\# 3, 5, 6, 13, 14, 15, 16, 17, 18, 19
2 - (6)   12

Practice Problems for Exam 1

1. The following numbers x_i, $i = 1, ..., 17$, represent a sample of size $n = 17$ from a given population.

3.0273	2.8598	3.2407	3.3058	3.0626	3.1718
2.5795	2.4375	2.4639	2.8931	2.8598	3.4566
3.1399	3.2174	2.9072	2.6108	3.2846	

   (a) Compute the sample median and fourth spread.

   (b) Knowing that

$$\sum_{i=1}^{17} x_i = 50.5182, \qquad \sum_{i=1}^{17} x_i^2 = 151.6426,$$

   compute the sample mean and the variance.

   (c) Draw a box plot of the data.

$$\text{Geom. Series} = \frac{1^{st}\text{ Term}}{1-\text{Ratio}}$$

2. A box in a certain supply room contains four 40W light bulbs, five 60W bulbs, and six 75W bulbs.
   (a) Suppose that three bulbs are randomly selected. What is the probability that exactly two of the selected bulbs are rated 75W?
   (b) Suppose now that bulbs are to be selected one by one until a 75W bulb is found. What is the probability that it is necessary to examine at least six bulbs?

Case 1: With replacement $\Rightarrow$ $X = \#$ of trials $\sim \text{Geom}\left(\underbrace{\frac{6}{15}}_{=p}\right)$

prob. that choose 75W

$= \frac{6}{15} \Rightarrow P(X \geq 6) = \sum_{k=6}^{\infty} (1-p)^{k-1} \cdot p$

$= \frac{(1-p)^5 \cdot \cancel{p}}{1 - \cancel{(1-p)}} = (1-p)^5$

$P(X \geq 6) = P(\text{First Five} = F)$

$= (1-p)^5$

3. One percent of all individuals in a certain population are carriers of a particular disease. A diagnostic test for this disease has a 90% detection rate for carriers and a 5% detection rate for non-carriers. Suppose that the test is applied independently to two different blood samples from the same randomly selected individual.
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Without Replacement

$$P(X \geq 6) = P(\text{First Five} = F)$$

$$= \frac{9}{15} \cdot \frac{8}{14} \cdot \frac{7}{13} \cdot \frac{6}{12} \cdot \frac{5}{11}$$

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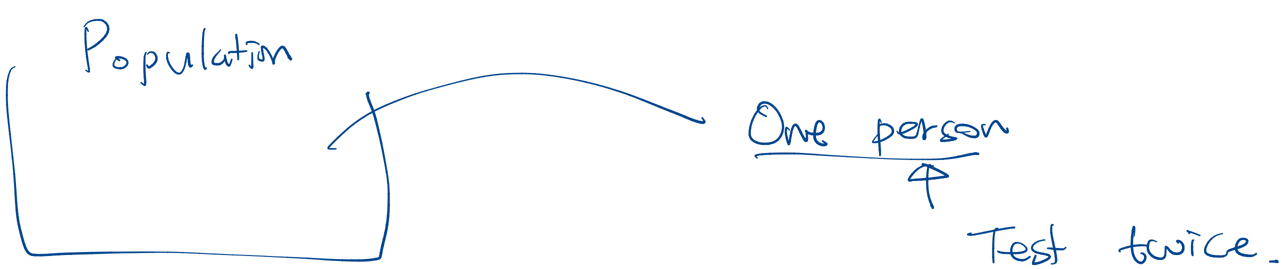


$A = \{\text{Carrier}\}$ $\quad P(A) = 0.01$

$B = \{\text{Both tests are Positive}\}$ $\quad P(B \mid A) = (0.9)^2$

$P(B \mid A^c) = (0.05)^2$

(a) $P(B) = P(B \mid A) \cdot P(A) + P(B \mid A^c) \cdot P(A^c)$

(b) $P(A \mid B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{P(B \mid A) \cdot P(A)}{P(B)}$

4. A toll bridge charges \$1.00 for passenger cars and \$2.50 for other vehicles. Suppose that during daytime hours, 60% of all vehicles are passenger cars. Suppose that 25 vehicles cross the bridge during a particular day.

   (a) Let X be the number of passenger cars among 25 vehicles. How is X distributed?

   (b) What is the expected number of passenger cars among 25 vehicles, that is, $\mathbb{E}[X]$?

   (c) Find the expression of the toll revenue from the 25 vehicles $h(X)$ in terms of X. Compute the expected revenue $\mathbb{E}[h(X)]$.

