Chapter 2. Probability

Math 3670 Summer 2024

Georgia Institute of Technology

Section 1.
Sample Spaces and Events

Experiments, Sample Spaces, and Events

An **experiment** is any activity or process whose outcome is subject to uncertainty.

Ex: Tossing a fair coin, roll a die, draw a card from a deck.

$\mathcal{S} = \{H, T\}$ $\qquad$ $\mathcal{S} = \{1, 2, 3, 4, 5, 6\}$

The **sample space** of an experiment, denoted by $\mathcal{S}$, is the set of all possible outcomes

An **event** is any collection (subset) of outcomes contained in the sample space $\mathcal{S}$.

An event is **simple** if it consists of exactly one outcome and **compound** if it consists of more than one outcome.

2 outcomes

$\mathcal{S} = \{H, T\}$ $\leftarrow$ 4 Events

$\mathcal{S} = \underbrace{\{1, 2, 3, 4, 5, 6\}}_{6 \text{ outcomes}}$ $\leftarrow$ 2^6 events

(empty set)

$$\underbrace{\{H\}, \{T\}, \{H, T\}, \phi, \{1\}, \{2\}, \{1, 2\}, \{3, 4, 6\}, \vdots}_{\text{Events}}$$

Roll a die 4 times

$$\mathcal{S} = \{\underbrace{(1,1,1,1), (1,1,1,2), \ldots}_{6^4 \text{ outcomes}}\}$$

6^4 outcomes, 2^{6^4} events

Example

Example (Two outcomes)

Consider an experiment examining a single fuse to see whether it is defective.

The sample space for this experiment is $S = \{D, N\}$

Examples of events are $\{D\}, \{N\}, \{N, D\}, \phi$

If we repeat the experiment 3 times, then the sample space is

Examples of events are $S = \{\underbrace{(D,D,D), (N,D,D), \ldots}_{2^3 = 8 \text{ outcomes}}\}$

2^8 events

Example

Example

Roll two dice.

The sample space $\mathcal{S} = \{(1,1),(1,2),\cdots\}$.

$\mathcal{S}$ $36 = 6^2$ outcomes

\# of events $= 2^{36}$

$=$ \# of subsets

How many outcomes in $\mathcal{S}$?

Let A be the event that the sum is 5. How many outcomes in A?

$$A = \{(a,b) : a+b=5\}$$

$a = 1, 2, \cdots 6$

$b = 1, 2, \cdots, 6$

$$= \{(1,4),(2,3),(3,2),(4,1)\}$$

Probability that sum is 5

= Prob of A.

Example

Example

Toss a fair coin once.

Then, the sample space is $\mathcal{S} = \{H, T\}$.

Suppose we repeat tossing a coin until it lands Tails.

Then, the sample space is

$S = \{ T, HT, HHT, HHHT, \ldots \}$

$A = \{ \#$ of trials is even $\}$

$= \{ HT, HHHT, HHHHHT, \ldots \}$

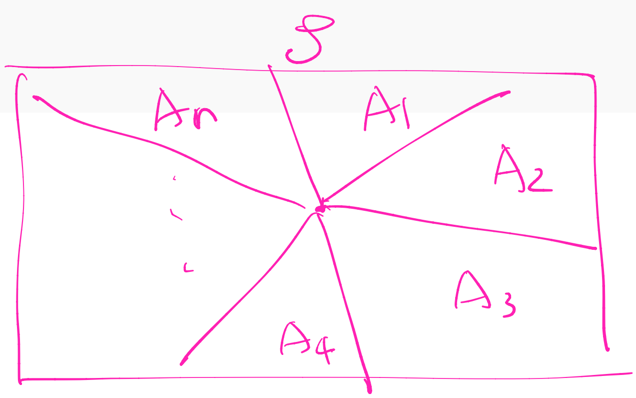
Some Relations from Set Theory

Definitions

outcomes

1. Union $A \cup B =$ $\{ x \in S : x \in A$ OR $x \in B \}$
2. Intersection $A \cap B =$ $\{ x \in S : x \in A$ AND $x \in B \}$
3. Complement $A' = A^c =$ $\{ x \in S : x \notin A \}$ Not Belong to.
4. We say A and B are mutually exclusive or disjoint if $A \cap B = \phi$
5. We say $A_1, A_2, \cdots, A_n$ are exhaustive if $A_1 \cup A_2 \cup \cdots \cup A_n = S$
6. If $A_1, A_2, \cdots, A_n$ are mutually exclusive and exhaustive, then

$A_1, \cdots, A_n$ are Partition.



