, \ldots, \vec{v}_k\}$ is linearly dependent.

$$\underbrace{\begin{bmatrix} \vec{v}_1 & \cdots & \vec{v}_k \end{bmatrix}}_{k=6} \Big\} \; n=5$$

maximum # of pivot. $= 5$

Fact 2. If any one or more of $\vec{v}_1, \ldots, \vec{v}_k$ is $\vec{0}$, then $\{\vec{v}_1, \ldots, \vec{v}_k\}$ is linearly dependent.

Section 1.8 : An Introduction to Linear Transforms

Chapter 1 : Linear Equations

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$\{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_p\}$ linearly indep.

$\Leftrightarrow$ $c_1 \cdot \vec{v}_1 + c_2 \cdot \vec{v}_2 + \cdots + c_p \vec{v}_p = \vec{0}$ implies $c_1 = \cdots = c_p = 0$

$\Leftrightarrow$ " has the only trivial solution

$\Leftrightarrow$ $A\vec{x} = \vec{0}$ " for $A = [\vec{v}_1, \cdots, \vec{v}_p]$, $\vec{x} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_p \end{bmatrix}$

$\Leftrightarrow$ $A\vec{x} = \vec{0}$ has no free variable.

$\Leftrightarrow$ Every column of A is pivot

1.8 : An Introduction to Linear Transforms

Topics

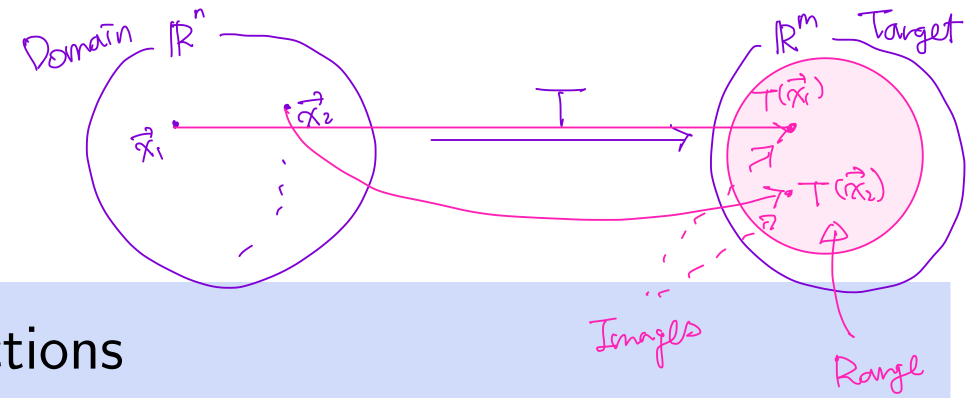
We will cover these topics in this section.

1. The definition of a linear transformation.
2. The interpretation of matrix multiplication as a linear transformation.

Objectives

For the topics covered in this section, students are expected to be able to do the following.

1. Construct and interpret linear transformations in $\mathbb{R}^n$ (for example, interpret a linear transform as a projection, or as a shear).
2. Characterize linear transforms using the concepts of
   - existence and uniqueness
   - domain, co-domain and range



From Matrices to Functions

rows

Let A be an $m \times n$ matrix. We define a function

columns

$$T : \mathbb{R}^n \to \mathbb{R}^m, \quad T(\vec{v}) = A\vec{v}$$

vector vector

$\in \mathbb{R}^m$ matrix multiplication.

$\vec{v} \in \mathbb{R}^n$

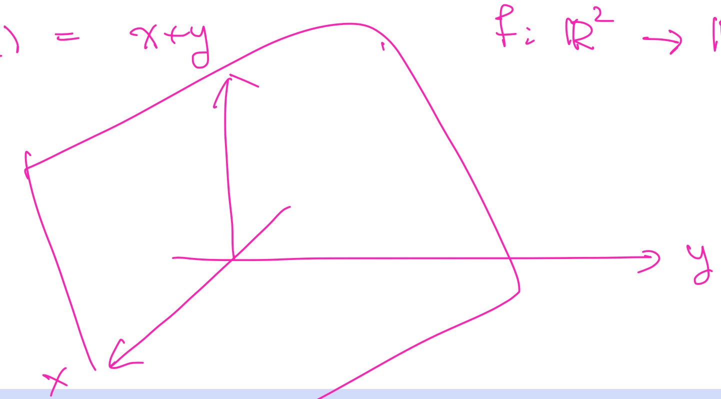
This is called a **matrix transformation**.

- The **domain** of T is $\mathbb{R}^n$. the set of inputs
- The **co-domain** or **target** of T is $\mathbb{R}^m$. the set where T maps to.
- The vector $T(\vec{x})$ is the **image** of $\vec{x}$ under T a vector
- The set of all possible images $T(\vec{x})$ is the **range**.

This gives us **another** interpretation of $A\vec{x} = \vec{b}$:

- set of equations
- augmented matrix
- matrix equation
- vector equation
- linear transformation equation

Ex) $f(x,y) = x+y$ $f: \mathbb{R}^2 \to \mathbb{R}$

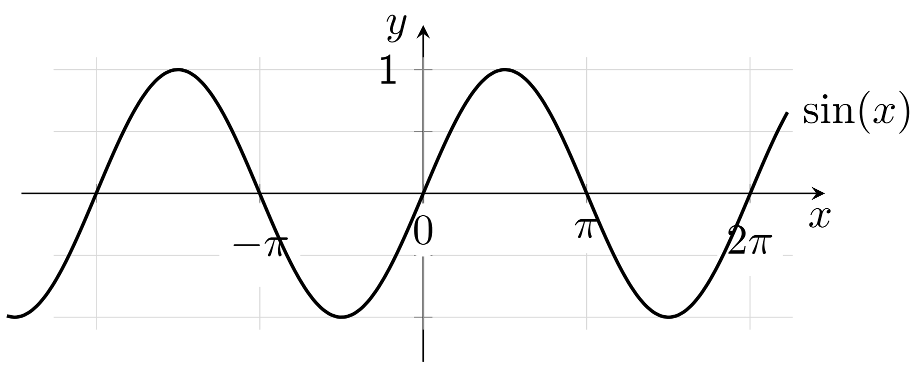


Functions from Calculus

Many of the functions we know have **domain** and **codomain** $\mathbb{R}$.We can express the **rule** that defines the function $\sin$ this way:

$$f : \mathbb{R} \to \mathbb{R} \qquad f(x) = \sin(x)$$

In calculus we often think of a function in terms of its graph, whose horizontal axis is the **domain**, and the vertical axis is the **codomain**.



This is ok when the domain and codomain are $\mathbb{R}$. It's hard to do when the domain is $\mathbb{R}^2$ and the codomain is $\mathbb{R}^3$. We would need five dimensions to draw that graph.

$T: \mathbb{R}^2 \to \mathbb{R}^3 \qquad T(\vec{v}) = A\vec{v}$

Example 1

$\begin{bmatrix} x \\ y \end{bmatrix} \longmapsto \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + y \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} x+y \\ y \\ x+y \end{bmatrix}$

Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$, $\vec{u} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 7 \\ 5 \\ 7 \end{bmatrix}$.

$\in \mathbb{R}^{3 \times 2}$

a) Compute $T(\vec{u})$. $= A \cdot \vec{u} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 3+4 \\ 4 \\ 3+4 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ 7 \end{bmatrix}$

b) Calculate $\vec{v} \in \mathbb{R}^2$ so that $T(\vec{v}) = \vec{b}$

$\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix}$

$\begin{bmatrix} x+y \\ y \\ x+y \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 5 \\ 7 \end{bmatrix}$

$y = 5$

$x = 2$

$\vec{v} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$

c) Give a $\vec{c} \in \mathbb{R}^3$ so there is no $\vec{v}$ with $T(\vec{v}) = \vec{c}$

$\mathbb{R}^2$

$\begin{bmatrix} 7 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \ldots$

or: Give a $\vec{c}$ that is not in the range of T.

or: Give a $\vec{c}$ that is not in the span of the columns of A.

$c_1 \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

"Range of T = Span of Columns in A"

Ex) $f(x) = x^2$ $\quad f(5x) = 25x^2, \quad 5f(x) = 5x^2$

$f(x) = 2x$ linear

$f(x) = x+1$ $\quad f(5x) = 5x+1$ Not linear

$5f(x) = 5\cdot(x+1)$

Linear Transformations

A function $T : \mathbb{R}^n \to \mathbb{R}^m$ is **linear** if

- $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$ for all $\vec{u}, \vec{v}$ in $\mathbb{R}^n$.
- $T(c\vec{v}) = cT(\vec{v})$ for all $\vec{v} \in \mathbb{R}^n$, and c in $\mathbb{R}$.

So if T is linear, then

$$\underbrace{T(c_1\vec{v}_1 + \cdots + c_k\vec{v}_k)}^{\text{linear combi.}} = \overbrace{c_1T(\vec{v}_1) + \cdots + c_kT(\vec{v}_k)}^{\text{lin. combi.}}$$

This is called the **principle of superposition**. The idea is that if we know $T(\vec{e}_1), \ldots, T(\vec{e}_n)$, then we know every $T(\vec{v})$.

Fact: Every matrix transformation T_A is linear.

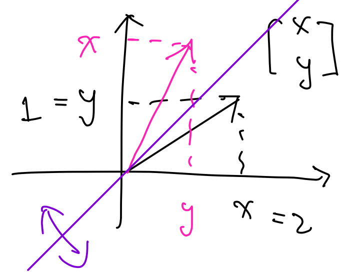
$T_A : \mathbb{R}^n \to \mathbb{R}^m$, $\quad T_A(\vec{v}) = A\vec{v}$.

Example 2

Suppose T is the linear transformation $T(\vec{x}) = A\vec{x}$. Give a short geometric interpretation of what $T(\vec{x})$ does to vectors in $\mathbb{R}^2$.

1) $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $\quad T: \mathbb{R}^2 \to \mathbb{R}^2$

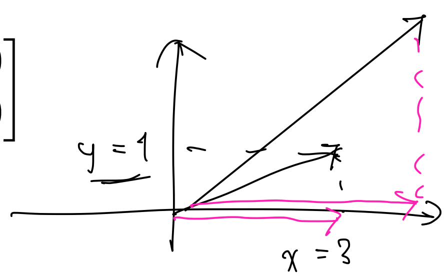
reflection.



$$\begin{bmatrix} x \\ y \end{bmatrix} \longmapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = x\begin{bmatrix} 0 \\ 1 \end{bmatrix} + y\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix}$$

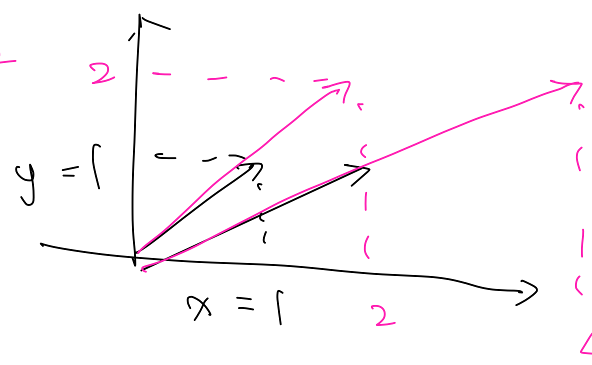
2) $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

projection



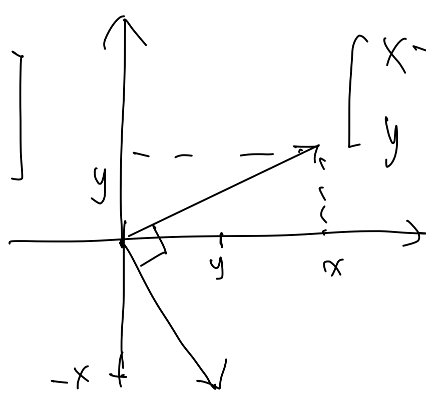
$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 0 \end{bmatrix}$$

3) $A = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$ for $k \in \mathbb{R}$ $\quad k = 2$



$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$$

4) $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$



$$\begin{bmatrix} x \\ y \end{bmatrix} \longmapsto \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ -x \end{bmatrix}$$

Example 3

What does T_A do to vectors in $\mathbb{R}^3$?

a) $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

b) $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

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$A \in \mathbb{R}^{m \times n}$

$T : \mathbb{R}^n \to \mathbb{R}^m$

$T(\vec{x}) = A \cdot \vec{x}$

Example 4

A linear transformation $T : \mathbb{R}^2 \mapsto \mathbb{R}^3$ satisfies $T(\vec{x}) = A\vec{x}$

$A \in \mathbb{R}^{3 \times 2}$

$A = [\vec{v}_1, \vec{v}_2]$

$\vec{v}_1, \vec{v}_2 \in \mathbb{R}^3$

$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}5\\-7\\2\end{bmatrix}, \qquad T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}-3\\8\\0\end{bmatrix}$$

What is the matrix that represents T?

