Section 3.1 : Introduction to Determinants

Chapter 3 : Determinants

Math 1554 Linear Algebra

Topics and Objectives

Topics

We will cover these topics in this section.

1. The definition and computation of a determinant
2. The determinant of triangular matrices

Objectives

For the topics covered in this section, students are expected to be able to do the following.

1. Compute determinants of $n \times n$ matrices using a cofactor expansion.
2. Apply theorems to compute determinants of matrices that have particular structures.

A Definition of the Determinant

Only for Square Matrices

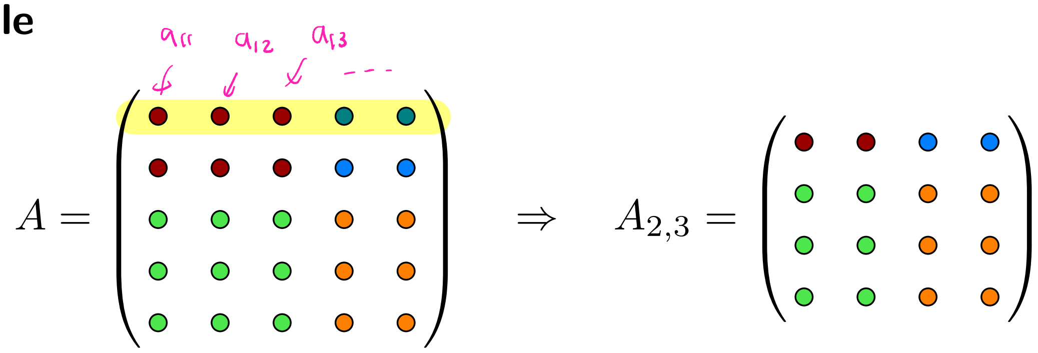
Suppose A is $n \times n$ and has elements a_{ij}.

1. If $n = 1$, $A = [a_{11}]$, and has determinant $\det A = a_{11}$.
2. Inductive case: for $n > 1$, n×n det using (n−1)×(n−1) determinant

$$\underset{n\times n}{\det A} = a_{11} \det \underset{(n-1)\times(n-1)}{\underline{A_{11}}} - a_{12} \det \underset{(n-1)\times(n-1)}{\underline{A_{12}}} + \cdots + (-1)^{1+n} a_{1n} \det \underline{A_{1n}}$$

where A_{ij} is the submatrix obtained by eliminating row i and column j of A.

Example



Example 1

Compute $\det \begin{bmatrix} \overset{\swarrow}{a} & \overset{\swarrow}{b} \\ c & d \end{bmatrix}$.

$$\overset{\text{def}}{=} a_{11} \cdot \det A_{11} - a_{12} \det A_{12}$$

$$= a \cdot \det [d] - b \cdot \det [c]$$

$$= ad - bc .$$

Recall

- $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{\underbrace{ad - bc}_{\overset{\shortparallel}{\det(A)}}} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

- $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ invertible $\iff ad - bc \neq 0$

  $\iff \det(A) \neq 0$.

Example 2

Compute $\det \begin{bmatrix} 1 & -5 & 0 \\ 2 & 4 & -1 \\ 0 & 2 & 0 \end{bmatrix} = \begin{vmatrix} 1 & -5 & 0 \\ 2 & 4 & -1 \\ 0 & 2 & 0 \end{vmatrix}$.

$$= a_{11} \det A_{11} - a_{12} \det A_{12} + a_{13} \det A_{13}$$

$$= 1 \cdot \det \begin{bmatrix} 4 & -1 \\ 2 & 0 \end{bmatrix} - (-5) \det \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix} + 0 \cdot \det \begin{bmatrix} 2 & 4 \\ 0 & 2 \end{bmatrix}.$$

$$= 1 \cdot (4 \cdot 0 - (-1) \cdot 2) - (-5) \cdot (2 \cdot 0 - (-1) \cdot 0) + 0 \cdot (2 \cdot 2 - 4 \cdot 0)$$

$$= 2.$$

$$\det A = a_{11}\underbrace{(-1)^{1+1}\det A_{11}}_{C_{11}} + a_{12}\underbrace{(-1)^{1+2}\det A_{12}}_{C_{12}} + a_{13}\underbrace{(-1)^{1+3}\det A_{13}}_{C_{13}} + \cdots + a_{1n}\underbrace{(-1)^{1+n}\det A_{1n}}_{C_{1n}}$$

Cofactors

$$= a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}$$

Cofactor Expansion along 1^{st} row.