No Overlap.

Some Relations from Set Theory

Example

Roll a die.

The sample space is $\mathcal{S} = \{1, 2, 3, 4, 5, 6\}$.

Let $A = \{1, 2, 3, 4\}$ and $B = \{2, 4, 6\}$.

Then,

$A \cup B =$ $\{1, 2, 3, 4, 6\}$

$A \cap B =$ $\{2, 4\}$

$A^c =$ $\{5, 6\}$

3:08

Exercise

(2.1-1) Four universities—1, 2, 3, and 4—are participating in a holiday basketball tournament.

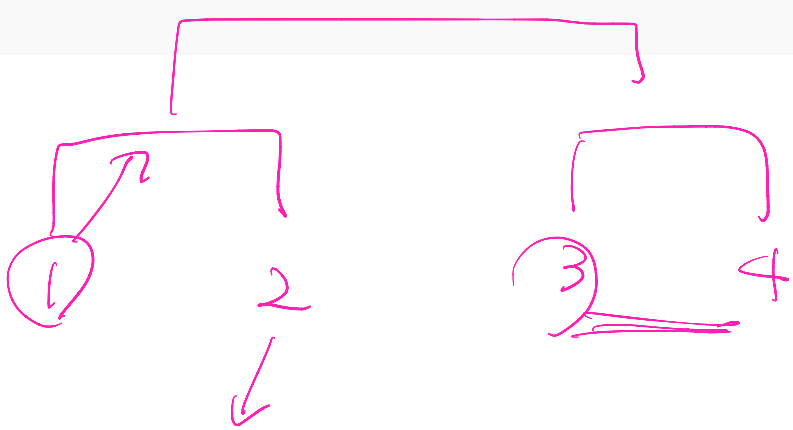
In the first round, 1 will play 2 and 3 will play 4.

Then the two winners will play for the championship, and the two losers will also play.

One possible outcome can be denoted by 1324 (1 beats 2 and 3 beats 4 in first-round games, and then 1 beats 3 and 2 beats 4).

1. List all outcomes in $\mathcal{S}$.
2. Let A denote the event that 1 wins the tournament. List outcomes in A.
3. Let B denote the event that 2 gets into the championship game. List outcomes in B.
4. What are the outcomes in $A \cup B$ and in $A \cap B$? What are the outcomes in A'?

#2



1 3 2 4

1 3 4 2

1 4 3 2

1 4 2 3

Section 2.
Axioms, Interpretations, and Properties of Probability

Recall: $\mathcal{S}$: Sample space

$A_1, A_2, \cdots, A_n$: Events

Exhaustive if $A_1 \cup A_2 \cup \cdots \cup A_n = \mathcal{S}$

Mutually Exclusive if No overlap.

$A_1 \cap A_2 = \phi$, $A_2 \cap A_3 = \phi, \cdots$

The Axioms of Probability

$\mathcal{S}$: Sample Space

$\mathbb{P} : \{\text{Events}\} \longrightarrow [0, 1]$

$\mathcal{S} \longmapsto \mathbb{P}(\mathcal{S}) = 1$

Axioms

1. For any event A, $\mathbb{P}(A) \geq 0$.
2. $\mathbb{P}(\mathcal{S}) = 1$.
3. If $A_1, A_2, A_3, \cdots$ is an infinite collection of disjoint events, then

// Mutually Exclusive.