Coefficients

$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = [\vec{v}_1, \vec{v}_2] \cdot \begin{bmatrix}1\\0\end{bmatrix} = 1 \cdot \vec{v}_1 + 0 \cdot \vec{v}_2 = \vec{v}_1 = \begin{bmatrix}5\\-7\\2\end{bmatrix}$$

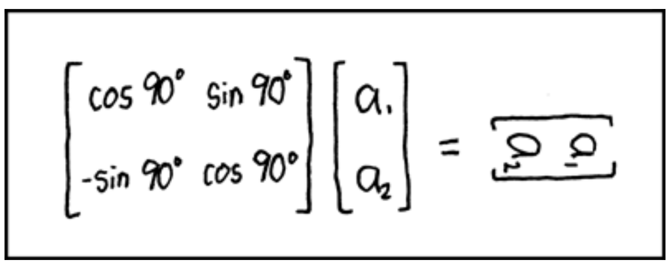
$$T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = [\vec{v}_1 \ \vec{v}_2] \begin{bmatrix}0\\1\end{bmatrix} = 0 \cdot \vec{v}_1 + 1 \cdot \vec{v}_2 = \vec{v}_2 = \begin{bmatrix}-3\\8\\0\end{bmatrix}$$

$$A = \begin{bmatrix}5 & -3\\-7 & 8\\2 & 0\end{bmatrix}.$$

Section 1.9 : Linear Transforms

Chapter 1 : Linear Equations

Math 1554 Linear Algebra



https://xkcd.com/184

1.9 : Matrix of a Linear Transformation

Topics
We will cover these topics in this section.

1. The **standard vectors** and the **standard matrix**.
2. Two and three dimensional transformations in more detail.
3. **Onto** and **one-to-one** transformations.

Objectives
For the topics covered in this section, students are expected to be able to do the following.

1. Identify and construct linear transformations of a matrix.
2. Characterize linear transformations as onto and/or one-to-one.
3. Solve linear systems represented as linear transforms.
4. Express linear transforms in other forms, such as as matrix equations or as vector equations.

Definition: The Standard Vectors

The **standard vectors** in $\mathbb{R}^n$ are the vectors $\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n$, where:

$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad \vec{e}_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

For example, in $\mathbb{R}^3$,

$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad \vec{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$A = [\ \vec{v}_1\ \cdots\ \vec{v}_n\ ]$$

$$A\vec{e}_1 = [\ \vec{v}_1\ \vec{v}_2\ \cdots\ \vec{v}_n\ ]\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = 1\cdot\vec{v}_1 + 0\cdot\vec{v}_2 + \cdots + 0\cdot\vec{v}_n$$

$$= \vec{v}_1$$

A Property of the Standard Vectors

Note: if A is an $m \times n$ matrix with columns $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$, then

$$A\vec{e}_i = \vec{v}_i, \text{ for } i = 1, 2, \ldots, n$$

So multiplying a matrix by $\vec{e}_i$ gives column i of A.

Example

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \vec{e}_2 = \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}, \quad \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

The Standard Matrix

Theorem

Let $T : \mathbb{R}^n \mapsto \mathbb{R}^m$ be a linear transformation. Then there is a unique matrix A such that

$$T(\vec{x}) = A\vec{x}, \qquad \vec{x} \in \mathbb{R}^n.$$

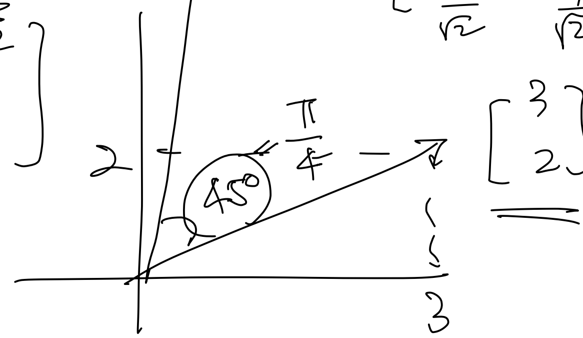
In fact, A is a $m \times n$, and its j^{th} column is the vector $T(\vec{e}_j)$.

$$A = \begin{bmatrix} T(\vec{e}_1) & T(\vec{e}_2) & \cdots & T(\vec{e}_n) \end{bmatrix}$$

The matrix A is the **standard matrix** for a linear transformation.

$$\begin{bmatrix}\cos\left(\frac{\pi}{4}\right) & -\sin\left(\frac{\pi}{2}\right)\\ \sin\left(\frac{\pi}{4}\right) & \cos\left(\frac{\pi}{4}\right)\end{bmatrix} = \begin{bmatrix}\frac{1}{\sqrt2} & -\frac{1}{\sqrt2}\\ \frac{1}{\sqrt2} & \frac{1}{\sqrt2}\end{bmatrix}$$

$$? = \begin{bmatrix}\frac{1}{\sqrt2} & -\frac{1}{\sqrt2}\\ \frac{1}{\sqrt2} & \frac{1}{\sqrt2}\end{bmatrix}\begin{bmatrix}3\\2\end{bmatrix}$$

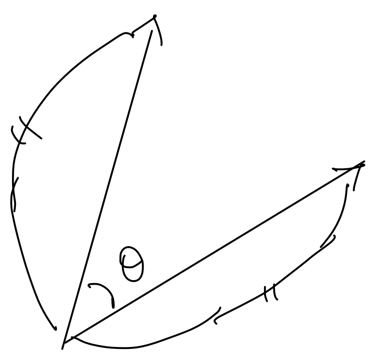


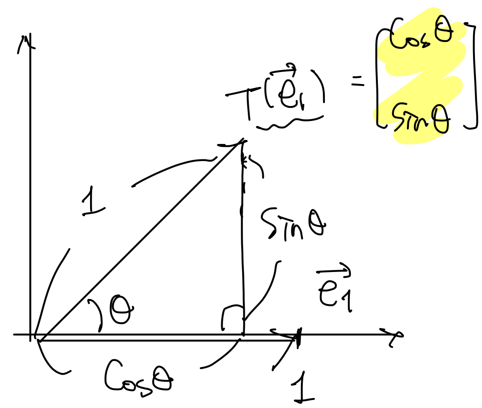
Rotations

Example 1

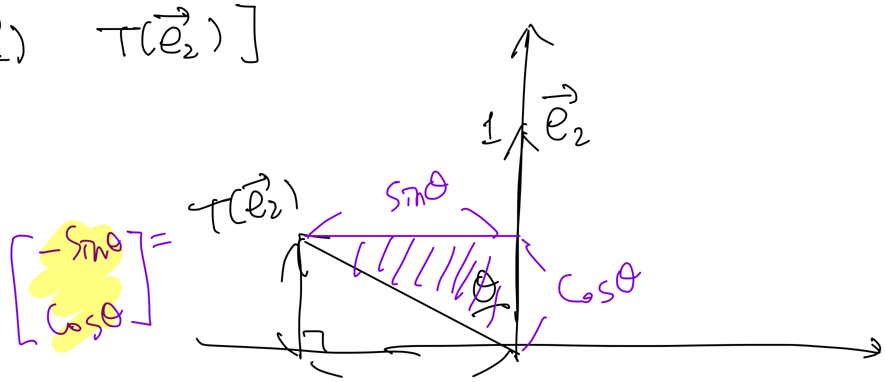
What is the linear transform $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$T(\vec{x}) = \vec{x} \text{ rotated counterclockwise by angle } \theta?$$





$$A = \begin{bmatrix} T(\vec{e}_1) & T(\vec{e}_2) \end{bmatrix}$$

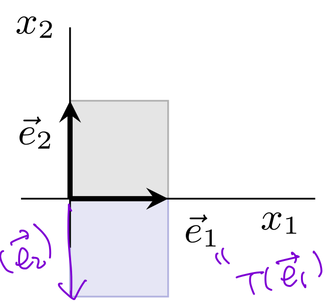
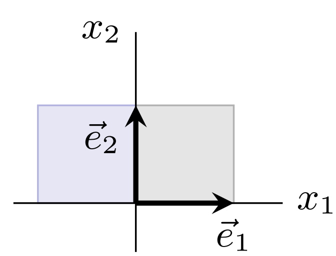


$$A = \begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}.$$

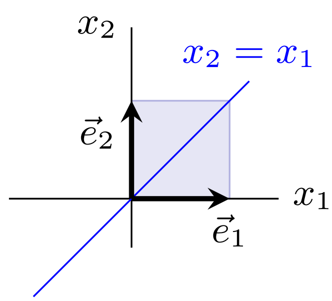
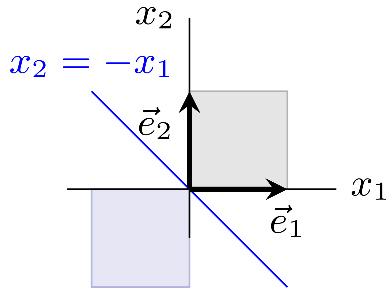
Standard Matrices in $\mathbb{R}^2$

- There is a long list of geometric transformations of $\mathbb{R}^2$ in our textbook, as well as on the next few slides (reflections, rotations, contractions and expansions, shears, projections, ...)
- Please familiarize yourself with them: you are expected to memorize them (or be able to derive them)

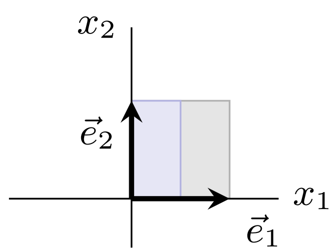
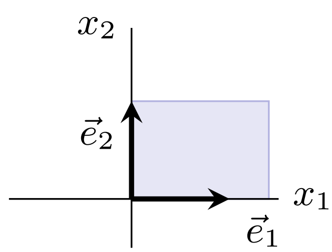
Two Dimensional Examples: Reflections

transformation	**image of unit square**	**standard matrix**
reflection through x_1-axis	 $-\vec{e}_2 = T(\vec{e}_2)$ $= T(\vec{e}_1)$	$\left(\underbrace{\begin{matrix}1\\0\end{matrix}}_{=\vec{e}_1}\quad\underbrace{\begin{matrix}0\\-1\end{matrix}}_{=-\vec{e}_2}\right)$
reflection through x_2-axis		$\begin{pmatrix}-1 & 0\\0 & 1\end{pmatrix}$

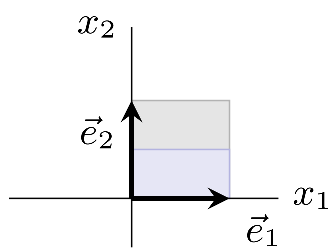
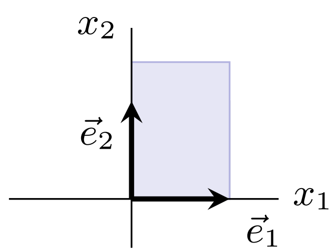
Two Dimensional Examples: Reflections

<table>
<tr><th>transformation</th><th>image of unit square</th><th>standard matrix</th></tr>
<tr><td>reflection through $x_2 = x_1$</td><td>
</td><td>$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$</td></tr>
<tr><td>reflection through $x_2 = -x_1$</td><td>
</td><td>$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$</td></tr>
</table>

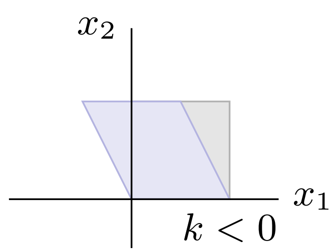
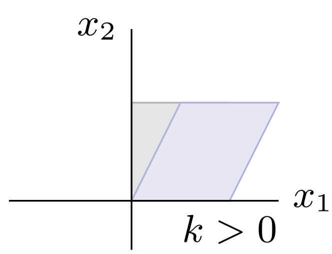
Two Dimensional Examples: Contractions and Expansions

transformation	**image of unit square**	**standard matrix**
Horizontal Contraction		$\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$. $\lvert k \rvert < 1$
Horizontal Expansion		$\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$, $k > 1$

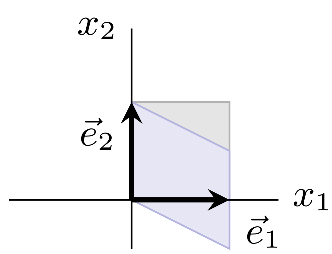
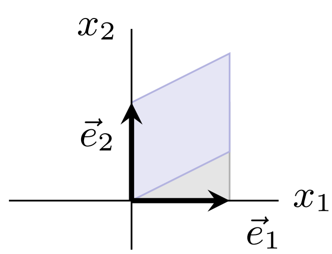
Two Dimensional Examples: Contractions and Expansions

transformation	**image of unit square**	**standard matrix**
Vertical Contraction		$\begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}$, $\lvert k \rvert < 1$
Vertical Expansion		$\begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}$, $k > 1$