Cofactors

Cofactors give us a more convenient notation for determinants.

Definition: Cofactor

The (i, j) cofactor of an $n \times n$ matrix A is

$$C_{ij} = (-1)^{i+j}\det A_{ij}$$

$(n-1)\times(n-1)$ matrix removing i^{th} row and j^{th} col. from A.

The pattern for the negative signs is

$$\begin{pmatrix} + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}$$

$\det(A) \overset{def}{=}$ Cofactor Expansion along 1^{st} row

$= a_{11}C_{11} + a_{12}C_{12} + \dots + a_{1n}C_{1n}$

$\overset{Thm}{=}$ Cofactor Expansion along i^{th} row

$=$ " j^{th} column.

Theorem

The determinant of a matrix A can be computed down any row or column of the matrix. For instance, down the j^{th} column, the determinant is

$$\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}.$$

This gives us a way to calculate determinants more efficiently.

Example

$\det(A) = a_{31}C_{31} + a_{32}C_{32} + \dots + a_{3n}C_{3n}$

along 3^{rd} row

$= a_{14}C_{14} + a_{24}C_{24} + \dots + a_{n4}C_{n4}$

along 4^{th} col.

Example 3

Compute the determinant of $\begin{bmatrix} 5 & 4 & 3 & 2 \\ 0 & 1 & 2 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & 3 \end{bmatrix}$. $= A$

$$\det(A) = \overset{a_{11}}{\overset{\shortparallel}{5}} \cdot (-1)^{1+1} \cdot \overset{C_{11}}{\det} \begin{bmatrix} 1 & 2 & 0 \\ -1 & 1 & 0 \\ 1 & 1 & 3 \end{bmatrix} \overset{0}{\overset{\shortparallel}{\left(+ a_{21} C_{21} + a_{31} C_{31} + a_{41} C_{41} \right)}}$$

$$= 5 \cdot \left(\overset{a_{33}}{\overset{\shortparallel}{3}} \cdot (-1)^{3+3} \det \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} \right)$$

$$= 5 \cdot 3 \cdot \left(1 \cdot 1 - 2(-1) \right) = 45 .$$

Example

$$\det\begin{bmatrix} 1 & & * & \\ 0 & 2 & & \\ 0 & 0 & 3 & \\ 0 & 0 & 0 & 4 \end{bmatrix} = \overset{a_{11}}{\overset{\|}{1}}\cdot(-1)^{1+1}\cdot\det\begin{bmatrix} 2 & (*) & \\ 0 & 3 & \\ 0 & 0 & 4 \end{bmatrix}$$

$$= 1\cdot 2\cdot(-1)^{1+1}\det\begin{bmatrix} 3 & * \\ 0 & 4 \end{bmatrix}$$

$$= 1\cdot 2\cdot 3\cdot 4 .$$

Triangular Matrices

Theorem

If A is a triangular matrix then

$$\det A = a_{11}a_{22}a_{33}\cdots a_{nn}.$$

Example 4

Compute the determinant of the matrix. Empty elements are zero.

$$\det\begin{bmatrix} 2 & 1 & & & & & \\ & 2 & 1 & & & \bigcirc & \\ & & 2 & 1 & & & \\ & & & 2 & 1 & & \\ & \bigcirc & & & 2 & 1 & \\ & & & & & 2 & 1 \\ & & & & & & 2 \end{bmatrix} = 2^7$$

$$A \xrightarrow{\qquad\qquad} REF$$

Row Operations

Q : A : RREF n×n .

$$\det(A) = 1 \text{ or } 0 .$$

In general A is 10×10

det A = Sum of 10 det. of 9×9

= $10 \cdot 9$ many 8×8 det.

