[1] $X_1, X_2, \cdots, X_7$: i.i.d. Poisson with $\lambda = 2$.

$W = \sum_1^7 X_i$.

$$M_X(t) = E[e^{tX}] = \sum_{k=0}^{\infty} e^{tk}\cdot e^{-2}\cdot \frac{2^k}{k!} = e^{-2}\sum_{k=0}^{\infty}\frac{(2\cdot e^t)^k}{k!}$$

$$= e^{-2}\cdot e^{2\cdot e^t} = e^{2\cdot(e^t-1)}$$

$$M_W(t) = (M_X(t))^7 = \left(e^{2(e^t-1)}\right)^7 = e^{14(e^t-1)}.$$

$W \sim$ Poisson (14).

[2] $X \sim N(1,4)$, $Y \sim N(2,5)$ indep. $W = X+Y$

$$M_X(t) = \exp\left(\mu t + \frac{\sigma^2}{2}t^2\right) = \int_{\mathbb{R}} e^{tx}\cdot\frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{|x-\mu|^2}{2\sigma^2}}\,dx$$

$$-\frac{|x-\mu|^2}{2\sigma^2} + tx = -\frac{1}{2\sigma^2}\left(x^2 - 2\mu x + \mu^2 - 2t\cdot\sigma^2 x\right)$$

$$= -\frac{1}{2\sigma^2}\left(|x-(\mu+t\sigma^2)|^2 + \mu^2 - (\mu+t\sigma^2)^2\right)$$

$$= -\frac{1}{2\sigma^2}|x-(\quad)|^2 - \frac{1}{2\sigma^2}\left(-2t\mu\sigma^2 - t^2\sigma^4\right)$$

$$M_W(t) = M_X(t)\cdot M_Y(t) = \exp\left(1\cdot t + \frac{4}{2}t^2\right)\cdot\exp\left(2\cdot t + \frac{5}{2}t^2\right)$$

$$= \exp\left((2+1)t + \frac{(4+5)}{2}t^2\right)$$

$W \sim N(2+1, 4+5)$

<u>CLT</u> $X_1, X_2, \cdots, X_n$: i.i.d. $E[X_1] = \mu$, $Var(X_1) = \sigma^2 < \infty$.

$\bar{X} = \frac{1}{n}(X_1 + \cdots + X_n)$ $\quad W_n = \dfrac{\bar{X}-\mu}{\sigma/\sqrt{n}} \Rightarrow N(0,1)$ as $n \to \infty$

$$\left(P(W_n \le x) \longrightarrow P(Z \le x)\right)$$

<u>LLN</u> $\bar{X} \to \mu$ in probability

(For any $\varepsilon > 0$, $\mathbb{P}(|\bar{X}-\mu| \ge \varepsilon) \to 0$ as $n \to \infty$)

⌜. $$P(|\bar{X}-\mu| \ge \varepsilon) \le \frac{Var(\bar{X})}{\varepsilon^2} = \frac{\sigma^2/n}{\varepsilon^2} = \frac{\sigma^2}{\varepsilon^2\cdot n} \to 0$$ ⌟

$\boxed{4}$ $X = \#$ of 6 appears $\sim \text{Bin}\left(720, \frac{1}{6}\right)$

$$P(135 \le X \le 150) = \sum_{k=135}^{150} \binom{720}{k} \left(\frac{1}{6}\right)^k \left(\frac{5}{6}\right)^{720-k}$$

$$P(135 \le X \le 150) = P\left(\frac{135-120}{\sqrt{100}} \le \frac{X-np}{\sqrt{np(1-p)}} \le \frac{150-120}{\sqrt{100}}\right)$$

(takes integer values $P(X=135) \neq 0$)

$$\simeq P(1.5 \le Z \le 3) = \Phi(3) - \Phi(1.5).$$

$$\frac{X-np}{\sqrt{np(1-p)}} \Rightarrow N(0,1), \quad np = 720 \cdot \frac{1}{6} = 120 \quad np(1-p) = 100$$

($\because$ $X = X_1 + \cdots + X_{720}$, $\underline{X_i \sim \text{Ber}\left(\frac{1}{6}\right)}$ $\mu = \frac{1}{6} = p$, $\sigma^2 = \frac{1}{6} \cdot \frac{5}{6} = p(1-p)$

$\frac{X}{n}$: sample mean, $\frac{\frac{X}{n} - p}{\sqrt{p(1-p)}/\sqrt{n}} \Rightarrow N(0,1)$

