MATH 461 MIDTERM 1 PRACTICE SOLUTION

1. Practice problems

Q1) Consider a "thick" coin with three possible outcomes of a toss (Heads, Tails, and Edge) for which Heads and Tails are equally likely, but Heads is five times as likely than Edge. What is the probability of Heads?

Solution. Let H, T, E stand for Heads, Tails, and Edge respectively. Then, $\mathbb{P}(H) = \mathbb{P}(T) = p$ for some p. $\mathbb{P}(E) = \frac{1}{5}p$. We must have $\mathbb{P}(H) + \mathbb{P}(T) + \mathbb{P}(E) = 1$ or $p + p + \frac{1}{5}p = 1$ or $p = \frac{5}{11}$.

Q2) Suppose A, B and C are events with $\mathbb{P}(A) = 0.43, \mathbb{P}(B) = 0.40, \mathbb{P}(C) = 0.32, \mathbb{P}(A \cap B) = 0.29, \mathbb{P}(A \cap C) = 0.22, \mathbb{P}(B \cap C) = 0.20$ and $\mathbb{P}(A \cap B \cap C) = 0.15$. Find $\mathbb{P}(A^c \cap B^c \cap C^c)$.

Solution. $\mathbb{P}(A \cup B \cup C) = \mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(C) - \mathbb{P}(A \cap B) - \mathbb{P}(A \cap C) - \mathbb{P}(B \cap C) + \mathbb{P}(A \cap B \cap C) = .43 + .40 + .32 - .29 - .22 - .20 + .15 = 0.59$ and $\mathbb{P}(A^c \cap B^c \cap C^c) = 1 - \mathbb{P}(A \cup B \cup C) = 1 - 0.59 = 0.41$.

Q3) If $\mathbb{P}(A) = 0.4, \mathbb{P}(B) = 0.5, \mathbb{P}(B \mid A) = 0.75$, find $\mathbb{P}(A \mid B)$ and $\mathbb{P}(A \mid B^c)$.

Solution. $\mathbb{P}(A \cap B) = \mathbb{P}(B \mid A)\mathbb{P}(A) = 0.75 \times 0.4 = 0.30$ and $\mathbb{P}(A \mid B) = \mathbb{P}(A \cap B)/\mathbb{P}(B) = 0.3/0.5 = 0.6$. Now $\mathbb{P}(B^c) = 1 - \mathbb{P}(B) = 1 - 0.5 = 0.5, \mathbb{P}(A \cap B^c) = \mathbb{P}(A) - \mathbb{P}(A \cap B) = 0.4 - 0.3 = 0.1$ and $\mathbb{P}(A \mid B^c) = \mathbb{P}(A \cap B^c)/\mathbb{P}(B^c) = 0.1/0.5 = 0.2$.

Q4) During the first week of the semester, 80% of customers at a local convenience store bought either beer or potato chips (or both). 60% bought potato chips. 30% of the customers bought both beer and potato chips. Are events a randomly selected customer bought potato chips and a randomly selected customer bought beer independent?

Solution. Let B be the event that the customer bought Beer and C be the event that the customer bought chips. Then $\mathbb{P}(B \cup C) = 0.8, \mathbb{P}(C) = 0.6, \mathbb{P}(B \cap C) = 0.3$. Then $\mathbb{P}(B \cap C^c) = \mathbb{P}(B \cup C) - \mathbb{P}(C) = 0.8 - 0.6 = 0.2$ and $\mathbb{P}(B) = \mathbb{P}(B \cap C) + \mathbb{P}(B \cap C^c) = 0.3 + 0.2 = 0.5$. Now $\mathbb{P}(B)\mathbb{P}(C) = 0.5 \times 0.6 = 0.3 = \mathbb{P}(B \cap C)$. Thus the events are independent.

Q5) Seventy percent of the light aircraft that disappear while in flight in Neverland are subsequently discovered. Of the aircraft that are discovered, 60% have an emergency locator, whereas 90% of the aircraft not discovered do not have such a locator. Suppose a light aircraft that has just disappeared has an emergency locator. What is the probability that it will not be discovered?