$\vdots$

= $10 \cdot 9 \cdot 8 \cdot \ldots \cdot 1$ of 1×1 matrices

Computational Efficiency

Note that computation of a co-factor expansion for an $N \times N$ matrix requires roughly $N!$ multiplications.

- A 10×10 matrix requires roughly $10! = 3.6$ million multiplications
- A 20×20 matrix requires $20! \approx 2.4 \times 10^{18}$ multiplications

Co-factor expansions may not be practical, but determinants are still useful.

- We will explore other methods for computing determinants that are more efficient.
- Determinants are very useful in multivariable calculus for solving certain integration problems.

Section 3.2 : Properties of the Determinant

Chapter 3 : Determinants

Math 1554 Linear Algebra

“A problem isn’t finished just because you’ve found the right answer.”
- Yōko Ogawa

We have a method for computing determinants, but without some of the strategies we explore in this section, the algorithm can be very inefficient.

Topics and Objectives

Topics

We will cover these topics in this section.

- The relationships between row reductions, the invertibility of a matrix, and determinants.

Objectives

For the topics covered in this section, students are expected to be able to do the following.

1. Apply properties of determinants (related to row reductions, transpose, and matrix products) to compute determinants.
2. Use determinants to determine whether a square matrix is invertible.

Swap	$R_1 \leftrightarrow R_2$	Sign Change
Replacement	$R_3 \to R_3 - 2R_2$	Doesn't change
Scaling	$R_7 \to 5 \cdot R_7$	$\det \to 5 \cdot \det$

Row Operations

- We saw how determinants are difficult or impossible to compute with a cofactor expansion for large N.
- Row operations give us a more efficient way to compute determinants.

Theorem: Row Operations and the Determinant

Let A be a square matrix.

1. If a multiple of a row of A is added to another row to produce B, then $\det B = \det A$.
2. If two rows are interchanged to produce B, then $\det B = -\det A$.
3. If one row of A is multiplied by a scalar k to produce B, then $\det B = k \det A$.

$$\begin{bmatrix} - & a & - \\ - & b & - \\ - & c & - \end{bmatrix} \longrightarrow \begin{bmatrix} - & b & - \\ - & c & - \\ - & a & - \end{bmatrix}$$

$\det = 5$ $\qquad\qquad$ $\det = 5$

Example 1 Compute $\begin{vmatrix} 1 & -4 & 2 \\ -2 & 8 & -9 \\ -1 & 7 & 0 \end{vmatrix} = 15$

$$\begin{bmatrix} 1 & -4 & 2 \\ -2 & 8 & -9 \\ -1 & 7 & 0 \end{bmatrix} \xrightarrow[\substack{R_2 \to R_2 + 2R_1 \\ R_3 \to R_3 + R_1}]{} \begin{bmatrix} 1 & -4 & 2 \\ 0 & 0 & -5 \\ 0 & 3 & 2 \end{bmatrix}$$

$$\xrightarrow[R_2 \Leftrightarrow R_3]{} \underbrace{\begin{bmatrix} 1 & -4 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & -5 \end{bmatrix}}_{\det = 1 \cdot 3 \cdot (-5) = -15}$$

$$A \longrightarrow \cdots \longrightarrow \text{Triangula Mat.}$$

$\det(A) = 0$

0 on diagonal

$\det = 0$

when? have non-pivot column.

Pivot in Every Column $\Longleftrightarrow$ $\det(A) \neq 0$

$\Longleftrightarrow$ A is invertible.