$$\mathbb{P}(A_1 \cup A_2 \cup \cdots) = \sum_{i=1}^{\infty} \mathbb{P}(A_i).$$

$$\mathbb{P}(\underbrace{A_1 \cup A_2 \cup \cdots}_{\text{disjoint}}) = \sum_{i=1}^{\infty} \mathbb{P}(A_i)$$

The Axioms of Probability

Properties

1. $\mathbb{P}(\varnothing) = 0$.
2. If $A_1, A_2, \cdots, A_n$ is a finite collection of disjoint events, then

$$\mathbb{P}(A_1 \cup A_2 \cup \cdots \cup A_n) = \sum_{i=1}^{n} \mathbb{P}(A_i).$$

3. In particular, $\mathbb{P}(A) + \mathbb{P}(A^c) = 1$.
4. For any two events, $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$.

① $A_1 = A_2 = \cdots = \phi$ disjoint

$$\mathbb{P}(A_1 \cup A_2 \cup \cdots) = \mathbb{P}(\phi \cup \phi \cup \cdots) = \underline{\mathbb{P}(\phi)}$$

$$= \mathbb{P}(\phi) + \mathbb{P}(\phi) + \cdots$$

$$\Rightarrow \quad \mathbb{P}(\phi) = 0.$$

② $A_1, \cdots, A_n$ given $\quad A_{n+1} = A_{n+1} = \cdots = \phi$

$$\mathbb{P}(A_1 \cup A_2 \cup \cdots) = \mathbb{P}(A_1 \cup A_2 \cup \cdots \cup A_n)$$

$$= \sum_{i=1}^{\infty} \mathbb{P}(A_i) \quad \text{by Axiom} \quad = \sum_{i=1}^{n} \mathbb{P}(A_i)$$

The Axioms of Probability

Properties

1. $\mathbb{P}(\varnothing) = 0$.
2. If $A_1, A_2, \cdots, A_n$ is a finite collection of disjoint events, then
$$\mathbb{P}(A_1 \cup A_2 \cup \cdots \cup A_n) = \sum_{i=1}^{n} \mathbb{P}(A_i).$$
3. In particular, $\mathbb{P}(A) + \mathbb{P}(A^c) = 1$.
4. For any two events, $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$.

③ $A_1 = A$, $A_2 = A^c$: Disjoint ?

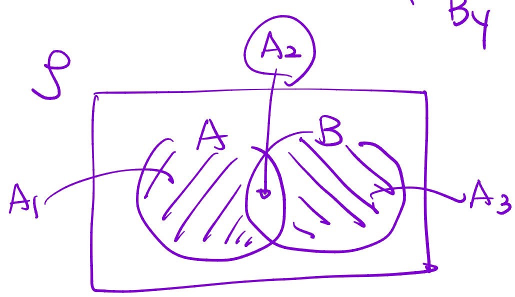
$(A_1 \cap A_2 = A \cap A^c = \phi)$

By ②, $P(A \cup A^c) = P(A) + P(A^c)$

$$\underbrace{P(A \cup A^c)}_{\parallel} \quad 1 = P(S)$$

↑ By Axiom 2.

④



$P(\underline{A \cup B})$

$= P(A_1 \cup A_2 \cup A_3)$

$= P(A_1) + P(A_2) + P(A_3)$

$$P(A) = P(A_1 \cup A_2) = P(A_1) + P(A_2)$$

$$P(B) = P(\underbrace{A_2 \cup A_3}_{\text{disjoint}}) = P(A_2) + P(A_3)$$

$$P(A \cup B) = \underbrace{P(A_1) + P(A_2)}_{= P(A)} + \underbrace{P(A_2) + P(A_3)}_{= P(B)} - \underbrace{P(A_2)}_{= P(A \cap B)}$$

Example

Example (Two outcomes)

Consider an experiment examining a single fuse to see whether it is defective.

The sample space for this experiment is $\mathcal{S} = \{N, D\}$.