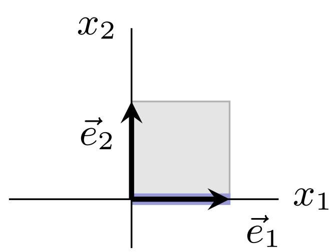
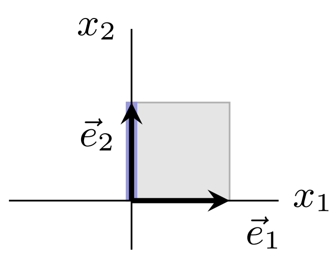
Two Dimensional Examples: Shears

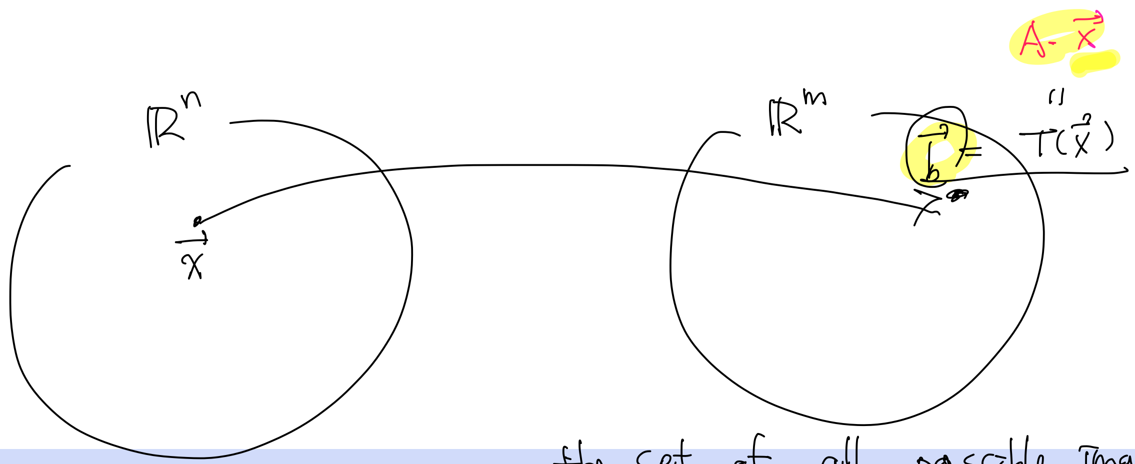
transformation	image of unit square	standard matrix
Horizontal Shear(left)		$\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$, $k < 0$
Horizontal Shear(right)		$\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$, $k > 0$

Two Dimensional Examples: Shears

transformation	**image of unit square**	**standard matrix**
Vertical Shear(down)		$\begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix}$, $k < 0$
Vertical Shear(up)		$\begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix}$, $k > 0$

Two Dimensional Examples: Projections

transformation	**image of unit square**	**standard matrix**
Projection onto the x_1-axis		$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$
Projection onto the x_2-axis		$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$



Onto

the set of all possible images

$\parallel$

Range = Target

$\Updownarrow$

Definition

A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is **onto** if for all $\vec{b} \in \mathbb{R}^m$ there is a $\vec{x} \in \mathbb{R}^n$ so that $T(\vec{x}) = \vec{b}$.

Onto is an **existence property:** for any $\vec{b} \in \mathbb{R}^m$, $A\vec{x} = \vec{b}$ has a solution.

Examples

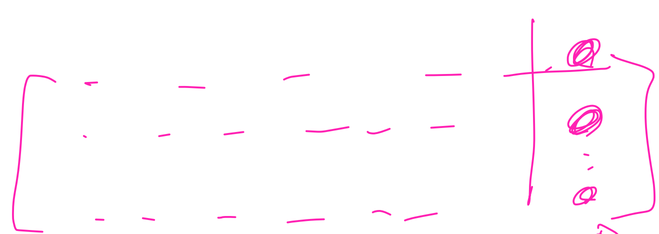
- A rotation on the plane is an onto linear transformation.
- A projection in the plane is not onto.

Useful Fact

T is onto if and only if its standard matrix A has a pivot in every row.

$\Updownarrow$

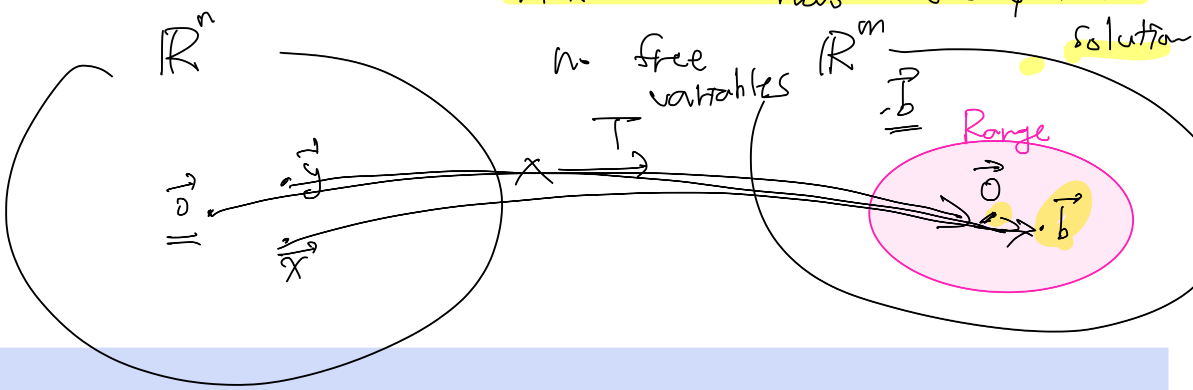
$[\, A \mid \vec{b} \,]$ consistent for any $\vec{b}$



Not a pivot column

T is 1-1 iff ($\Leftrightarrow$) $T(\vec{x}) = \vec{0}$ implies $\vec{x} = \vec{0}$

$A\vec{x} = \vec{0}$ has the only trivial solution



One-to-One

Definition

A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is **one-to-one** if for all $\vec{b} \in \mathbb{R}^m$ there is at most one (possibly no) $\vec{x} \in \mathbb{R}^n$ so that $T(\vec{x}) = \vec{b}$.

One-to-one is a uniqueness property, it does not assert existence for all $\vec{b}$.

Examples

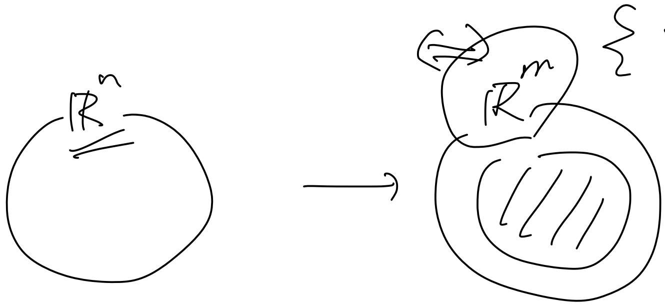
- A rotation on the plane is a one-to-one linear transformation.
- A projection in the plane is not one-to-one.

Useful Facts

- T is one-to-one if and only if the only solution to $T(\vec{x}) = 0$ is the zero vector, $\vec{x} = \vec{0}$.
- T is one-to-one if and only if the standard matrix A of T has no free variables.

$\Leftrightarrow$ Columns of A are linearly independent.

$\Leftrightarrow \{T(e_1), ---, T(e_n)\}$ lin. indep.



$T: \mathbb{R}^n \to \mathbb{R}^m$ is one to one $\Rightarrow$ $n \le m$

$\Updownarrow$ $\cancel{m < n}$

$A \in \mathbb{R}^{m \times n}$ no free variable

Example

Complete the matrices below by entering numbers into the missing entries so that the properties are satisfied. **If it isn't possible to do so, state why**.

row column $\mapsto$ Every column has a pivot

a) A is a 2×3 standard matrix for a one-to-one linear transform.

$$A = \begin{pmatrix} 1 & 0 & \\ 0 & & 1 \end{pmatrix}$$

max pivot $= 2$ need 3 $\Rightarrow$ N.P.

b) B is a 3×2 standard matrix for an onto linear transform.

row column $\mapsto$ Every row $= 3$ pivot.

$$B = \begin{pmatrix} 1 & \\ & \\ & \end{pmatrix}$$

max pivot $= 2$ $\Rightarrow$ N.P.

c) C is a 3×3 standard matrix of a linear transform that is one-to-one ~~and onto~~.

Possible Not onto

$$C = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix} \quad \left(\vec{b} \right)$$

$C\vec{x} = \vec{b}$

$[\, C \mid \vec{b} \,]$

$\in \mathbb{R}^{m\times n}$

Every row in A has pivot

$\Updownarrow$

$A\vec{x} = \vec{b} \in \mathbb{R}^m$

Theorem

For a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ with standard matrix A these are equivalent statements.

1. T is onto. $\Leftrightarrow$ for any $\vec{b} \in \mathbb{R}^m$, there exists $\vec{x} \in \mathbb{R}^n$ s.t. $T(\vec{x}) = \vec{b}$
2. The matrix A has columns which span $\mathbb{R}^m$.
3. The matrix A has m pivotal columns.

Theorem

For a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ with standard matrix A these are equivalent statements.

1. T is one-to-one. $T(\vec{x}) = \vec{0}$ implies $\vec{x} = \vec{0}$
2. The unique solution to $T(\vec{x}) = \vec{0}$ is the trivial one.
3. The matrix A linearly independent columns.
4. Each column of A is pivotal.

Additional Examples

1. Construct a matrix $A \in \mathbb{R}^{2\times 2}$, such that $T(\vec{x}) = A\vec{x}$, where T is a linear transformation that rotates vectors in $\mathbb{R}^2$ counterclockwise by $\pi/2$ radians about the origin, then reflects them through the line $x_1 = x_2$.
2. Define a linear transformation by

$$T(x_1, x_2) = (3x_1 + x_2, 5x_1 + 7x_2, x_1 + 3x_2)$$

   Is T one-to-one? Is T onto?

Section 2.1 : Matrix Operations

Chapter 2 : Matrix Algebra

Math 1554 Linear Algebra

Topics and Objectives

Topics
We will cover these topics in this section.

1. Identity and zero matrices
2. Matrix algebra (sums and products, scalar multiplies, matrix powers)
3. Transpose of a matrix

Objectives
For the topics covered in this section, students are expected to be able to do the following.

1. **Apply** matrix algebra, the matrix transpose, and the zero and identity matrices, to **solve** and **analyze** matrix equations.

Definitions: Zero and Identity Matrices

1. A **zero matrix** is any matrix whose every entry is zero.

$$0_{2\times 3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad 0_{2\times 1} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

2. The $n \times n$ **identity matrix** has ones on the main diagonal, otherwise all zeros.

$$I = I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Note: any matrix with dimensions $n \times n$ is **square**. Zero matrices need not be square, identity matrices must be square.

$$5\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix} = \begin{bmatrix}5\cdot 1 & 5\cdot 2\\5\cdot 3 & 5\cdot 4\end{bmatrix}.$$

Sums and Scalar Multiples = Component-wisely.

Suppose $A \in \mathbb{R}^{m\times n}$, and $a_{i,j}$ is the element of A in row i and column j.

1. If A and B are $m \times n$ matrices, then the elements of $A+B$ are $a_{i,j}+b_{i,j}$.
2. If $c \in \mathbb{R}$, then the elements of cA are $ca_{i,j}$.

$$\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix} + \begin{bmatrix}5 & 6\\7 & 8\end{bmatrix} = \begin{bmatrix}1+5 & 2+6\\3+7 & 4+8\end{bmatrix}$$

For example, if

$$\begin{bmatrix}1 & 2 & 3\\4 & 5 & 6\end{bmatrix} + c\begin{bmatrix}7 & 4 & 7\\0 & 0 & k\end{bmatrix} = \begin{bmatrix}15 & 10 & 17\\4 & 5 & 16\end{bmatrix}$$

What are the values of c and k?

$\begin{bmatrix}1\\2\end{bmatrix} + \begin{bmatrix}3 & 4\end{bmatrix}$ is not defined.

Properties of Sums and Scalar Multiples

Scalar multiples and matrix addition have the expected properties.