5. Two six-sided dice are tossed independently. Let X be the maximum of the two outcomes. (For example, if two outcomes are 1 and 5, then $X = 5$.)

   (a) Let $p(x)$ be the PMF of X. Find $p(3)$ and $p(4)$.

   (b) Let $F(x)$ be the CDF of X. Find $F(0)$ and $F(3.2)$.

$$\boxed{F(0) = \mathbb{P}(X \leq 0) = 0}$$

$X = 1, 2, 3, 4, 5, 6$

1: $(1,1)$

2: $(1,2)$, $(2,2)$, $(2,1)$

3: $(1,3)$, $(2,3)$, $(3,3)$, $(3,2)$, $(3,1)$

4: $(2,4)$, $(3,4)$, $(4,4)$, $(4,3)$, $(4,2)$, $(4,1)$, $(1,4)$

5: 9

6: 11

$$F(3.2) = \mathbb{P}(X \leq 3.2) = \frac{1+3+5}{36}$$

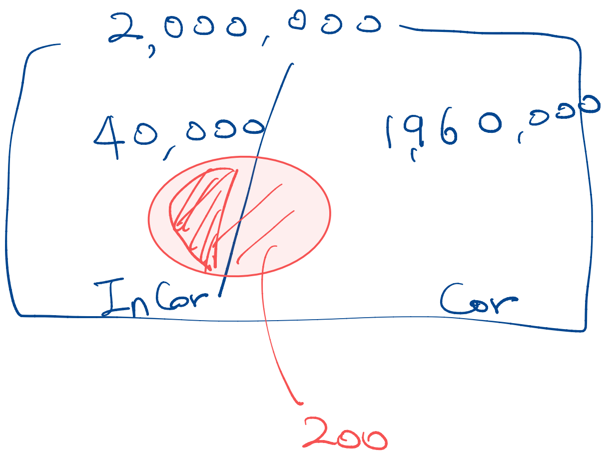
$$p(x) = \begin{cases} \frac{1}{36}, & x=1 \\ \frac{3}{36}, & x=2 \\ \vdots & \vdots \\ \frac{11}{36}, & x=6 \\ 0, & \text{otherwise} \end{cases}$$

$$p(x) = \begin{cases} \frac{2x-1}{36}, & x=1,\dots,6 \\ 0, & \text{otherwise} \end{cases}$$

6. In Atlanta there are 2,000,000 families. Among them 40,000 do not report correctly their incomes. The IRS selects a sample of 200 families and controls their tax returns. Let X be the number of incorrect reports among these 200 families.

(a) What is the probability distribution of X? Write a formula for the probability that $X = 4$.

(b) Use a binomial approximation to compute the average and variance of X. Justify the approximation.



$$\mathbb{P}(X=4) = \frac{\binom{40,000}{4}\binom{1960,000}{196}}{\binom{2M}{200}}$$

$$\frac{4}{200} = 0.02$$

$$X \sim HG(200, 40,000, 2M) \approx Bin\left(200, \frac{40,000}{2M}\right)$$

0.02

2M large compare to 200

$$\mathbb{E}[X] = 200 \cdot (0.02)$$

$$Var(X) \approx 200 \cdot (0.02) \cdot (1 - 0.02)$$

$$\frac{2M - 200}{2M - 1} \approx 1$$

7. The following numbers x_i, $i = 1, 2, \ldots, 19$ represent a sample of size $n = 19$ from a given population.

0.985	0.645	0.118	0.894	0.784	0.101	
0.253	−0.321	1.832	0.378	0.679	0.681	
0.753	0.006	0.624	−1.104	0.273	0.126	0.618

(a) Compute the sample median and the fourth spread.

(b) Knowing that $\sum_{i=1}^{19} x_i = 8.325$ and $\sum_{i=1}^{19} x_i^2 = 10.063$, compute the sample mean and variance.

(c) Draw a box plot of the data.

(d) Are there an outlier or an extreme outlier?

8. In a university every semester 20 students enroll for the honors calculus class for the first time. Each of them as a probability $p = 0.9$ of passing the exam. If they do not pass the exam, they will enroll again in the same class the following semester. In this case, it is observed that the probability of passing the exam is $q = 0.6$, that is, 60% of the students that fail the exam for the first time will pass it at the second time. If they fail the exam for the second time, they have to drop the class.