Suppose $\mathbb{P}(D) = 0.01$. What is $\mathbb{P}(N)$? $P(N) = \underline{0.99}$

If we repeat the experiment 3 times, then what is the probability that there are 2 defective ones?

$$S = \{\underbrace{NNN, NDN, \ldots}_{8 = 2^3 \text{ outcomes}}\}$$

$$P(\text{Exactly } 2 \text{ Def.})$$

$$= P(\{DDN, DND, NDD\})$$

$$= P(DDN) + P(DND) + P(NDD)$$

$$= 3 \cdot P(DDN) = 3\, P(D) \cdot P(D) \cdot P(N)$$

$$= 3 \cdot (0.01) \cdot (0.01) \cdot (0.99)$$

Example

$\mathcal{S} = \{H, T\}$

$\mathbb{P}(H) = \mathbb{P}(T) = \frac{1}{2}$

Example

Toss a fair coin once.

Suppose we repeat tossing a coin until it lands Tails. $\mathcal{S} = \{T, HT, HHT, \cdots\}$

Let E_k be the event that the number of trials (tossing) is k, for $k = 1, 2, \cdots$.

Then,

$$\mathbb{P}(E_1) + \mathbb{P}(E_2) + \cdots = 1 = \left(\tfrac{1}{2}\right)^1 + \left(\tfrac{1}{2}\right)^2 + \cdots = \frac{1^{st}\text{ term}}{1 - \text{Ratio}} = \frac{\frac{1}{2}}{1-\frac{1}{2}}$$

Find the probability that it will take three or more flips of the coin to observe Tails.

$$= 1 - \mathbb{P}(E_1) - \mathbb{P}(E_2) = 1 - \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$$

$E_1 = \{\text{toss } 1 \text{ time}\} = \{T\}$

$E_2 = \{\text{toss } 2 \text{ times}\} = \{HT\}$

$\vdots$

$= \{HHT\}$

① $E_1 \cup E_2 \cup \cdots = \mathcal{S} \rightarrow$ Exhaustive

② $E_1, E_2, \cdots$ Disjoint $\rightarrow$ Mutually Exclusive.

$\Rightarrow \mathbb{P}(E_1) + \mathbb{P}(E_2) + \cdots = 1$.

Q: $\underline{P(E_1) + P(E_3) + P(E_5) + \cdots} = \left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^5 + \cdots = \frac{\frac{1}{2}}{1-\frac{1}{4}} = \frac{2}{3}$

$P(E_1) = P(T) = \frac{1}{2}$

$P(E_2) = P(H \cdot T) = P(H)\,P(T) = \left(\frac{1}{2}\right)^2$

$\vdots$

$P(E_k) = \left(\frac{1}{2}\right)^k$

Example

Example

In a certain residential suburb, 60% of all households get Internet service from the local cable company, 80% get television service from that company, and 50% get both services from that company.

If a household is randomly selected, what is the probability that it gets at least one of these two services from the company, and what is the probability that it gets exactly one of these services from the company?

$A = \{\text{Internet}\}$

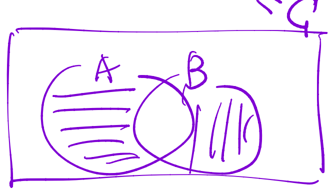
$B = \{\text{TV}\}$

$P(A) = 0.6, \quad P(B) = 0.8, \quad P(A \cap B) = 0.5$

$P(\text{at least one I, TV}) = P(A \cup B)$

$= P(A) + P(B) - P(A \cap B) = 0.9.$

$P(\underbrace{\text{Exactly 1}}_{C}) =$



$C \cup (A \cap B) = A \cup B$

disjoint

$$P(C) + P(A \cap B) = P(A \cup B)$$

$$P(C) = P(A \cup B) - P(A \cap B)$$

$$= 0.9 - 0.5 = 0.4.$$

Equally Likely Outcomes

In many experiments consisting of N outcomes, it is reasonable to assign equal probabilities to all N simple events.

$\mathcal{S} = \{a_1, a_2, \dots, a_N\}$

For example,

$A_1 = \{a_1\}, \quad A_2 = \{a_2\}, \dots$

1. Tossing a fair coin or fair die once or twice (or any fixed number of times), or
2. Selecting one or several cards from a well-shuffled deck of 52.