If $r, s \in \mathbb{R}$ are scalars, and A, B, C are $m \times n$ matrices, then

1. $A + 0_{m \times n} = A$
2. $(A + B) + C = A + (B + C)$
3. $r(A + B) = rA + rB$
4. $(r + s)A = rA + sA$
5. $r(sA) = (rs)A$

$A \in \mathbb{R}^{m \times \underline{n}}$ $\qquad \vec{x} \longmapsto A\vec{x} \in \mathbb{R}^m$

$\vec{x} \in \mathbb{R}^n$

$\underset{\in \mathbb{R}^{m\times n}}{A} \cdot \underset{\in \mathbb{R}^{n \times p}}{B} \in \mathbb{R}^{m \times p}$

$$A \cdot B = A \cdot [\vec{x}_1 \ \ \vec{x}_2 \ \cdots \ \vec{x}_p]$$
$$= [\underset{\in \mathbb{R}^m}{A\vec{x}_1} \ \ \underset{\in \mathbb{R}^m}{A\vec{x}_2} \ \cdots \ \underset{\in \mathbb{R}^m}{A\vec{x}_p}] \in \mathbb{R}^{m \times p}$$

Matrix Multiplication

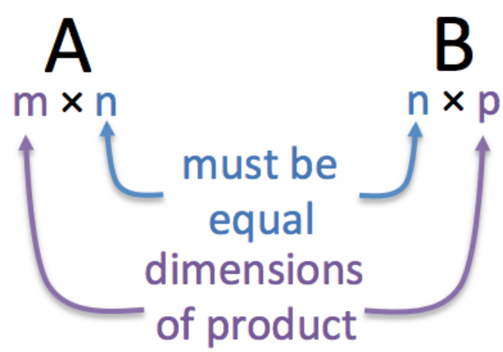
match

Definition

Let A be a $m \times n$ matrix, and B be a $n \times p$ matrix. The product is AB a $m \times p$ matrix, equal to

$$AB = A\left[\vec{b}_1 \quad \cdots \quad \vec{b}_p\right] = \left[A\vec{b}_1 \quad \cdots \quad A\vec{b}_p\right]$$

Note: the dimensions of A and B determine whether AB is defined, and what its dimensions will be.



Dot product

Ex $\begin{bmatrix} 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 4 \end{bmatrix} = 1 \cdot 3 + 2 \cdot 4$

$$\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \cdot \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} = a_1 b_1 + \cdots + a_n \cdot b_n$$

$\mathbb{R}^{m\times n} \to A \cdot B \in \mathbb{R}^{n\times p}$

dot product $\in \mathbb{R}^{m\times p}$

$$\begin{bmatrix} \vec{a}_1 \\ \vec{a}_2 \\ \vdots \\ \vec{a}_m \end{bmatrix} \cdot \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \cdots & \vec{b}_p \end{bmatrix} = \begin{array}{c} \begin{bmatrix} \vec{a}_1\cdot\vec{b}_1 & \vec{a}_1\cdot\vec{b}_2 & \cdots & \vec{a}_1\cdot\vec{b}_p \\ \vdots & \ddots & & \\ & & \vec{a}_i\cdot\vec{b}_j & \cdots \\ \vec{a}_m\cdot\vec{b}_1 & \cdots & \vdots & \end{bmatrix} \begin{array}{l} \\ \\ \leftarrow i\text{th} \\ \\ \end{array} \\ \qquad\qquad j\text{th} \end{array}$$

Row Column Rule for Matrix Multiplication

The Row Column Rule is a convenient way to calculate the product AB that many students have encountered in pre-requisite courses.

Row Column Method

If $A \in \mathbb{R}^{m\times n}$ has rows $\vec{a}_i$, and $B \in \mathbb{R}^{n\times p}$ has columns $\vec{b}_j$, each element of the product $C = AB$ is $c_{ij} = \vec{a}_i \cdot \vec{b}_j$.

Example

Compute the following using the row-column method.

$$C = \underset{2\times 2}{A}\overset{2\times 3}{B} = \begin{pmatrix} 2 & 0 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 3 & 0 & 1 \\ 4 & 5 & 6 \end{pmatrix} = \begin{pmatrix} 6 & 0 & * \\ * & -5 & * \end{pmatrix} \in \mathbb{R}^{2\times 3}$$

$A \in \mathbb{R}^{m \times n}$ $\Rightarrow$ $A \cdot B \in \mathbb{R}^{m \times p}$ defined

$B \in \mathbb{R}^{n \times p}$

$\underset{n \times p}{B} \cdot \underset{m \times n}{A}$ not defined unless $p = m$

$A \cdot B \in \mathbb{R}^{m \times m}$

$B \cdot A \in \mathbb{R}^{n \times n}$ have same size

Properties of Matrix Multiplication

Let A, B, C be matrices of the sizes needed for the matrix multiplication to be defined, and A is a $m \times n$ matrix. $m = n$

1. (Associative) $(AB)C = A(BC)$
2. (Left Distributive) $A(B + C) = AB + AC$
3. (Right Distributive) $\cdots$ $(A+B) \cdot C = AC + BC$
4. (Identity for matrix multiplication) $I_m A = A I_n = A$

Warnings:

1. (non-commutative) In general, $AB \neq BA$.
2. (non-cancellation) $AB = AC$ does not mean $B = C$.
3. (Zero divisors) $AB = 0$ does not mean that either $A = 0$ or $B = 0$.

In $\mathbb{R}$, $a \cdot b = 0 \Rightarrow a = 0$ or $b = 0$

$A, B \in \mathbb{R}^{2 \times 2}$ $A \cdot B = 0$ and $A \neq 0$, $B = 0$

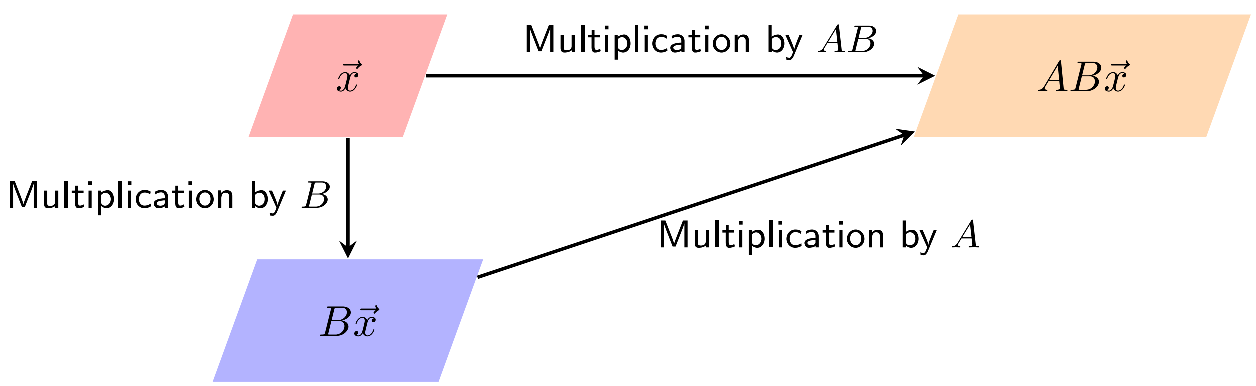
$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

The Associative Property

The associative property is $(AB)C = A(BC)$. If $C = \vec{x}$, then

$$(AB)\vec{x} = A(B\vec{x})$$

Schematically:



The matrix product $AB\vec{x}$ can be obtained by either: multiplying by matrix AB, or by multiplying by B then by A. This means that matrix multiplication corresponds to **composition of the linear transformations**.

Example

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

Give an example of a 2×2 matrix $\underline{\underline{B}}$ that is non-commutative with A.

Find B s.t. $AB \neq BA$

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \neq \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} a+c & b+d \\ 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} a & a \\ c & c \end{bmatrix}$$

Additional Examples

True or false:

1. For any I_n and any $A \in \mathbb{R}^{n\times n}$, $(I_n + A)(I_n - A) = I_n - A^2$.

2. For any A and B in $\mathbb{R}^{n\times n}$, $(A+B)^2 = A^2 + B^2 + 2AB$.

The Transpose of a Matrix

A^T is the matrix whose columns are the rows of A.

Example

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 0 & 2 & 0 \end{bmatrix}^T = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 0 \\ 4 & 2 \\ 5 & 0 \end{bmatrix}$$

Properties of the Matrix Transpose

1. $(A^T)^T = A$
2. $(A+B)^T = A^T + B^T$
3. $(rA)^T = r \cdot A^T$
4. $\boxed{(AB)^T = B^T \cdot A^T}$

Pf:

$A \in \mathbb{R}^{m \times n}$ $\quad$ $B \in \mathbb{R}^{n \times p}$ $\Rightarrow A \cdot B \in \mathbb{R}^{m \times p}$

$\downarrow$ $\quad$ $\downarrow$

$\underbrace{A^T \in \mathbb{R}^{n \times m} \quad B^T \in \mathbb{R}^{p \times n}}$

$\Downarrow$

$A^T \cdot B^T$ not defined

$B^T \cdot A^T \in \mathbb{R}^{p \times n}$

$$A^3 = \overbrace{A \cdot A \cdot A}^{3 \text{ times}}$$

Matrix Powers

For any $n \times n$ matrix and positive integer k, A^k is the product of k copies of A.

$$A^k = AA \dots A$$

Example: Compute C^8.

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$C^8 = \begin{bmatrix} 1^8 & 0 & 0 \\ 0 & 2^8 & 0 \\ 0 & 0 & 2^8 \end{bmatrix}$$

$$C^2 = C \cdot C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1^2 & 0 & 0 \\ 0 & \cancel{4}\,2^2 & 0 \\ 0 & 0 & 2^2\cancel{4} \end{bmatrix}$$

Pattern.

$$C^3 = C^2 \cdot C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix}$$

$\ddots$

Example

Define

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 8 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Which of these operations are defined, and what are the dimensions of the result?

1. $A + 3C$ N. D.
2. $A(AB)^T$ $\underbrace{A \cdot B}_{2\times 2,\ 2\times 3} \in \mathbb{R}^{2\times 3}$ $(AB)^T \in \mathbb{R}^{3\times 2}$ $\underbrace{A}_{2\times 2} \cdot \underbrace{(AB)^T}_{3\times 2}$ Not defined.
3. $A + ABCB^T \in \mathbb{R}^{2\times 2}$

2×2 2×2

$A \cdot B \cdot C \cdot B^T$

2×2 2×3 3×3 3×2

2×3

2×3

$\Rightarrow 2\times 2$

Section 2.2 : Inverse of a Matrix

Chapter 2 : Matrix Algebra

Math 1554 Linear Algebra

"Your scientists were so preoccupied with whether or not they could, they didn't stop to think if they should."

- Spielberg and Crichton, Jurassic Park, 1993 film

The algorithm we introduce in this section **could** be used to compute an inverse of an $n \times n$ matrix. At the end of the lecture we'll discuss some of the problems with our algorithm and why it can be difficult to compute a matrix inverse.

Topics and Objectives

Topics

We will cover these topics in this section.

1. Inverse of a matrix, its algebraic properties, and its relation to solving systems of linear equations.
2. Elementary matrices and their role in calculating the matrix inverse.

Objectives

For the topics covered in this section, students are expected to be able to do the following.

1. Apply the formal definition of an inverse, and its algebraic properties, to solve and analyze linear systems.
2. Compute the inverse of an $n \times n$ matrix, and use it to solve linear systems.
3. Construct elementary matrices.

Motivating Question

Is there a matrix, A, such that $\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} A = I_3$?

The Matrix Inverse

Definition

$A \in \mathbb{R}^{n\times n}$ is **invertible** (or **non-singular**) if there is a $C \in \mathbb{R}^{n\times n}$ so that

$$AC = CA = I_n. = \begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{bmatrix}$$

If there is, we write $C = A^{-1}$.

Facts

① $A \cdot C = I_n \Rightarrow C \cdot A = I_n$

② For a given A, find $\underline{\underline{C}}$ unique

s.t. $A \cdot C = I$

The Inverse of a 2×2 Matrix

There's a formula for computing the inverse of a 2×2 matrix.

Theorem

The 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is non-singular (Invertible) if and only if $ad - bc \neq 0$, and then

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \boxed{\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

4 variables

4 equations.

linear

Example

State the inverse of the matrix below.

$$\begin{bmatrix} \overset{=a}{2} & \overset{=b}{5} \\ \underset{=c}{-3} & \underset{=d}{-7} \end{bmatrix}$$

$ad - bc = 2 \cdot (-7) - (5) \cdot (-3) = 1 \neq 0$

Invertible

$$\begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix}^{-1} = \frac{1}{\underset{= ad-bc}{1}} \begin{bmatrix} -7 & -5 \\ -(-3) & 2 \end{bmatrix} = \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix}$$

$$A\vec{x} = \vec{b}$$
$$A^{-1}\cdot A\vec{x} = A^{-1}\cdot\vec{b}$$
$$\vec{x} = I_n\cdot\vec{x} = A^{-1}\cdot\vec{b}$$

The Matrix Inverse

$$A\cdot A^{-1} = A^{-1}\cdot A = I_n$$

Theorem

$A \in \mathbb{R}^{n\times n}$ has an inverse if and only if for all $\vec{b} \in \mathbb{R}^n$, $A\vec{x} = \vec{b}$ has a unique solution. And, in this case, $\vec{x} = A^{-1}\vec{b}$.