Invertibility

Important practical implication: If A is reduced to echelon form, by r interchanges of rows and columns, then

$$|A| = \begin{cases} (-1)^r \times (\text{product of pivots}), & \text{when } A \text{ is invertible} \\ 0, & \text{when } A \text{ is singular.} \end{cases}$$

Recall

remove ith row, jth col from A

$A \in \mathbb{R}^{n \times n}$, $C_{ij} = (-1)^{i+j} \det A_{ij}$

- $\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$

  $= a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$

  Cofactor Exansion

- For a triangular $A \in \mathbb{R}^{n \times n}$

  $\det(A) = \prod_{i=1}^{n} a_{ii} = a_{11} \cdot a_{22} \cdots a_{nn}$

- $A \underbrace{\longrightarrow \cdots \longrightarrow}_{\text{row operations}}$ REF (upper triangular)

  swap flips the sign
  replacement doesn't change det
  scalar multiple on a row = scalar multiple on det.

Example 2 Compute the determinant

$$\det(A) = \begin{vmatrix} 0 & 1 & 2 & -1 \\ 2 & 5 & -7 & 3 \\ 0 & 3 & 6 & 2 \\ -2 & -5 & 4 & 2 \end{vmatrix}$$

no change

$R_4 \to R_2 + R_4$

$$\boxed{=} \begin{vmatrix} 0 & 1 & 2 & -1 \\ \boxed{2} & 5 & -7 & 3 \\ 0 & 3 & 6 & 2 \\ 0 & 0 & -3 & 5 \end{vmatrix}$$

Cofactor expansion along 1^{st} col.

$$= \underset{a_{21}}{\underset{\parallel}{2}} \cdot (-1)^{2+1} \det \begin{bmatrix} 1 & 2 & -1 \\ 3 & 6 & 2 \\ 0 & -3 & 5 \end{bmatrix}$$

$R_2 \to R_2 - 3R_1$

$$\boxed{=} -2 \cdot \begin{vmatrix} 1 & 2 & -1 \\ 0 & 0 & 5 \\ 0 & -3 & 5 \end{vmatrix}$$

flip sign

$R_2 \leftrightarrow R_3$

$$= 2 \begin{vmatrix} 1 & 2 & -1 \\ 0 & -3 & 5 \\ 0 & 0 & 5 \end{vmatrix} = 2 \cdot \underbrace{1 \cdot (-3) \cdot 5}_{\text{product of diagonals}} .$$

triangular

Note Cofactor factor exp. works for both rows, columns

$\Rightarrow$ Can do row / column operation

$$\det\begin{bmatrix} | & | & | \\ v_1 & v_2 & v_3 \\ | & | & | \end{bmatrix} = -\det\begin{bmatrix} | & | & | \\ v_2 & v_1 & v_3 \\ | & | & | \end{bmatrix}$$

$$= \det\begin{bmatrix} | & | & | \\ v_1 & v_1+v_2 & v_3 \\ | & | & | \end{bmatrix}$$

Properties of the Determinant

For any square matrices A and B, we can show the following.

1. $\det A = \det A^T$.
2. A is invertible if and only if $\det A \neq 0$.
3. $\det(AB) = \det A \cdot \det B$. ✓

Note

- $\underbrace{\det(A^T \cdot A)}_{=} \geq 0$ for $A \in \mathbb{R}^{n\times n}$

$$= \det(A^T) \cdot \underline{\det(A)}$$

$$= (\det(A))^2$$

$$= \det(A^2)$$

- In general, $AB \neq BA$

but, $\det(AB) = \det(BA)$

- A invertible, B invertible $\Leftrightarrow$ AB invertible

- AB invertible $\Leftrightarrow$ BA invertible

Additional Example (if time permits)

Use a determinant to find all values of λ such that matrix C is not invertible.

$$C = \overset{=A}{\begin{pmatrix} 5 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}} - \lambda I_3 = \begin{bmatrix} 5-\lambda & 0 & 0 \\ 0 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{bmatrix}$$