If $\mathcal{S}$ consists of N outcomes, and an event E has m outcomes, then

$$\mathbb{P}(E) = \frac{m}{N} = \frac{\#\text{ of outcomes in } E}{\text{'' in } \mathcal{S}}$$

$A_1, A_2, \dots$: Mutually Exclusive & Exhaustive.

$$\Rightarrow \begin{cases} P(A_1) + P(A_2) + \dots + P(A_N) = 1 \\ P(A_1) = P(A_2) = \dots = P(A_N) \end{cases}$$

$$\Rightarrow P(A_1) = P(A_2) = \dots = \frac{1}{N}$$

Example

Example

You have six unread mysteries on your bookshelf and six unread science fiction books.

The first three of each type are hardcover, and the last three are paperback.

Consider randomly selecting one of the six mysteries and then randomly selecting one of the six science fiction books.

What is the probability that both selected books are paperbacks? $P(A) = \frac{9}{36}$

$\underbrace{m_1 \quad m_2 \quad m_3}_{H} \quad \underbrace{\cdots \quad m_6}_{P} \qquad \underbrace{s_1 \quad s_2}_{H} \quad \underbrace{\cdots \quad s_6}_{P}$

$S = \{ \underbrace{(m_1, s_1), (m_1, s_2), \cdots}_{6 \times 6 = 36 \text{ outcomes}} \}$

$A = \{ \text{Both } P \}$

$= \{ \underbrace{(m_4, s_4)\ (m_4, s_5), \cdots}_{3 \times 3 = 9 \text{ outcomes}} \}$

Exercise

(2.2-12) Consider randomly selecting a student at a certain university.

Let A denote the event that the selected individual has a Visa credit card and B be the analogous event for a MasterCard.

Suppose that $\mathbb{P}(A) = .5, \mathbb{P}(B) = .4$, and $\mathbb{P}(A \cap B) = .25$.

1. Compute the probability that the selected individual has at least one of the two types of cards.
2. What is the probability that the selected individual has neither type of card?
3. Describe, in terms of A and B, the event that the selected student has a Visa card but not a MasterCard, and then calculate the probability of this event.

Section 3.
Counting Techniques

Counting

When the various outcomes of an experiment are equally likely (the same probability is assigned to each simple event), the task of computing probabilities reduces to counting.

Letting N denote the number of outcomes in a sample space and $\mathcal{N}(A)$ represent the number of outcomes contained in an event A,

$$\mathbb{P}(A) = \frac{\mathcal{N}(A)}{N}.$$

Product Rule

Product Rule

If the first element or object of an ordered pair can be selected in n_1 ways,

and for each of these n_1 ways the second element of the pair can be selected in n_2 ways,

then the number of pairs is $n_1 n_2$.

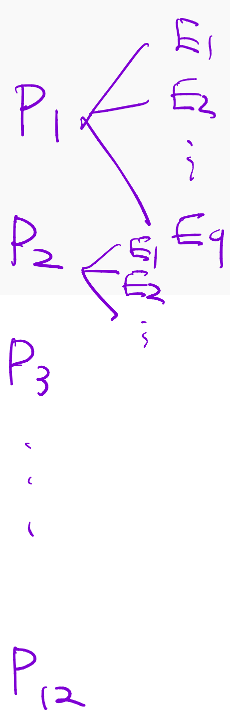
Product Rule

Example

A homeowner doing some remodeling requires the services of both a plumbing contractor and an electrical contractor.

If there are 12 plumbing contractors and 9 electrical contractors available in the area, in how many ways can the contractors be chosen?

$12 \times 9 = 108$ ways



Product Rule

Product Rule

Suppose a set consists of ordered collections of k elements (k-tuples) and that there are n_1 possible choices for the first element; for each choice of the first element, there are n_2 possible choices of the second element; ... ; for each possible choice of the first $k-1$ elements, there are n_k choices of the k-th element.

Then there are $n_1 n_2 \cdots n_k$ possible k-tuples.