$\frac{X-np}{\sqrt{np(1-p)}} =$)

$$P(135 \le X \le 150) = P(134.5 \le X \le 150.5)$$

$\| $

$$\sum_{k=135}^{150} P(X=k) \qquad = P\left(\frac{134.5-120}{\sqrt{100}} \le \frac{X-np}{\sqrt{np(1-p)}} \le \frac{150.5-120}{\sqrt{100}}\right)$$

$$\simeq P(1.45 \le Z \le 3.05)$$

$$= \Phi(3.05) - \Phi(1.45)$$

$\boxed{6}$ $\overline{X}$ $\mu = \underset{\sim}{80}$, $\sigma^2 = 60$, $n = 15$

$$P(75 < \overline{X} < 85) = P(|\overline{X} - \mu| < 5)$$

$$= 1 - P(|\overline{X} - \mu| \ge \underline{5})$$

$$\ge 1 - \frac{\text{Var}(\overline{X})}{5^2} = 1 - \frac{\sigma^2/n^{=4}}{5^2} = \frac{21}{25}$$

$$\text{Var}(\overline{X}) = \underline{\text{Var}}\left(\frac{1}{n}(X_1 + \cdots + X_n)\right)$$

$$= \frac{1}{n^2}\text{Var}(X_1 + \cdots + X_n) = \frac{1}{n^2} \cdot (\overbrace{\text{Var}(X_1) + \cdots + \text{Var}(X_n)}^{n}) = \frac{1}{n}\text{Var}(X_1)$$

$$= \frac{\sigma^2}{n}.$$

$\boxed{7}$ (a) $f_{W_1}(t) = 10 \cdot (1-t)^9$, $f_{W_{10}}(t) = 10 \cdot t^9$

(b) $\mathbb{E}[W_1] = \frac{1}{11}$, $\mathbb{E}\, W_{10} = \frac{10}{11}$.

$\boxed{8}$ $f_{Y_3} = r \cdot \binom{n}{r} \; F(t)^{r-1} \, (1-F(t))^{n-r} \cdot f(t)$

$= 3 \cdot \binom{5}{3} \cdot (1-e^{-t})^2 \, (e^{-t})^2 \cdot e^{-t}$

$= 30 \cdot (1-e^{-t})^2 \cdot e^{-3t}$

$U = e^{-Y_3}$

$\mathbb{P}(U \leq t) = \mathbb{P}(e^{-Y_3} \leq t) = \mathbb{P}(-Y_3 \leq \ln t)$

$= \mathbb{P}(Y_3 \geq -\ln t) = 1 - \underbrace{F_{Y_3}(-\ln t)}$

$f_U(t) = f_{Y_3}(-\ln t) \cdot \frac{1}{t}$.

$= 30 \cdot (1-t)^2 \cdot t^3 \cdot \frac{1}{t} = 30 \cdot (1-t)^2 \cdot t^2$.

$\boxed{9}$ Likelihood function $= f(x_1) \cdot f(x_2) \cdot f(x_3) \cdot f(x_4)$

$= \left(\frac{2+\theta(2-x_1)}{6}\right) \cdot \left(\frac{2+\theta(2-x_2)}{6}\right) \cdots$

$\begin{pmatrix} x_1 = 3 \\ x_2 = 2 \\ x_3 = 3 \\ x_4 = 1 \end{pmatrix}$

$= \left(\frac{2-\theta}{6}\right) \cdot \frac{2}{6} \cdot \left(\frac{2-\theta}{6}\right)\left(\frac{2+\theta}{6}\right)$

$= \frac{2}{64} \underbrace{(2-\theta)^2 \cdot (2+\theta)}_{F(\theta)}$.

$\theta = \{-1, 0, 1\}$

$F(-1) = 3^2 \cdot 1 = 9$

$F(0) = 2^3 = 8$

$F(1) = 1^2 \cdot 3 = 3$

$\therefore \underbrace{\hat{\theta}_{MLE} = -1}$.

$\boxed{10}$ $n=16$ $N(\mu, \underbrace{25}_{=\sigma^2})$ $\overline{X} = 73.8$ $\alpha = 0.05$

$\Downarrow$

$$\overline{X} - z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}} \leq \mu \leq \overline{X} + z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}} \qquad z_{\frac{\alpha}{2}} = 1.96.$$

$$73.8 - 1.96 \cdot \frac{5}{4} \leq \mu \leq 73.8 + 1.96 \cdot \frac{5}{4}.$$

Pf:

$$\mathbb{P}\left(-z_{\frac{\alpha}{2}} \leq \underset{\frac{\overline{X}-\mu}{\sigma/\sqrt{n}}}{\underset{\shortparallel}{Z}} \leq z_{\frac{\alpha}{2}}\right) = 1-\alpha$$

$\Rightarrow$ Solve for μ

$$\mathbb{P}\left(\overline{X} - z_{\frac{\alpha}{2}} \cdot \frac{\sigma}{\sqrt{n}} \leq \mu \leq \overline{X} + z_{\frac{\alpha}{2}} \cdot \frac{\sigma}{\sqrt{n}}\right) = 1-\alpha. \quad \lrcorner$$