Example

Solve the linear system.

$$3x_1 + 4x_2 = 7$$
$$5x_1 + 6x_2 = 7$$

$$\underbrace{\begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}}_{A}\cdot\underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 7 \\ 7 \end{bmatrix}}_{\vec{b}}$$

① A is invertible? Yes

$$ad - bc = 3\cdot 6 - 4\cdot 5 = -2 \neq 0.$$

② $$A^{-1} = \frac{1}{-2}\cdot\begin{bmatrix} 6 & -4 \\ -5 & 3 \end{bmatrix}$$

③ $$\vec{x} = A^{-1}\cdot\vec{b} = -\frac{1}{2}\begin{bmatrix} 6 & -4 \\ -5 & 3 \end{bmatrix}\cdot\begin{bmatrix} 7 \\ 7 \end{bmatrix} = \begin{bmatrix} -7 \\ 7 \end{bmatrix}$$

Properties of the Matrix Inverse

A and B are invertible $n \times n$ matrices.

1. $(A^{-1})^{-1} = A$
2. $(AB)^{-1} = B^{-1}A^{-1}$ (Non-commutative!)
3. $(A^T)^{-1} = (A^{-1})^T$

Exercise

$A \cdot (A^{-1}) = I$

$\underbrace{(A \cdot \underbrace{B) \cdot (B^{-1}}_{I} \cdot A^{-1})}_{I} = I$

Example

True or false: $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$.

$((AB) \cdot C)^{-1} = C^{-1} \cdot (AB)^{-1}$

$= C^{-1} \cdot B^{-1} \cdot A^{-1}$

$\underbrace{(AB\underbrace{C) \cdot (C^{-1}}_{I} B^{-1} \cdot A^{-1})}_{I} = I$

$$A[\vec{x}_1\ \vec{x}_2 \cdots \vec{x}_n] = I = [\vec{e}_1\ \vec{e}_2\ \cdots\ \vec{e}_n]$$

$$\underbrace{A[\vec{x}_1\ \vec{x}_2 \cdots \vec{x}_n]}_{= [A\vec{x}_1\ A\vec{x}_2\ \cdots\ A\vec{x}_n]}$$

$$\left.\begin{array}{l} A\vec{x}_1 = \vec{e}_1 \\ A\vec{x}_2 = \vec{e}_2 \\ \vdots \\ A\vec{x}_n = \vec{e}_n \end{array}\right\}$$

$\left[A \,\middle|\, \begin{matrix}1\\0\\0\\ \vdots\\0\end{matrix}\right], \left[A \,\middle|\, \begin{matrix}0\\1\\0\\ \vdots\\0\end{matrix}\right], \cdots, \left[A \,\middle|\, \begin{matrix}0\\ \vdots\\0\\1\end{matrix}\right]$ n linear systems

An Algorithm for Computing A^{-1}

If $A \in \mathbb{R}^{n\times n}$, and $n > 2$, how do we calculate A^{-1}? Here's an algorithm we can use:

1. Row reduce the augmented matrix $(A \mid I_n)$
2. If reduction has form $(I_n \mid B)$ then A is invertible and $B = A^{-1}$. Otherwise, A is not invertible.

Example

Compute the inverse of $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 0 & 0 & 1 \end{bmatrix}$.

$$\left[A \,\middle|\, \begin{matrix}1 & 0 & & 0\\0 & 1 & & \vdots\\0 & 0 & \cdots & 0\\ \vdots & \vdots & & \\0 & 0 & & 1\end{matrix}\right] \xrightarrow{r.o.} \left[\begin{matrix}1 & & 0\\ & 1 & \\ 0 & & 1\end{matrix} \,\middle|\, \underbrace{?}_{A^{-1}}\right]$$

$$[A \mid I] \xrightarrow[r.o.]{} [I \mid ?]$$

$?$ ↑ A^{-1}

possible $\Rightarrow$ A is invertible

$A \xrightarrow{r.o} I$: A and I are row equivalent.

$$\left[A \,\middle|\, \underbrace{\vec{e}_1\ \vec{e}_2 \cdots \vec{e}_n}_{= I}\right] = [A \mid I] \xrightarrow[\text{row operation}]{} [\text{RREF} \mid *]$$

all pivot, $= I$

Every (row / column) has pivot

$A \in \mathbb{R}^{n\times n}$ is invertible $\overset{\text{(definition)}}{\iff}$ there exists $C \in \mathbb{R}^{n\times n}$ s.t. $A \cdot C = I = C \cdot A$

$\iff A \cdot [\vec{x_1}\ \ \vec{x_2}\ \cdots\ \vec{x_n}] = [\vec{e_1}\ \ \vec{e_2}\ \cdots\ \vec{e_n}]$ $\iff$ For any $\vec{b} \in \mathbb{R}^n$

$$\left\{\begin{aligned} A\vec{x_1} &= \vec{e_1} \\ A\vec{x_2} &= \vec{e_2} \\ &\vdots \\ A\vec{x_n} &= \vec{e_n} \end{aligned}\right. \Longleftarrow \text{Consistent}$$

$A\vec{x} = \vec{b}$ Consistent

$b = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$ solution $b_1 \cdot \vec{x_1} + \cdots + b_n \cdot \vec{x_n}$

Why Does This Work?

We can think of our algorithm as simultaneously solving n linear systems:

$$\begin{aligned} A\vec{x}_1 &= \vec{e}_1 \\ A\vec{x}_2 &= \vec{e}_2 \\ &\vdots \\ A\vec{x}_n &= \vec{e}_n \end{aligned}$$

Each column of A^{-1} is $A^{-1}\vec{e}_i = \vec{x}_i$.

Over the next few slides we explore another explanation for how our algorithm works. This other explanation uses elementary matrices.

$$I_n \xrightarrow{\text{1 row operation}} \boxed{\;} \leftarrow \text{Elementary matrices}$$

$$I_3 \xrightarrow[R_2 \leftrightarrow R_3]{} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}}$$

Elementary Matrices

An elementary matrix, E, is one that differs by I_n by one row operation. Recall our elementary row operations:

1. swap rows
2. multiply a row by a non-zero scalar
3. add a multiple of one row to another

We can represent each operation by a matrix multiplication with an **elementary matrix**.

$$A \in \mathbb{R}^{3\times 3}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \xrightarrow[R_2 \leftrightarrow R_3]{} \underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 7 & 8 & 9 \\ 4 & 5 & 6 \end{bmatrix}}$$

$$= \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}}$$

Example

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} \xrightarrow[R_2 \to R_2 + 2\cdot R_1]{} \begin{bmatrix}1&0&0\\2&1&0\\0&0&1\end{bmatrix} = E$$

Suppose

$$E\begin{bmatrix}1&1&1\\-2&1&0\\0&0&1\end{bmatrix}=\begin{bmatrix}1&1&1\\0&3&2\\0&0&1\end{bmatrix}$$

By inspection, what is E? How does it compare to I_3?

A is invertible $\iff$ A is row equivalent to I

$\Updownarrow$

A is a product of elementary matrices

$$A \xrightarrow[E_1]{(1)} E_1 \cdot A \xrightarrow[E_2]{(2)} E_2 E_1 A \;\cdots \xrightarrow[E_k]{(k)} I$$

$$A = E_1^{-1} E_2^{-1} \cdots E_k^{-1} \Longleftarrow \underbrace{(E_k \cdots E_3 E_2 E_1)}_{A^{-1}} A = I$$

Theorem

Returning to understanding why our algorithm works, we apply a sequence of row operations to A to obtain I_n:

$$(E_k \cdots E_3 E_2 E_1)A = I_n$$

Thus, $E_k \cdots E_3 E_2 E_1$ is the inverse matrix we seek.

Our algorithm for calculating the inverse of a matrix is the result of the following theorem.

Theorem

Matrix A is invertible if and only if it is row equivalent to the identity. In this case, the any sequence of elementary row operations that transforms A into I, applied to I, generates A^{-1}.

Using The Inverse to Solve a Linear System

- We could use A^{-1} to solve a linear system,

$$A\vec{x} = \vec{b}$$

  We would calculate A^{-1} and then:

- As our textbook points out, A^{-1} is seldom used: computing it can take a very long time, and is prone to numerical error.
- So why did we learn how to compute A^{-1}? Later on in this course, we use elementary matrices and properties of A^{-1} to derive results.
- A recurring theme of this course: just because we **can** do something a certain way, doesn't that we **should**.

Section 2.3 : Invertible Matrices

Chapter 2 : Matrix Algebra

Math 1554 Linear Algebra

"A synonym is a word you use when you can't spell the other one."
- Baltasar Gracián

The theorem we introduce in this section of the course gives us many ways of saying the same thing. Depending on the context, some will be more convenient than others.

Topics and Objectives

Topics
We will cover these topics in this section.

1. The invertible matrix theorem, which is a review/synthesis of many of the concepts we have introduced.

Objectives
For the topics covered in this section, students are expected to be able to do the following.

1. Characterize the invertibility of a matrix using the Invertible Matrix Theorem.
2. Construct and give examples of matrices that are/are not invertible.

Motivating Question
When is a square matrix invertible? Let me count the ways!

The Invertible Matrix Theorem

Invertible matrices enjoy a rich set of equivalent descriptions.

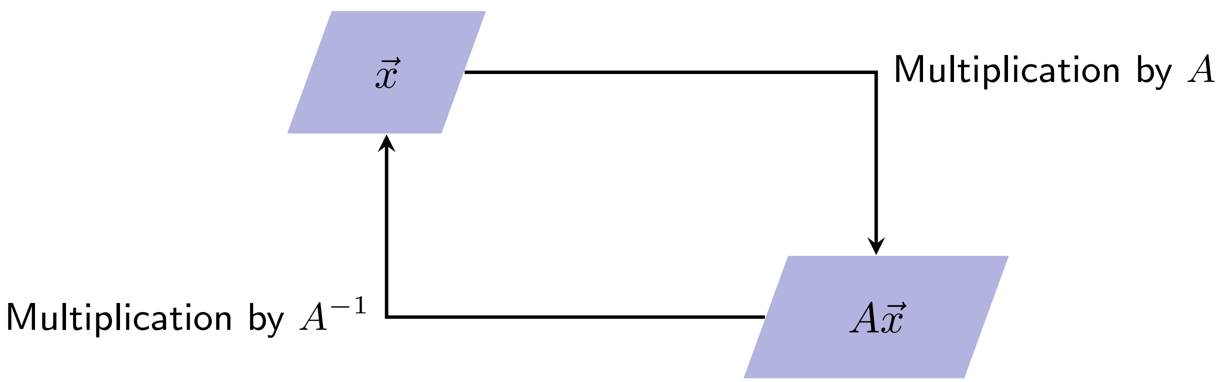
Theorem

Let A be an $n \times n$ matrix. These statements are all equivalent.

a) *A is invertible.*
b) *A is row equivalent to I_n.*
c) *A has n pivotal columns. (All columns are pivotal.)*
d) *$A\vec{x} = \vec{0}$ has only the trivial solution.* $\Leftrightarrow$ $T(\vec{x}) = A \cdot \vec{x}$ is 1-1
e) *The columns of A are linearly independent.*
f) *The linear transformation $\vec{x} \mapsto A\vec{x}$ is one-to-one.*
g) *The equation $A\vec{x} = \vec{b}$ has a solution for all $\vec{b} \in \mathbb{R}^n$.*
h) *The columns of A span $\mathbb{R}^n$.*
i) *The linear transformation $\vec{x} \mapsto A\vec{x}$ is onto.*
j) *There is a $n \times n$ matrix C so that $CA = I_n$. (A has a left inverse.)*
k) *There is a $n \times n$ matrix D so that $AD = I_n$. (A has a right inverse.)*
l) *A^T is invertible.*

Invertibility and Composition

The diagram below gives us another perspective on the role of A^{-1}.



The matrix inverse A^{-1} transforms Ax back to $\vec{x}$. This is because:

$$A^{-1}(A\vec{x}) = (A^{-1}A)\vec{x} =$$

The Invertible Matrix Theorem: Final Notes

- Items j and k of the invertible matrix theorem (IMT) lead us directly to the following theorem.

Theorem

If A and B are $n \times n$ matrices and $AB = I$, then A and B are invertible, and $B = A^{-1}$ and $A = B^{-1}$.

- The IMT is a set of equivalent statements. They divide the set of all square matrices into two separate classes: invertible, and non-invertible.
- As we progress through this course, we will be able to add additional equivalent statements to the IMT (that deal with determinants, eigenvalues, etc).

Example 1

Is this matrix invertible?

$$\begin{array}{c} 0 \\ 0 \end{array} \begin{bmatrix} 1 & 0 & -2 \\ 3 & 1 & -2 \\ -5 & -1 & 9 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & 0 & -2 \\ 0 & \boxed{1} & \boxed{4} \\ 0 & \boxed{-1} & \boxed{-1} \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 4 \\ 0 & 0 & 3 \end{bmatrix}$$

Invertible.

Ex

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 0 \\ 0 & 8 & 1 \end{bmatrix}$$

A Invertible Columns lin. indep.