   (a) What is the probability that among the students who enroll for the first time in a given year exactly 4 students fail the exam?

   (b) What is the average number of students who fail the exam for the first time?

   (c) What is the probability that among the students who enroll for the first time in a given year exactly 4 students will fail the exam twice?

   (d) What is the average number of students who fail the exam twice?

9. (Continued) Every semester, the student attending the class will be formed by the 20 students that just enrolled plus the students that failed the exam in the previous semester. Suppose that 3 students failed the exam in the previous semester.

   (a) Choosing a student in the class at random, what is the probability that the student already tried the exam in the previous semester?

   (b) Choose one of the 23 students in the class and observe that the student passes the exam, what is the probability that the student already tried the exam in the previous semester?

10. In a bowl, there are 10 red balls, 20 green balls, and 30 blue balls. You randomly chose 9 out of them without replacement.

   (a) Find the probability that 4 of the 9 chosen balls are red.

   (b) Find the probability that 6 of the 9 chosen balls are red or green.

   (c) Find the probability that 3 of the 9 chosen balls are red, 3 are blue, and 3 are green.

11. A customer purchases 10 bulbs from a store. We know that each bulb has the probability $p = 0.99$ to be working independently from the others.

(a) What is the probability that all bulbs are working?

(b) What is the probability that exactly 4 out of the 10 bulbs are working?

(c) What is the expected value of the number of working bulbs? and its variance?

$$q = 0.1 = \mathbb{P}(B^c \mid A)$$

12. (Continued) A working bulb has a probability of breaking down during the first month of use equal to $q = 0.1$. Let X be the random variable that counts the number of bulbs still working after a month of use among the 10 bulbs initially purchased.

(a) What is the probability that $X = 8$? What is the PMF of X?

(b) Given that after a month one of the bulbs is checked and found not working, what is the probability that it was working when purchased?
(Hint: Let A be the event that the bulb was working when purchased and B the event that the bulb is working after a month of use.)

(a) $X \sim \mathrm{Bin}(10, \underline{r})$ $\quad (0.9)(0.99) = r$

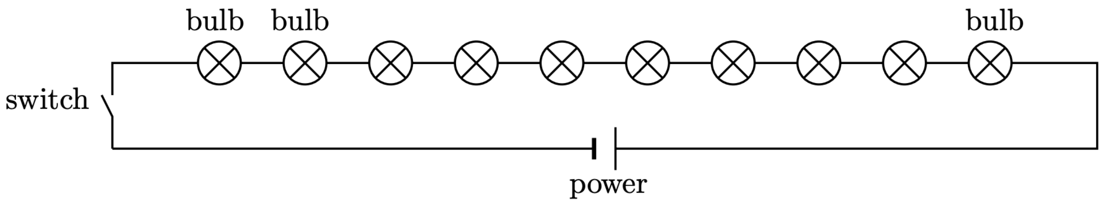
$$\mathbb{P}(X=8) = \binom{10}{8} r^8 \cdot (1-r)^2$$

(b)
$$\underline{\mathbb{P}(A \mid B^c)} = \frac{\mathbb{P}(B^c \mid A)\cdot \mathbb{P}(A)}{\mathbb{P}(B^c)} \quad = 1 - \mathbb{P}(B)$$

$$= \frac{(0.1)\cdot(0.99)}{1 - (0.99)\cdot(0.9)}$$

$$\mathbb{P}(B^c) = \mathbb{P}(B^c \mid A)\mathbb{P}(A) + \underline{\mathbb{P}(B^c \mid A^c)}\cdot \mathbb{P}(A^c) = (0.1)\cdot(0.99) + 1\cdot(0.01)$$

13. (Continued) Our customer uses 10 bulbs in a room where they are connected in series as shown in the figure. (Each bulb has the probability $p = 0.99$ to be working independently from the others as before.)



When the switch is closed, the bulbs light up only if they are all working.

(a) What is the probability that the light will not go on when the switch is closed?

(b) Given that the light does not go on when the switch is closed, what is the probability that the first bulb is not working?

(Hint: Let A_i the event that the i-th bulb is working. Express the events that the light does not go on and that the first bulb is not working in term of A_i.)

(c) Given that the light does not go on when the switch is closed, what is the probability that **only** the first bulb is not working?

14. (Continued) Assume now that you have two rooms (Room 1 and Room 2) identical to the one described above, i.e. each with ten bulbs connected in series. Each room has its own power supply and its own switch. Our costumer buys 20 bulbs and we know that exactly two of them are not working.