C is Not Invertible

$$\Longleftrightarrow \quad \det C = 0 \quad \Longleftrightarrow \quad \overset{(A-\lambda I)}{\overset{\|}{C}}\vec{x} = \vec{0} \text{ has nontrivial solutions}$$

$$\det C = \underset{a_{11}}{\underset{\|}{(5-\lambda)}} (-1)^{1+1} \cdot \det \begin{bmatrix} -\lambda & 1 \\ 1 & -\lambda \end{bmatrix}$$

$$= (5-\lambda)\left((-\lambda)^2 - 1^2\right)$$

$$= (5-\lambda)(\lambda+1)(\lambda-1) = 0$$

$$\Rightarrow \quad \lambda = 5 \text{ or } 1 \text{ or } -1$$

$$\Rightarrow \quad A\vec{x} = \lambda\vec{x} \text{ has nontrivial solutions when } \underbrace{\lambda = 5, 1, -1}_{\text{eigenvalue}}$$

$$\Rightarrow \quad \underbrace{\vec{x}_1 \neq 0, \quad \vec{x}_2 \neq 0, \quad \vec{x}_3 \neq 0}_{\text{eigenvectors}} \text{ such that}$$

$$A\vec{x}_1 = 5\vec{x}_1$$

$$A\vec{x}_2 = \vec{x}_2$$

$$A\vec{x}_3 = -\vec{x}_3$$

Additional Example (if time permits)

Determine the value of

$$\det A = \det\left(\begin{pmatrix}0 & 2 & 0\\ 1 & 1 & 2\\ 1 & 1 & 3\end{pmatrix}^8\right).$$

$$= \left(\det \underbrace{\begin{pmatrix}0 & 2 & 0\\ 1 & 1 & 2\\ 1 & 1 & 3\end{pmatrix}}_{}\right)^8$$

$$= \left(2 \cdot (-1)^{1+2} \cdot \underbrace{\det\begin{pmatrix}1 & 2\\ 1 & 3\end{pmatrix}}_{1\cdot 3 - 1\cdot 2 = 1}\right)^8$$

$$= 2^8.$$

Section 3.3 : Volume, Linear Transformations

Chapter 3 : Determinants

Math 1554 Linear Algebra

Topics and Objectives

"Geometric Meaning of Determinant".

Topics

We will cover these topics in this section.

1. Relationships between area, volume, determinants, and linear transformations.

Objectives

For the topics covered in this section, students are expected to be able to do the following.

1. Use determinants to compute the area of a parallelogram, or the volume of a parallelepiped, possibly under a given linear transformation.

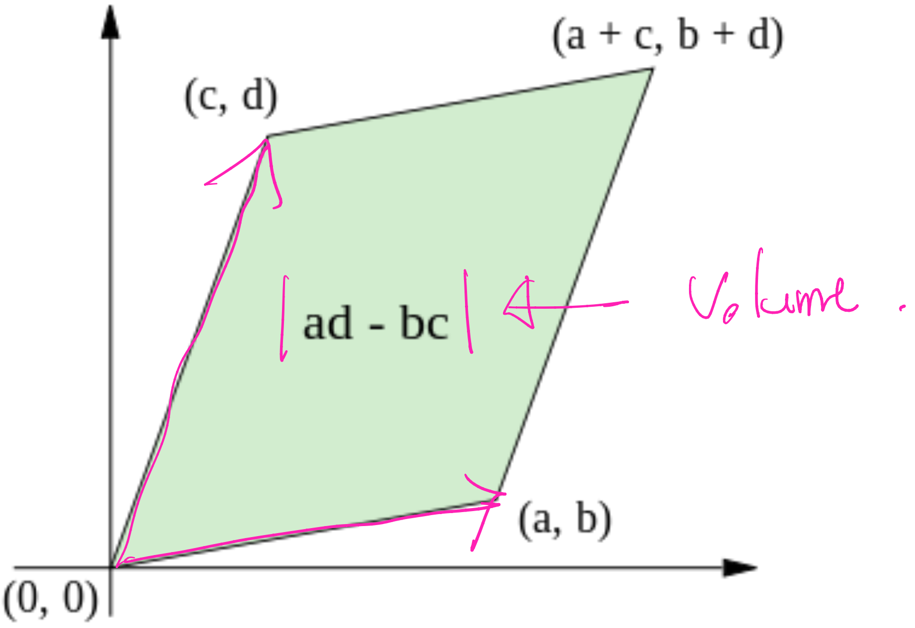
Students are not expected to be familiar with Cramer's rule.