Example 2

If possible, fill in the missing elements of the matrices below with numbers so that each of the matrices are singular. If it is not possible to do so, state why.

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & * & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

N. I.

N. P.

lin. dep.

$\Downarrow$

Not invertible.

Section 2.4 : Partitioned Matrices

Chapter 2 : Matrix Algebra

Math 1554 Linear Algebra

"Mathematics is not about numbers, equations, computations, or algorithms. Mathematics is about understanding."
- William Paul Thurston

Multiple perspectives of the same concept is a theme of this course; each perspective deepens our understanding. In this section we explore another way of representing matrices and their algebra that gives us another way of thinking about them.

Topics and Objectives

Topics

We will cover these topics in this section.

1. Partitioned matrices (or block matrices)

Objectives

For the topics covered in this section, students are expected to be able to do the following.

1. Apply partitioned matrices to solve problems regarding matrix invertibility and matrix multiplication.

What is a Partitioned Matrix?

Example

This matrix:

$$\left[\begin{array}{ccc|cc} 3 & 1 & 4 & 1 & 0 \\ 1 & 6 & 1 & 0 & 1 \\ \hline 0 & 0 & 0 & 4 & 2 \end{array}\right]$$

can also be written as:

identity

$$\begin{bmatrix} \begin{bmatrix} 3 & 1 & 4 \\ 1 & 6 & 1 \end{bmatrix} & \boxed{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} \\ \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 4 & 2 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{bmatrix}$$

zero

We partitioned our matrix into four **blocks**, each of which has different dimensions.

Another Example of a Partitioned Matrix

Example: The reduced echelon form of a matrix. We can use a partitioned matrix to

$$\begin{bmatrix} 1 & 0 & 0 & 0 & * & \cdots & * \\ 0 & 1 & 0 & 0 & * & \cdots & * \\ 0 & 0 & 1 & 0 & * & \cdots & * \\ 0 & 0 & 0 & 1 & * & \cdots & * \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix} = \begin{bmatrix} I_4 & F \\ 0 & 0 \end{bmatrix}$$

This is useful when studying the **null space** of A, as we will see later in this course.

Row Column Method

Recall that a row vector times a column vector (of the right dimensions) is a scalar. For example,

$$\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = 1\cdot 1 + 1\cdot 0 + 1\cdot 2$$

This is the **row column** matrix multiplication method from Section 2.1.

Theorem

Let A be $m \times n$ and B be $n \times p$ matrix. Then, the (i, j) entry of AB is

$$\text{row}_i\, A \cdot \text{col}_j\, B.$$

This is the **Row Column Method** for matrix multiplication.

Partitioned matrices can be multiplied using this method, as if each block were a scalar (provided each block has appropriate dimensions).

$$\begin{array}{c} \\ 4 \end{array} \left(\begin{array}{c|c} (A) & (B) \\ \hline (C) & (D) \end{array}\right) \left(\begin{array}{c|c} (P) & (Q) \\ \hline (R) & (S) \end{array}\right) 4$$

4 4

$$= \left(\begin{array}{c|c} AP + BR & \cdot \\ \hline \cdot & \cdot \end{array}\right)$$

$$4\left\{\begin{pmatrix} \overset{\in\mathbb{R}^{2\times 2}}{A} & \overset{\in\mathbb{R}^{2\times 2}}{B} \\ C & D \end{pmatrix}\right. \cdot \begin{pmatrix} P & Q \\ R & S \end{pmatrix}\Bigg\} 4 = \begin{pmatrix} A\cdot P + B\cdot R & AQ+BS \\ C\cdot P + DR & CQ+DS \end{pmatrix}$$

$$\underbrace{\qquad}_{4} \qquad \underbrace{\qquad}_{4} \qquad \downarrow \in \mathbb{R}^{2\times 2}$$

Example of Row Column Method

Recall, using our formula for a 2×2 matrix, $\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}^{-1} = \boxed{\frac{1}{ac}\begin{bmatrix} c & -b \\ 0 & a \end{bmatrix}}$.

Example: Suppose $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times n}$, and $C \in \mathbb{R}^{n\times n}$ are invertible matrices. Construct the inverse of $\begin{bmatrix} A & B \\ 0 & C \end{bmatrix}$.

$$2n\left\{\begin{pmatrix} \overset{\in\mathbb{R}^{n\times n}}{A} & \overset{\in\mathbb{R}^{n\times n}}{B} \\ \underset{\in\mathbb{R}^{n\times n}}{0} & \underset{\in\mathbb{R}^{n\times n}}{C} \end{pmatrix}^{-1}\right. = \begin{pmatrix} & \\ & \end{pmatrix}$$

$$\underbrace{\qquad\qquad}_{2n}$$

$$\begin{pmatrix} P & Q \\ R & S \end{pmatrix} = \begin{pmatrix} A^{-1} & -A^{-1}BC^{-1} \\ 0 & C^{-1} \end{pmatrix}$$

$$= \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \cdot \begin{pmatrix} P & Q \\ R & S \end{pmatrix} = \begin{pmatrix} I_n & 0 \\ 0 & I_n \end{pmatrix} = I_{2n}$$

$$= \begin{pmatrix} \underset{I_n}{\boxed{AP}} + \underset{=0}{BR} & \underset{=0}{\boxed{AQ+BS}} \\ \underset{=0}{CR} & C\cdot S \end{pmatrix}$$

$AQ + B\cdot C^{-1} = 0$

$P = A^{-1}$, $Q = A^{-1}AQ = -A^{-1}B\cdot C^{-1}$

$S = C^{-1}$, $R = 0$

Section 2.5 : Matrix Factorizations

Chapter 2 : Matrix Algebra

Math 1554 Linear Algebra

"Mathematical reasoning may be regarded rather schematically as the exercise of a combination of two facilities, which we may call intuition and ingenuity." - Alan Turing

The use of the LU Decomposition to solve linear systems was one of the areas of mathematics that Turing helped develop.

Topics and Objectives

Topics

We will cover these topics in this section.

1. The LU factorization of a matrix
2. Using the LU factorization to solve a system
3. Why the LU factorization works

Objectives

For the topics covered in this section, students are expected to be able to do the following.

1. Compute an LU factorization of a matrix.
2. Apply the LU factorization to solve systems of equations.
3. Determine whether a matrix has an LU factorization.

Motivation

- Recall that we **could** solve $A\vec{x} = \vec{b}$ by using

$$\vec{x} = A^{-1}\vec{b}$$

- This requires computation of the inverse of an $n \times n$ matrix, which is especially difficult for large n.
- Instead we could solve $A\vec{x} = \vec{b}$ with Gaussian Elimination, but this is not efficient for large n
- There are more efficient and accurate methods for solving linear systems that rely on matrix factorizations.

Matrix Factorizations

- A **matrix factorization**, or **matrix decomposition** is a factorization of a matrix into a product of matrices.
- Factorizations can be useful for solving $A\vec{x} = \vec{b}$, or understanding the properties of a matrix.
- We explore a few matrix factorizations throughout this course.
- In this section, we factor a matrix into **lower** and into **upper** triangular matrices.

Triangular Matrices

- A rectangular matrix A is **upper triangular** if $a_{i,j} = 0$ for $i > j$. Examples:

$$\begin{pmatrix} 1 & 5 & 0 \\ 0 & 2 & 4 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

- A rectangular matrix A is **lower triangular** if $a_{i,j} = 0$ for $i < j$. Examples:

$$\begin{pmatrix} 1 & 0 & 0 \\ 3 & 2 & 0 \end{pmatrix}, \quad \begin{pmatrix} 3 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 2 \\ 1 \\ 2 \end{pmatrix}$$

Ask: Can you name a matrix that is both upper and lower triangular?

The LU Factorization

Row operations (only replacement) $\Rightarrow$ Echelon form $= U$

$L =$ the product of elementary matrices.

any size.

Theorem

If A is an $m \times n$ matrix that can be row reduced to echelon form without row exchanges, then $A = LU$. L is a lower triangular $m \times m$ matrix with 1's on the diagonal, U is an **echelon** form of A.

upper triangular.

Example: If $A \in \mathbb{R}^{3\times 2}$, the LU factorization has the form:

$$\underbrace{\begin{pmatrix} * & * \\ * & * \\ * & * \end{pmatrix}}_{2} = A = LU = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ * & 1 & 0 \\ * & * & 1 \end{pmatrix}}_{L} \begin{pmatrix} * & * \\ 0 & * \\ 0 & 0 \end{pmatrix} = U \in \mathbb{R}^{3\times 2}$$

3

Square matrix with $\mathbb{R}^{3\times 3}$ # of rows in A.

Same size.

Why We Can Compute the LU Factorization

Suppose A can be row reduced to echelon form U without interchanging rows. Then,

$$E_p \cdots E_1 A = U$$

where the E_j are matrices that perform elementary row operations. They happen to be lower triangular and invertible, e.g.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

Therefore,

$$A = \underbrace{E_1^{-1} \cdots E_p^{-1}}_{=L} U = LU.$$

Using the LU Decomposition

Goal: given A and $\vec{b}$, solve $A\vec{x} = \vec{b}$ for $\vec{x}$.

Algorithm: construct $A = LU$, solve $A\vec{x} = LU\vec{x} = \vec{b}$ by:

1. Forward solve for $\vec{y}$ in $L\vec{y} = \vec{b}$.
2. Backwards solve for x in $U\vec{x} = \vec{y}$.

Example: Solve the linear system whose LU decomposition is given.

$$A = LU = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} 2 \\ 3 \\ 2 \\ 0 \end{pmatrix}$$

Solve $A\vec{x} = \vec{b}$ $\quad$ ($L \cdot \underbrace{\boxed{U \cdot \vec{x}}}_{=\vec{y}} = \vec{b}$)

① For $U \cdot \vec{x} = \vec{y}$, Solve $L\vec{y} = \vec{b}$

$$\begin{pmatrix} y_1 \\ 2 + y_2 \\ 2 + y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 2 \\ 0 \end{pmatrix} \longrightarrow$$

$y_1 = 2$

$2 + y_2 = 3$, $y_2 = 1$

$2 + y_3 = 2$, $y_3 = 0$

$y_4 = 0$

② Solve $U \cdot \vec{x} = \vec{y}$

$$\begin{pmatrix} x_1 \\ 2x_2 \\ \cancel{x_3}\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$x_3 = 0$

$x_2 = \frac{1}{2}$

$x_1 = 2$

An Algorithm for Computing LU

To compute the LU decomposition:

1. Reduce A to an echelon form U by a sequence of row replacement operations, if possible.
2. Place entries in L such that the same sequence of row operations reduces L to I.

Note that

- In MATH 1554, the only row replacement operation we can use is to *replace a row with a multiple of a row above it*.
- More advanced linear algebra courses address this limitation.

Example: Compute the LU factorization of A.

$$A = \begin{pmatrix} 4 & -3 & -1 & 5 \\ -16 & 12 & 2 & -17 \\ 8 & -6 & -12 & 22 \end{pmatrix}$$

① Find U:

$$\begin{pmatrix} 4 & -3 & -1 & 5 \\ -16 \to 0 & 12 & 2 & -17 \\ 8 \to 0 & -6 & -12 & 22 \end{pmatrix} = A$$

① $R_2 \to R_2 + 4R_1$
② $R_3 \to R_3 - 2R_1$

$$\longrightarrow \begin{pmatrix} 4 & -3 & -1 & 5 \\ 0 & 0 & -2 & 3 \\ 0 & 0 & \underbrace{-10}_{\to 0} & 12 \end{pmatrix}$$

③ $R_3 \to R_3 - 5R_2$

$$\longrightarrow \begin{pmatrix} 4 & -3 & -1 & 5 \\ 0 & 0 & -2 & 3 \\ 0 & 0 & 0 & -3 \end{pmatrix} = U$$

$$\underbrace{E_3 E_2 E_1}A = U$$

$$A = \underbrace{E_1^{-1} E_2^{-1} E_3^{-1}}_{= L} U$$

$$E_1^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1}$$

$$E_2^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ +2 & 0 & 1 \end{pmatrix}^{-1}$$

$$E_3^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & +5 & 1 \end{pmatrix}^{-1}$$

$$E_1^{-1} E_2^{-1} E_3^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 2 & 5 & 1 \end{pmatrix} = L$$

Summary

- To solve $A\vec{x} = L\overset{\vec{y}}{\boxed{U\vec{x}}} = \vec{b}$,
  1. Forward solve for $\vec{y}$ in $L\vec{y} = \vec{b}$.
  2. Backwards solve for $\vec{x}$ in $U\vec{x} = \vec{y}$.

y_1, y_2, $\vdots$

x_n, x_{n-1}, $\vdots$

- To compute the LU decomposition:
  1. Reduce A to an echelon form U by a sequence of row replacement operations, if possible.
  2. Place entries in L such that the same sequence of row operations reduces L to I. ← Product of elementary matrices
- The textbook offers a different explanation of how to construct the LU decomposition that students may find helpful.
- Another explanation on how to calculate the LU decomposition that students may find helpful is available from MIT OpenCourseWare: www.youtube.com/watch?v=rhNKncraJMk

Section 2.8 : Subspaces of $\mathbb{R}^n$

Chapter 2 : Matrix Algebra

Math 1554 Linear Algebra

Topics and Objectives

Topics

We will cover these topics in this section.