   (a) When both switches are closed, what is the probability that the light will go on only in Room 1? (Hint: Let Y be the number of non-working bulbs in Room 1.)

   (b) When both switches are closed, what is the probability that the light will not go on in all the two rooms, i.e. no light in Room 1 and no light in Room 2?

15. (Continued) Assume that 5 out of the 20 bulbs are not working.

   (a) What is the probability that exactly 3 of the non-working bulbs are in Room 1? What is the probability that exactly 2 of the non-working bulbs are in Room 1?

   (b) When both switches are closed, what is the probability that the light will go on in Room 1?

$$\frac{\binom{5}{0}\binom{15}{10}}{\binom{20}{10}}$$

$$P(\text{Shipped} \cap \text{Def}) = P(\text{Def} \mid \text{Shipped}) \cdot P(\text{Shipped})$$

16. A company produces personal computers and has two programs, program A and program B, to check if they work. The quality control center decides to operate as follows:

(a) If the computer fails Test A (the test done using program A) then it is discarded.

(b) If the computer passes Test A it is checked with Test B (the test done using program B). If it fails Test B it is discarded.

(c) The computers that pass Test B are shipped to be sold.

It is observed that 95% of the computer tested with Test A pass it and 99 of those tested with Test B pass it. After a large quality review it is found that

(a) 0.1% of the shipped computer were defective.

(b) 0.2% of the computer discarded because they failed Test B were working. $0.002 = \mathbb{P}(\text{Working} \mid \text{Fail B}) = \mathbb{P}(\text{Working} \mid \text{Fail A})$

(c) 0.2% of the computer discarded because they failed Test A were working.

Find the following.

(a) What is the percentage of shipped computer on the total produced?

(b) Given that a computer is working, what is the probability that it is discarded?

(c) What is the probability that a shipped computer is not working? $A \cap B$

(a) $\mathbb{P}(\text{Shipped}) = \mathbb{P}(\text{Pass test A \& B})$

$= (0.95) \cdot (0.99)$

$= \mathbb{P}(A) \cdot \mathbb{P}(B \mid A)$

Fail A or Fail B

(b) $\mathbb{P}(\text{Discarded} \mid \text{Working})$

$= \mathbb{P}(\text{Fail A} \mid \text{Working}) + \mathbb{P}(\text{Fail B} \mid \text{Working})$

$= \dfrac{\mathbb{P}(\text{Working} \mid \text{Fail A}) \cdot \mathbb{P}(\text{Fail A})}{\mathbb{P}(\text{Working})} + \ldots$

$\mathbb{P}(\text{Working}) = \mathbb{P}(\text{Working} \mid \text{Shipped})\,\mathbb{P}(\text{Shipped})$

$+ \mathbb{P}(\text{Working} \mid \text{Fail A})\,\mathbb{P}(\text{Fail A})$

$+ \mathbb{P}(\quad " \quad \mid \text{Fail B})\,\mathbb{P}(\text{Fail B})$

17. (Continued) A customer buys computer from our company. He needs 5 working computers for a critical job. He decides to buy N computers with $N > 5$ because he wants that the probability that at least 5 among the N he bought are working to be higher than $1 - 10^4$.

    (a) How large should N be?

    (b) Another customer buys 3 computers every day. What is the probability that he will find the first non-working computer after exactly 10 days.

    (c) Write an expression for the expected number of days he will wait before buying the first non-working computer and try to evaluate it.

A , B , C

letter with 10 alphabets

Q: $\mathbb{P}($ AAA exists $)$

$$\begin{cases} \mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B) \\ \mathbb{P}(A \cup B \cup C) = \ldots \\ \mathbb{P}(A \cap B) = \mathbb{P}(A \mid B)\,\mathbb{P}(B) \\ \qquad\qquad = \mathbb{P}(B \mid A)\,\mathbb{P}(A) \\ \mathbb{P}(A \mid B) = \dfrac{\mathbb{P}(B \mid A)\,\mathbb{P}(A)}{\mathbb{P}(B)} \end{cases}$$

$Ber(p)$ $\quad X = \begin{cases} 1 & \text{w.p. } (p) \\ 0 & \text{w.p. } (1-p) \end{cases}$

$Geom(p)$ # of trials until 1st S.

$Bin(n, p)$ # of S in n trials

$NB(r, p)$ # of F until r^{th} S

$HG(n, M, N)$ Similar to Bin
"without replacement".