"Matrix as a collection of n column vectors".

"Signed Area of Parallelogram" $n = 2$ Case.

Determinants, Area and Volume

In $\mathbb{R}^2$, determinants give us the area of a parallelogram.

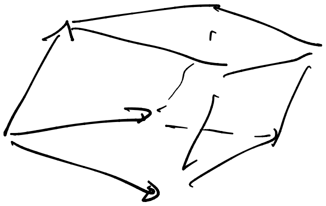


$$\text{area of parallelogram} = \det\begin{pmatrix} a & c \\ b & d \end{pmatrix} = ad - bc.$$

Det is a function from

$\{v_1, \cdots, v_n\} \longrightarrow$ Numbers.

$A \in \mathbb{R}^{n\times n}$ = a collection of vectors (columns)

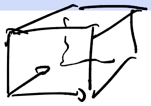


a parallopiped spanned by vectors

$|\det(A)| = \text{Vol}(\checkmark)$

Determinants as Area, or Volume

I_n = collection of standard vector



Theorem

The volume of the parallelpiped spanned by the columns of an $n \times n$ matrix A is $|\det A|$.

Key Geometric Fact (which works in any dimension). The area of the parallelogram spanned by two vectors $\vec{a}, \vec{b}$ is equal to the area spanned by $\vec{a}, c\vec{a} + \vec{b}$, for any scalar c.

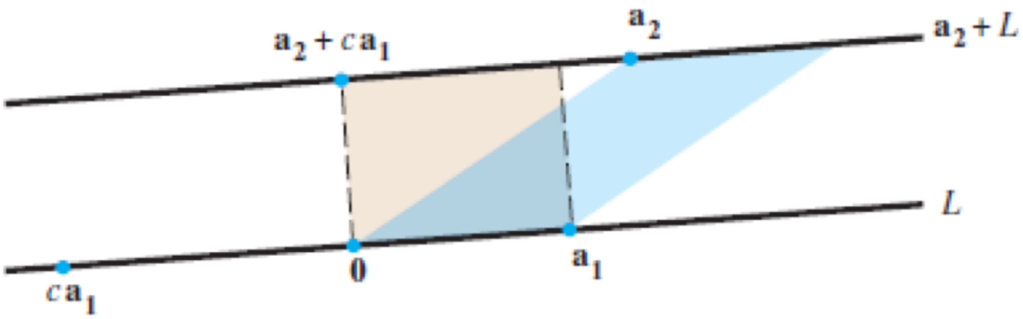
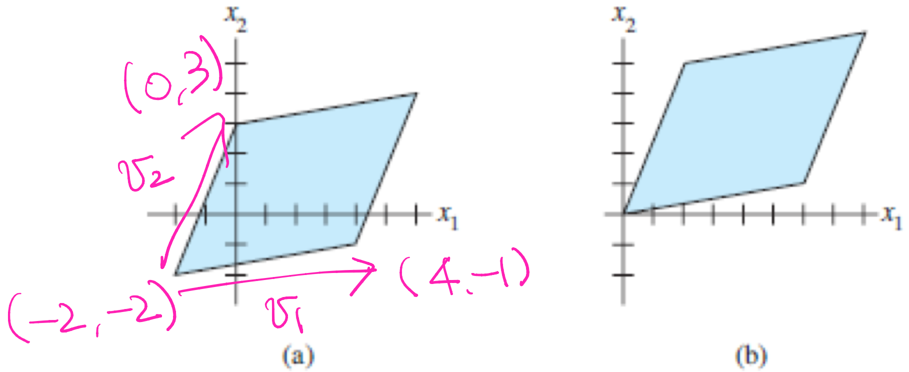


FIGURE 2 Two parallelograms of equal area.