1. Subspaces, Column space, and Null spaces
2. A basis for a subspace.

Objectives

For the topics covered in this section, students are expected to be able to do the following.

1. Determine whether a set is a subspace.
2. Determine whether a vector is in a particular subspace, or find a vector in that subspace.
3. Construct a basis for a subspace (for example, a basis for Col(A))

Motivating Question

Given a matrix A, what is the set of vectors $\vec{b}$ for which we can solve $A\vec{x} = \vec{b}$?

Subsets of $\mathbb{R}^n$

No structure.

Definition

A **subset of** $\mathbb{R}^n$ is any collection of vectors that are in $\mathbb{R}^n$.

$\text{Span}(\{\vec{v}_1, \vec{v}_2\}) = \{c_1\vec{v}_1 + c_2\vec{v}_2 : c_1, c_2 \in \mathbb{R}\}$

↑ a subset of $\mathbb{R}^n$.

a subspace.

Subspaces in $\mathbb{R}^n$

smallest subspace.

Smallest subset? ϕ or $\{\vec{0}\}$

Definition

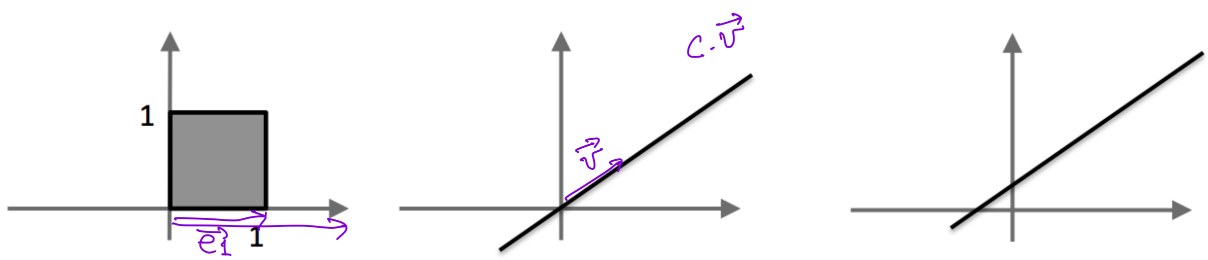
A subset H of $\mathbb{R}^n$ is a **subspace** if it is closed under scalar multiplies and vector addition. That is: for any $c \in \mathbb{R}$ and for $\vec{u}, \vec{v} \in H$,

1. $c\,\vec{u} \in H$
2. $\vec{u} + \vec{v} \in H$

H is a subspace. $\Rightarrow \vec{0} \in H$

Note that condition 1 implies that the zero vector must be in H.

Example 1: Which of the following subsets could be a subspace of $\mathbb{R}^2$?



a) the unit square

b) a line passing through the origin

c) a line that doesn't pass through the origin

$\vec{e}_1 \in H$

$2\cdot\vec{e}_1 \notin H$

Not a subspace.

$c\vec{v} + c'\vec{v}$
$= (c+c')\vec{v}$

$c'\cdot(c\,\vec{v}) = (c'\cdot c)\vec{v}$

A subspace

$\vec{0} \notin H$

Not a space.

The Column Space and the Null Space of a Matrix

Recall: for $\vec{v}_1, \dots, \vec{v}_p \in \mathbb{R}^n$, that $\text{Span}\{\vec{v}_1, \dots, \vec{v}_p\}$ is:

$= \{ c_1\vec{v}_1 + \cdots + c_p\vec{v}_p : c_1, \dots, c_p \in \mathbb{R} \}$

$(c_1\vec{v}_1 + \cdots + c_p\vec{v}_p) + (c_1'\vec{v}_1 + \cdots + c_p'\vec{v}_p)$

$= (c_1+c_1')\vec{v}_1 + \cdots + (c_p+c_p')\vec{v}_p \in \text{Span}\{\vec{v}_1, \dots, \vec{v}_p\} \leftarrow$ a subspace.

This is a **subspace**, spanned by $\vec{v}_1, \dots, \vec{v}_p$.

Definition

Given an $m \times n$ matrix $A = \begin{bmatrix} \vec{a}_1 & \cdots & \vec{a}_n \end{bmatrix}$

1. The **column space of** A, $\text{Col}\, A$, is the subspace of $\mathbb{R}^m$ spanned by $\vec{a}_1, \dots, \vec{a}_n$. columns
2. The **null space of** A, $\text{Null}\, A$, is the subspace of $\mathbb{R}^n$ spanned by the set of all vectors $\vec{x}$ that solve $A\vec{x} = \vec{0}$.
solutions

Column space of A = Span {Columns}

= Range of $T(\vec{x}) = A\vec{x}$

Null space of A = Span {Solution to $A\vec{x} = \vec{0}$}

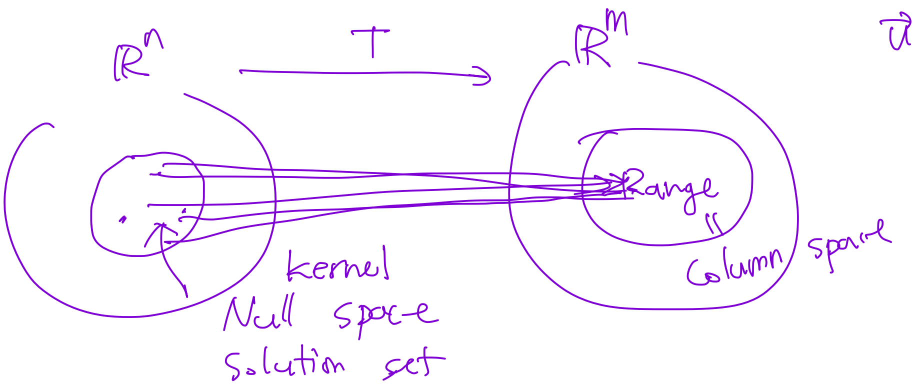
$\parallel$

Solution set = {Solutions to $A\vec{x} = \vec{0}$}

$\vec{u}, \vec{v}$, solution

$\vec{u} + \vec{v}$ solution

$c \cdot \vec{u}$ "



Example

Is $\vec{b}$ in the column space of A?

$$A = \begin{bmatrix} 1 & -3 & -4 \\ -4 & 6 & -2 \\ -3 & 7 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & -4 \\ 0 & -6 & -18 \\ 0 & 0 & 0 \end{bmatrix}, \quad \vec{b} = \begin{pmatrix} 3 \\ 3 \\ -4 \end{pmatrix}$$

① $\vec{b} \in \underbrace{\mathrm{Col}(A)}_{= \text{Span of Columns}}$?

$\Leftrightarrow$ $\vec{b}$ = lin. combi. of Columns

$\Leftrightarrow$ $A\vec{x} = \vec{b}$ is consistent

$$[\ A \mid \vec{b}\ ]$$

$\vec{v} = \begin{pmatrix} -5\lambda \\ -3\lambda \\ \lambda \end{pmatrix} \in \mathrm{Null}(A)$?

$\Leftrightarrow$ $\vec{v}$ is a solution to $A\vec{x} = \vec{0}$

$\Leftrightarrow$ $A\vec{v} = \vec{0}$

$$\Leftrightarrow \begin{bmatrix} 1 & -3 & -4 \\ 0 & -6 & -18 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -5 \\ -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Example 2 (continued)

Using the matrix on the previous slide: is $\vec{v}$ in the null space of A?

$$\vec{v} = \begin{pmatrix} -5\lambda \\ -3\lambda \\ \lambda \end{pmatrix}, \quad \lambda \in \mathbb{R}$$

Basis

① Lin. indep.

② Span H ⟶ ✓

> **Definition**
>
> A **basis** for a subspace H is a set of linearly independent vectors in H that span H.

Example

defined by equation $A\vec{x} = \vec{0}$

The set $H = \{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \in \mathbb{R}^4 \mid x_1 + 2x_2 + x_3 + 5x_4 = 0 \}$ is a subspace.

$A\vec{x} = \vec{0}$

Solution set

a) H is a null space for what matrix A?

$\begin{bmatrix} 1 & 2 & 1 & 5 \end{bmatrix}$

b) Construct a basis for H.

$$\text{Null}(A) = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2x_2 - x_3 - 5x_4 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \right\}$$

$$= \left\{ x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -5 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$= \text{Span} \underbrace{\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}}_{\text{lin. indep.}}$$

basis

9/25/23

Def

H is a subspace of $\mathbb{R}^n$ if $\begin{cases} H \subseteq \mathbb{R}^n \quad (\text{subset}) \\ \vec{u}, \vec{v} \in H \text{ implies } \vec{u}+\vec{v} \in H \\ \vec{u} \in H,\ c \in \mathbb{R} \text{ implies } c \cdot \vec{u} \in H \end{cases}$

Examples

$\{\vec{0}\}$, $\mathbb{R}^n$, $\text{Span}\{\vec{v}_1, \cdots, \vec{v}_p\}$ are subspaces.

$A \in \mathbb{R}^{m \times n}$, $A = [\vec{a}_1, \cdots, \vec{a}_n]$, $\vec{a}_1, \cdots, \vec{a}_n \in \mathbb{R}^m$

$\text{Col}(A) = \text{Span}\{\vec{a}_1, \cdots, \vec{a}_n\} \subseteq \mathbb{R}^m$

$\text{Null}(A) = \{\vec{x} \in \mathbb{R}^n : A\vec{x} = 0\} \subseteq \mathbb{R}^n$

Basis

H is a subspace

$\mathcal{B} \subset H$ is a basis if $\begin{cases} \text{Span}\,\mathcal{B} = H \\ \mathcal{B} \text{ is linearly indep.} \end{cases}$

$\mathcal{B} = \{\vec{v}_1, \cdots, \vec{v}_n\}$

$\text{Span}\,\mathcal{B} = \text{Span}\{\vec{v}_1, \cdots, \vec{v}_n\}$

Example

Construct a basis for Null$A$ and a basis for Col$A$.

free

$$A = \begin{bmatrix} -3 & 6 & -1 & 0 \\ 1 & -2 & 2 & 0 \\ 2 & -4 & 5 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

① $\text{Null}(A) = \{ \vec{x} \in \mathbb{R}^4 : A\vec{x} = \vec{0} \}$

$x_1 - 2x_2 = 0$ $\quad$ x_2, x_4 free

$x_3 = 0$

$$= \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2x_2 \\ x_2 \\ 0 \\ x_4 \end{bmatrix} \right\} = \left\{ x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} : x_2, x_4 \in \mathbb{R} \right\}$$

$$= \text{Span} \{ \underbrace{\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}}_{\text{lin. indep}} \}$$

$\Rightarrow \mathcal{B} = \left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is basis for Null(A).

Example

Construct a basis for Null$A$ and a basis for Col$A$.

$$A = \begin{bmatrix} -3 & 6 & -1 & 0 \\ 1 & -2 & 2 & 0 \\ 2 & -4 & 5 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathrm{Col}(A) = \mathrm{Span}\{ \underbrace{\vec{a_1}, \vec{a_2}, \vec{a_3}, \vec{a_4}}_{\text{Not lin. indep.}} \}$$

$$= \mathrm{Span}\{ \vec{a_1}, \vec{a_2}, \vec{a_3} \}$$

$$= \mathrm{Span}\{ \underbrace{\vec{a_1}, \vec{a_3}}_{\text{lin. indep}} \}$$

$$\mathcal{B} = \{ \vec{a_1}, \vec{a_3} \} = \left\{ \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix} \right\}$$

is a basis for Col (A)

Note $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ is Not.

Additional Example

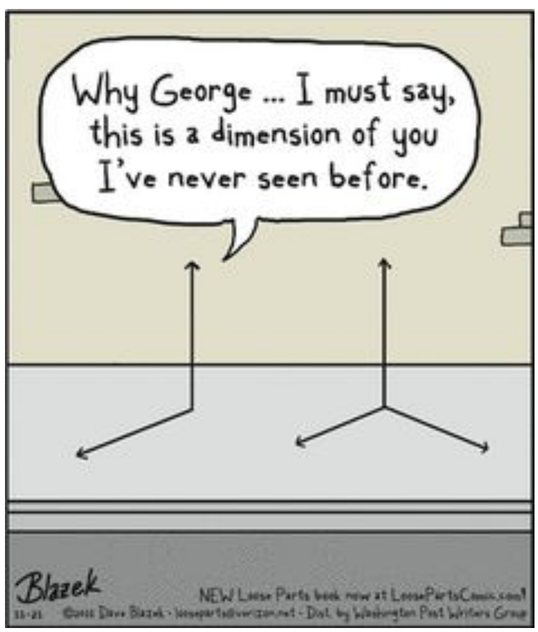
Let $V = \left\{ \begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{R}^2 \mid ab = 0 \right\}$.