Solution. Let D = Discovered and L = Locator. Then $\mathbb{P}(D) = 0.70, \mathbb{P}(D^c) = 1 - 0.70 = 0.30, \mathbb{P}(L \mid D) = 0.60, \mathbb{P}(L^c \mid D^c) = 0.90$. Need

$$\mathbb{P}(D^c \mid L) = \frac{\mathbb{P}(L \mid D^c)\mathbb{P}(D^c)}{\mathbb{P}(L \mid D^c)\mathbb{P}(D^c) + \mathbb{P}(L \mid D)\mathbb{P}(D)} = \frac{0.1 \times 0.3}{0.1 \times 0.3 + 0.6 \times 0.7} = \frac{3}{45} = \frac{1}{15}.$$

Q6) Suppose X is a random variable taking values in $S = \{0, 1, 2, 3, \ldots\}$ with PMF $p(0) = \mathbb{P}(X = 0) = c$, $p(k) = \mathbb{P}(X = k) = \frac{1}{2^k k!}, k = 1, 2, 3, \ldots$. Find the value of c that would make this a valid probability model. Find $\mathbb{E}[X], \mathbb{E}[2^X]$.

Solution. We must have $\sum_{k=0}^{\infty} f(k) = 1$ or $p + \sum_{k=1}^{\infty} \frac{1}{2^k k!} = 1$ or $p = 1 - \sum_{k=1}^{\infty} \frac{1}{2^k k!}$. Since $\sum_{k=0}^{\infty} a^k/k! = e^a$, we have $\sum_{k=1}^{\infty} \frac{1}{2^k k!} = \sum_{k=0}^{\infty} \frac{1}{2^k k!} - 1 = e^{1/2} - 1$. Thus $p = 1 - (e^{1/2} - 1) = 2 - e^{1/2}$.

$\mathbb{E}(X) = \sum_{k=0}^{\infty} k f(k) = \sum_{k=1}^{\infty} k \cdot \frac{1}{2^k k!} = \sum_{k=1}^{\infty} \frac{1}{2^k (k-1)!} = \sum_{\ell=0}^{\infty} \frac{1}{2^{\ell+1} \ell!} = \frac{1}{2} e^{1/2}$.

$\mathbb{E}(2^X) = \sum_{k=0}^{\infty} 2^k f(k) = \sum_{k=0}^{\infty} 2^k \cdot \frac{1}{2^k k!} = \sum_{k=0}^{\infty} \frac{1}{k!} = e$.

Q7) A certain basketball player knows that on average he will successfully make 78% of his free throw attempts. Assuming all throw attempts are independent. What are the expectation and the variance of the number of successful throws in 1020 attempts.

Solution. Let X be the number of successful throws, then it is binomial with $n = 1020$ and $p = 0.78$. Thus, the expectation is $np = 1020 \times 0.78$ and the variance is $np(1 - p) = 1020 \times 0.78 \times 0.22$.

Q8) In each of the following cases, is it appropriate to use Binomial model? If yes, what are the values of its parameters n and p (if known)? If no, explain why Binomial model is not appropriate or what model to use if you can.

(a) A fair coin is tossed 3 times. X = number of Heads.

Yes. $n = 3, p = 0.50$.

(b) A box contains 40 parts, 10 of which are defective. A person takes 7 parts out of the box with replacement. X = number of defective parts selected.

Yes. $n = 7, p = 10/40 = 0.25$.

(c) Seven members of the same family are tested for a particular food allergy. X = number of family members who are allergic to this particular food.

Yes if we can assume independence, No if we cannot.

(d) The bus numbered 1 arrives timely at Transit plaza 80% of the time. X = number of buses that arrives late before you see the first timely bus.

No. If we assume the buses are independent, then $X+1$ can be modeled by Geometric(0.8).

(e) Suppose that 5% of tax returns have arithmetic errors. 25 tax returns are selected at random. X = number of arithmetic errors in those 25 tax returns.

Yes. $n = 25, p = 0.05$.

Q9) Suppose you roll two 5 faced dice, with faces labeled 1,2,3,4,5, and each equally likely to appear on top. Let X denote the smaller of the two numbers that appear. If both dice show the same number, then X is equal to that common number. Find the PMF of X. Compute $\mathbb{P}(X \le 3|X > 1)$ and $\mathbb{E}[5X-2]$.