Example 1

Calculate the area of the parallelogram determined by the points $(-2,-2),(0,3),(4,-1),(6,4)$



(a) (b)

FIGURE 5 Translating a parallelogram does not change its area.

$$v_1 = \begin{bmatrix} 4 \\ -1 \end{bmatrix} - \begin{bmatrix} -2 \\ -2 \end{bmatrix} = \begin{bmatrix} 6 \\ 1 \end{bmatrix}$$

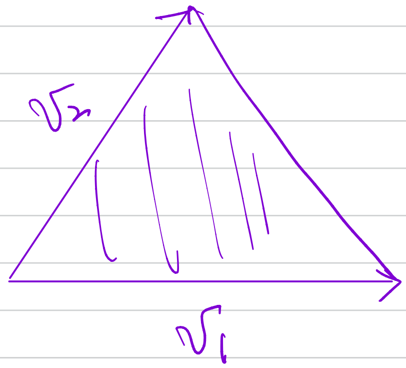
$$v_2 = \begin{bmatrix} 0 \\ 3 \end{bmatrix} - \begin{bmatrix} -2 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

$$\text{Area} = \left| \det \begin{bmatrix} v_1 & v_2 \end{bmatrix} \right|$$

$$= \left| \det \begin{bmatrix} 6 & 2 \\ 1 & 5 \end{bmatrix} \right| = \left| 6 \cdot 5 - 2 \cdot 1 \right|$$