1. Give an example of a vector that is in V.
2. Give an example of a vector that is not in V.
3. Is the zero vector in V?
4. Is V a subspace?

Section 2.9 : Dimension and Rank

Chapter 2 : Matrix Algebra

Math 1554 Linear Algebra



Topics and Objectives

Topics

We will cover these topics in this section.

1. Coordinates, relative to a basis.
2. Dimension of a subspace.
3. The Rank of a matrix

Objectives

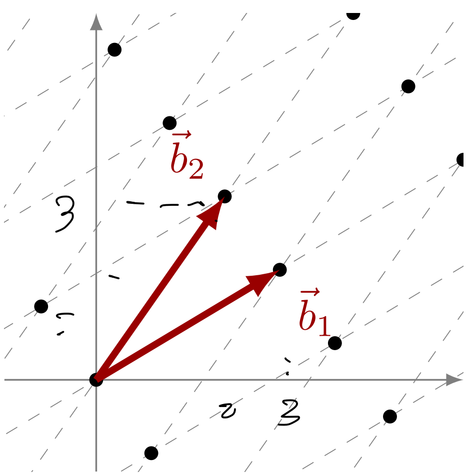
For the topics covered in this section, students are expected to be able to do the following.

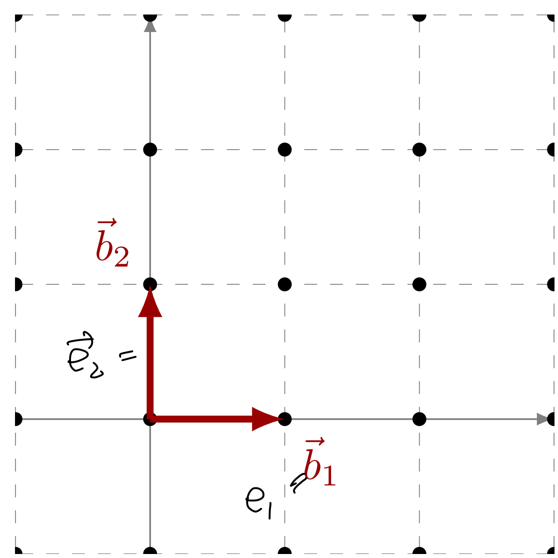
1. Calculate the coordinates of a vector in a given basis.
2. Characterize a subspace using the concept of dimension (or cardinality).
3. Characterize a matrix using the concepts of rank, column space, null space.
4. Apply the Rank, Basis, and Matrix Invertibility theorems to describe matrices and subspaces.

Choice of Basis

Key idea: There are many possible choices of basis for a subspace. Our choice can give us dramatically different properties.

Example: sketch $\vec{b}_1 + \vec{b}_2$ for the two different coordinate systems below.





Ex: $H = \mathbb{R}^2$

$\mathcal{B}_1 = \{ \vec{e_1}, \vec{e_2} \}$

$\mathcal{B}_2 = \left\{ \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right\}$

Coordinates

lin. indep
$\Downarrow$
unique

Every $\vec{v} \in H$ can be written as a linear combination of $\mathcal{B}$

Definition

$\Rightarrow$

Let $\mathcal{B} = \{\vec{b}_1, \ldots, \vec{b}_p\}$ be a basis for a subspace H. If $\vec{x}$ is in H, then **coordinates** of $\vec{x}$ **relative** $\mathcal{B}$ are the weights (scalars) $c_1, \ldots, c_p$ so that

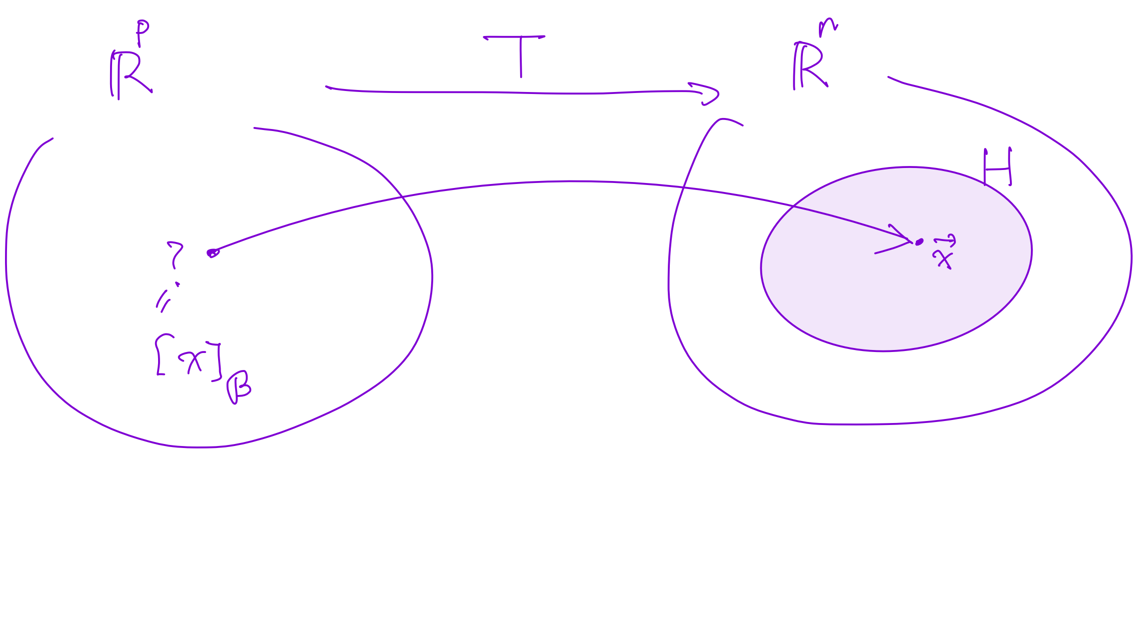
$$\vec{x} = c_1\vec{b}_1 + \cdots + c_p\vec{b}_p \quad = A \cdot [x]_{\mathcal{B}}$$

And

$$[x]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_p \end{bmatrix}$$

is the **coordinate vector of $\vec{x}$ relative to $\mathcal{B}$**, or the $\mathcal{B}$-**coordinate vector of** $\vec{x}$

$$A = [\underbrace{\vec{b}_1}_{\in \mathbb{R}^n}, \ldots, \underbrace{\vec{b}_p}_{\in \mathbb{R}^n}] \in \mathbb{R}^{n \times p}$$



Example 1

Let $\vec{v}_1 = \begin{bmatrix}1\\0\\1\end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix}1\\1\\1\end{bmatrix}$, and $\vec{x} = \begin{bmatrix}5\\3\\5\end{bmatrix}$. Verify that $\vec{x}$ is in the span of $\mathcal{B} = \{\vec{v}_1, \vec{v}_2\}$, and calculate $[\vec{x}]_{\mathcal{B}}$.

$\vec{x} = c_1 \vec{v}_1 + c_2 \vec{v}_2$?

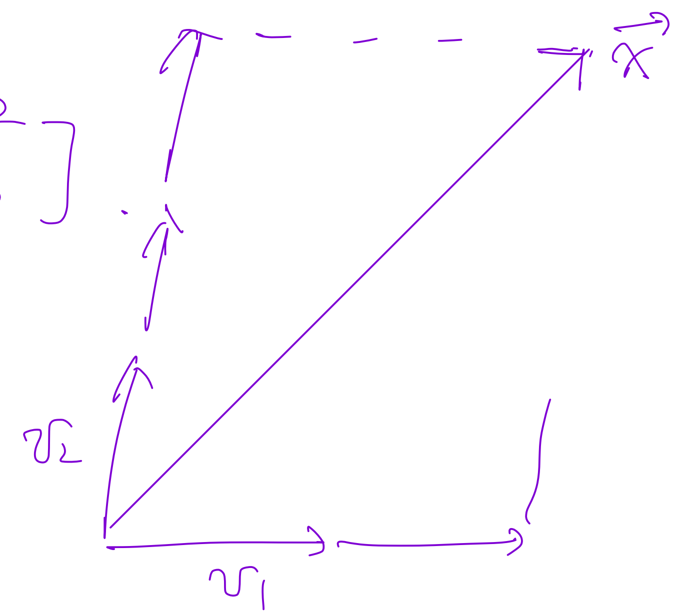
$\left[\begin{array}{cc|c} \vec{v}_1 & \vec{v}_2 & \vec{x} \end{array}\right]$ is consistent ?

$$\left[\begin{array}{cc|c} 1 & 1 & 5 \\ 0 & 1 & 3 \\ 1 & 1 & 5 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 1 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right]$$

Consistent

$c_1 = 2 \qquad c_2 = 3$

$[x]_{\mathcal{B}} = \begin{bmatrix}2\\3\end{bmatrix}$.



Dimension

Definition

The **dimension** (or cardinality) of a non-zero subspace H, $\dim H$, is the number of vectors in a basis of H. We define $\dim\{0\} = 0$.

Theorem

Any two choices of bases $\mathcal{B}_1$ and $\mathcal{B}_2$ of a non-zero subspace H have the same dimension.

Examples:

1. $\dim \mathbb{R}^n =$ n $\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$ basis for $\mathbb{R}^n$
2. $H \overset{= \text{Null}(A)}{=} \{(x_1, \ldots, x_n) \ : \ x_1 + \cdots + x_n = 0\}$ has dimension $n-1$
3. $\dim(\text{Null}\, A)$ is the number of free variable
4. $\dim(\text{Col}\, A)$ is the number of pivots

#2. $\boxed{x_1 + \cdots + x_n} = 0$ $\quad \underbrace{[1 \ \cdots \ 1]}_{A} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = 0$

$\dim(\text{Null}(A))$ = # of free variables.

$= n-1$.

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ \vdots \end{bmatrix}, \ldots, \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \right\}$$

Rank

Definition

The **rank** of a matrix A is the dimension of its column space.

Example 2: Compute rank(A) and dim(Nul(A)).

$$\begin{bmatrix} 2 & 5 & -3 & -4 & 8 \\ 4 & 7 & -4 & -3 & 9 \\ 6 & 9 & -5 & 2 & 4 \\ 0 & -9 & 6 & 5 & -6 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} 2 & 5 & -3 & -4 & 8 \\ 0 & -3 & 2 & 5 & -7 \\ 0 & 0 & 0 & 4 & -6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\underbrace{\dim(\mathrm{Col}(A))}_{\text{Span}\{\text{Columns in } A\}} = \text{rank}(A) = \#\text{ of pivot columns}$

$\dim(\underbrace{\mathrm{Null}(A)}) = \#\text{ of free var.} = \#\text{ of non-pivot columns}$

To find a basis, parametric vector form

Rank, Basis, and Invertibility Theorems

Theorem (Rank Theorem)

\# of pivots

$\dim(\mathrm{Col}(A))$

\# of free var.

If a matrix A has n columns, then $\text{Rank}\, A + \dim(\text{Nul}\, A) = n$.

$= \#$ of columns

Theorem (Basis Theorem)

Any two bases for a subspace have the same dimension.

Theorem (Invertibility Theorem)

IMT

Let A be a $n \times n$ matrix. These conditions are equivalent.

1. *A is invertible.*
2. *The columns of A are a basis for $\mathbb{R}^n$.*
3. $\text{Col}\, A = \mathbb{R}^n$. = Range of T $\Leftrightarrow$ T is onto.
4. $\text{rank}\, A = \dim(\text{Col}\, A) = n$. ← full rank.
5. $\text{Null}\, A = \{\vec{0}\}$.

$\Leftrightarrow$ Columns in A lin. indep.

$\Leftrightarrow$ T 1-1

# of pivots $\leq \min\{\#$ of Col, # of rows$\}$ $= 2$

$$A = \begin{bmatrix} * & * & * \\ * & * & * \end{bmatrix}$$

Examples

If possible give an example of a 2×3 matrix A, that is in RREF and has the given properties.

a) rank$(A) = 3$ Not possible.

b) rank$(A) = 2$ $\Downarrow$ 2 pivots

$$\begin{bmatrix} 1 & 0 & * \\ 0 & 1 & * \end{bmatrix}$$

any numbers.

c) dim(Null(A)) $= 2$ $\Downarrow$ 2 non pivot & 1 pivot

$$\begin{bmatrix} 1 & \boxed{*} & \boxed{*} \\ 0 & 0 & 0 \end{bmatrix}$$

anything

d) Null$A = \{0\}$ Not Possible.

$\Leftrightarrow$ T is 1-1

because dim (Null (A)) ≥ 1