Solution. Let Y be the number of the first die, and Z the number of the second die. Then,

$$\mathbb{P}(X=1) = \mathbb{P}(Y=1 \text{ or } Z=1) = \mathbb{P}(Y=1) + \mathbb{P}(Z=1) - \mathbb{P}(Y=Z=1) = 9/25,$$

$$\mathbb{P}(X=2) = \mathbb{P}(Y=2, Z \ge 2 \text{ or } Z=2, Y \ge 2) = 7/25,$$

$$\mathbb{P}(X=3) = \mathbb{P}(Y=3, Z \ge 3 \text{ or } Z=3, Y \ge 3) = 5/25,$$

$$\mathbb{P}(X=4) = \mathbb{P}(Y=4, Z \ge 4 \text{ or } Z=4, Y \ge 4) = 3/25,$$

$$\mathbb{P}(X=5) = \mathbb{P}(Y=Z=5) = 1/25.$$

Thus,

$$\mathbb{P}(X \le 3|X > 1) = \frac{\mathbb{P}(1 < X \le 3)}{\mathbb{P}(X > 1)} = \frac{\mathbb{P}(X=2) + \mathbb{P}(X=3)}{1 - \mathbb{P}(X=1)} = \frac{12}{25-9} = \frac{3}{4}$$

and

$$\mathbb{E}[5X-2] = \frac{1}{25}((5-2)\cdot 9 + (10-2)\cdot 7 + (15-2)\cdot 5 + (20-2)\cdot 3 + (25-2)\cdot 1) = 45.$$

Q10) Consider an urn containing 21 balls, of which 7 are red, 8 are green and 6 are blue. A sample of size 15 is to be drawn with (or without) replacement. What is the conditional probability that the first 5 balls drawn will be green given that the sample drawn contains exactly 6 green balls?

Solution 1 (with replacement). Let A be the event that first 5 balls are green, and B the event that exactly 6 green balls are drawn. If X is the number of Green balls drawn, it is binomial with $n = 15$

and $p = 8/21$. Then, $\mathbb{P}(B) = \mathbb{P}(X = 6)$ and

$$\mathbb{P}(A \cap B) = p^6(1-p)^9\binom{10}{1}.$$

Thus,

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} = \frac{\binom{10}{1}p^6(1-p)^9}{\binom{15}{6}p^6(1-p)^9} = \frac{\binom{10}{1}}{\binom{15}{6}}.$$

Solution 2 (without replacement). The number of ways to choose 15 balls from 21 balls with ordering is $\binom{21}{15}15!$. To compute the number of ways to choose 6 green balls and 9 non-green balls with ordering, we first choose the spots for green balls from 15 spots, $\binom{15}{6}$, then choose 6 green balls with order and 9 non-green balls with order, so that

$$\binom{15}{6} \cdot \left(\binom{8}{6}6!\right) \cdot \left(\binom{13}{9}9!\right).$$

Thus,

$$\mathbb{P}(B) = \left(\binom{15}{6} \cdot \left(\binom{8}{6}6!\right) \cdot \left(\binom{13}{9}9!\right).\right) / \left(\binom{21}{15}15!\right) = \frac{\binom{8}{6}\binom{13}{9}}{\binom{21}{15}}.$$

Similarly, if 6 balls are green and first five are green, the number of ways is

$$\binom{10}{1} \cdot \left(\binom{8}{6}6!\right) \cdot \left(\binom{13}{9}9!\right).$$

Therefore,

$$\mathbb{P}(A \cap B) = \binom{10}{1} \cdot \left(\binom{8}{6}6!\right) \cdot \left(\binom{13}{9}9!\right) / \left(\binom{21}{15}15!\right), \qquad \mathbb{P}(A|B) = \frac{\binom{10}{1}}{\binom{15}{6}}.$$

Solution 3 (without replacement). One can use Bayes formula. Since A is determined only by first 5 drawing,

$$\mathbb{P}(A) = \frac{8}{21} \cdot \frac{7}{20} \cdot \frac{6}{19} \cdot \frac{5}{18} \cdot \frac{4}{17}.$$

For B given A, we can consider 16 balls with 3 green balls and 13 non-green balls, and draw 10 balls with exactly 1 green ball. Thus,

$$\mathbb{P}(B|A) = \frac{\binom{3}{1}\binom{13}{9}}{\binom{16}{10}}.$$

Thus,

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(B|A)\mathbb{P}(A)}{\mathbb{P}(B)} = \frac{\binom{3}{1}\binom{13}{9}}{\binom{16}{10}} \frac{8}{21} \cdot \frac{7}{20} \cdot \frac{6}{19} \cdot \frac{5}{18} \cdot \frac{4}{17} \frac{\binom{21}{15}}{\binom{8}{6}\binom{13}{9}} = \frac{10}{\binom{15}{6}}.$$

Solution 4 (without replacement). Since we consider an event A given B, we think that we have only 15 ball and 6 of them are green. Under this situation, the probability that the first five are green is

$$\mathbb{P}(A|B) = \frac{6}{15} \cdot \frac{5}{14} \cdot \frac{4}{13} \cdot \frac{3}{12} \cdot \frac{2}{11} = \frac{10}{\binom{15}{6}}.$$

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