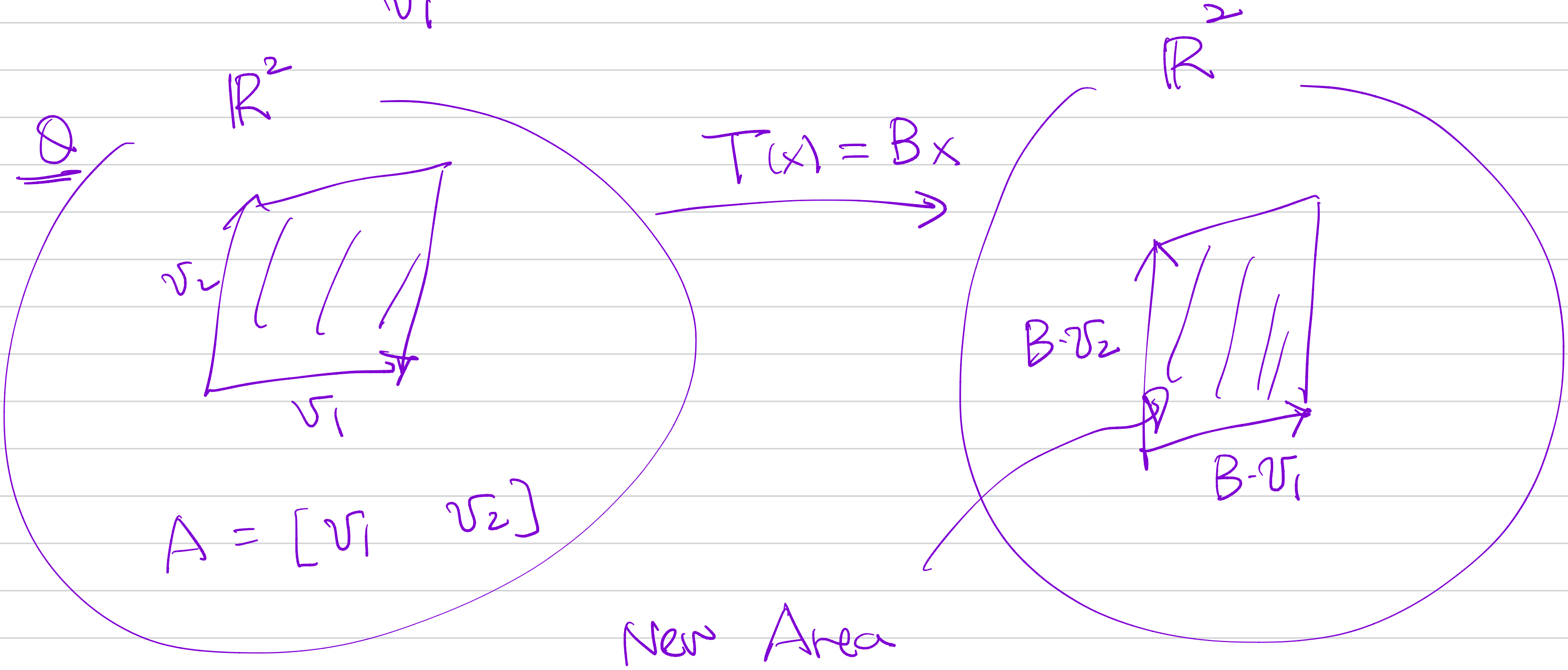
$$= 28.$$

Q



$$\text{Area} = \frac{1}{2}\left|\det[v_1, v_2]\right|$$

Q



$A = [v_1 \quad v_2]$

New Area

$$= \left|\ \det \underbrace{[Bv_1, Bv_2]}\right|$$

$$= \left|\det(B\cdot A)\right|$$

$$= \underbrace{\left|\det(B)\right|}\cdot\underbrace{\left|\det(A)\right|}_{\text{old}}$$

Linear Transformations

Theorem

If $T_A \ : \mathbb{R}^n \mapsto \mathbb{R}^n$, and S is some parallelogram in $\mathbb{R}^n$, then

$$\text{volume}\,(T_A(S)) = |\det(A)| \cdot \text{volume}(S)$$

An example that applies this theorem is given in this week's worksheets.

Example

$$\int_0^1 (\underbrace{2x}_{\overset{\|}{u}})^{100}\,dx = \int_0^2 u^{100}\ \overset{\det(A)}{\boxed{\underline{2}}}\ du$$

$$\boxed{A=2}$$

$$\begin{tikzcd} x \arrow[r, mapsto] & 2x \\ \mathbb{R}^1 & \mathbb{R}^1 \end{tikzcd}$$

In general

$$|\det(\ )| = \phi$$

Jacobian

Q: Find k so that A is singular.

$$A = \begin{pmatrix} 1 & -3 & k \\ 7 & 2 & -3 \\ -1 & 2 & 5 \end{pmatrix}$$

① Row Reduce → whether free var.

② $\underline{\det(A) = 0}$ Solve for k.

Cofactor Exp.

Row Operation.

$$\det(A) = \underset{a_{11}}{\underset{\shortparallel}{1}}\,(-1)^{1+1}\det\begin{pmatrix} 2 & -3 \\ 2 & 5 \end{pmatrix} + (-3)\,(-1)^{1+2}\det\begin{pmatrix} 7 & -3 \\ -1 & 5 \end{pmatrix}$$

$$+\, k\cdot(-1)^{1+3}\det\begin{pmatrix} 7 & 2 \\ -1 & 2 \end{pmatrix}$$

$$= (10 - (-6)) + 3\cdot(35 - 3) + k\,(14 - (-2))$$

$$= 16 + 96 + 16k = 0$$

$$1 + 6 + k = 0 \qquad \underline{k = -7}_{//}.$$

Suppose $\det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = 4$. Find the determinant of the matrices below.

$$A = \begin{pmatrix} g & h & i \\ a & b & c \\ d & e & f \end{pmatrix} \qquad B = \begin{pmatrix} a & b & c \\ 2d \cancel{+a} & 2e \cancel{+b} & 2f \cancel{+c} \\ g & h & i \end{pmatrix} \qquad C = \begin{pmatrix} a & a \cancel{-c} & c \\ d & d \cancel{-f} & f \\ g & g \cancel{-i} & i \end{pmatrix}$$

$$\det(A) = \boxed{4} \qquad \det(B) = \boxed{2 \cdot 4} = 8 \qquad \det(C) = \boxed{0}$$

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \longrightarrow \begin{pmatrix} d & e & f \\ a & b & c \\ g & h & i \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} g & h & i \\ a & b & c \\ d & e & f \end{pmatrix}$$