Product Rule

Example

A homeowner doing some remodeling requires the services of both a plumbing contractor and an electrical contractor.

There are 12 plumbing contractors and 9 electrical contractors available in the area.

Suppose the home remodeling job involves first purchasing several kitchen appliances.

They will all be purchased from the same dealer, and there are five dealers in the area.

How many ways to choose first an appliance dealer, then a plumbing contractor, and finally an electrical contractor?

Permutation

Baseball

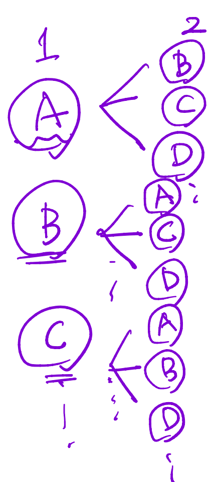
Question: If a Little League team has 15 players on its roster, how many ways are there to select 9 players to form a starting lineup?

1 2 3 4 . . . 9

$$\underbrace{15 \times 14 \times 13 \times \cdots \times 7}_{9 \text{ positions}}$$

$15 - 1 + 1$

$15 - 9 + 1$



Permutation

$\{a_1, a_2, \cdots, a_n\}$ distinct.

k with order

Permutation

An ordered subset is called a permutation.

The number of permutations of size k that can be formed from the n individuals or objects in a group will be denoted by $P_{k,n}$. Using the factorials,

$$P_{k,n} = n \times (n-1) \times \cdots \times (n-k+1).$$

$n \quad n-1 \quad \cdots \quad (n-k+1)$

k spots

k numbers

Permutation

10 TAs

Example

There are ten teaching assistants available for grading papers in a calculus course at a large university.

The first exam consists of four questions, and the professor wishes to select a different assistant to grade each question (only one assistant per question).

In how many ways can the assistants be chosen for grading?

1st 2nd 3rd 4th

$$P_{4,10} = 10 \times 9 \times 8 \times 7$$

Combination

Question: Each department has one representative on the college's student council.

suppose that three of the seven representatives are to be selected to attend a statewide convention.

How many ways are there to select the three?

7 → 3 Choose without order

X number of ways

Choose 3 out of 7 without order → give them order

A B C → □ □ □

$3 \times 2 \times 1 = 6 = 3!$

$X \cdot (3!) =$ Choose 3 out of 7 with order $= P_{3,7} = 7 \times 6 \times 5$

$$X = \frac{7 \times 6 \times 5}{3!} = 35.$$

Factorial: $n! = n \cdot (n-1) \cdot \cdots 3 \cdot 2 \cdot 1$

Combination

$$P_{k,n} = n \times (n-1) \times \cdots \times (n-k+1) = \frac{n \cdot (n-1) \cdots (n-k+1)(n-k) \cdots 2 \cdot 1}{(n-k)(n-k-1) \cdots 2 \cdot 1}$$

$$= \frac{n!}{(n-k)!}$$

Combination

An unordered subset is called a combination.

One way to denote the number of combinations is $C_{k,n}$, but we shall instead use notation that is quite common in probability books:

$$C_{k,n} = \binom{n}{k} = \frac{P_{k,n}}{k!} = \frac{n!}{(n-k)!\,k!}$$

"n choose k"

Combination

Every outcome is Equally Likely

Example

A university warehouse has received a shipment of 25 printers, of which 10 are laser printers and 15 are inkjet models.

If 6 of these 25 are selected at random to be checked by a particular technician,

what is the probability that exactly 3 of those selected are laser printers (so that the other 3 are inkjets)?

① $P(L = 3, I = 3) = \dfrac{\text{\# of outcomes in Event}}{\text{Total \# of outcomes}}$

$$= \frac{\binom{10}{3} \cdot \binom{15}{3}}{\binom{25}{6}} = \frac{\frac{10!}{3!(10-3)!} \cdot \frac{15!}{3!(15-3)!}}{\frac{25!}{6!(25-6)!}}$$

$$= \frac{\frac{10 \cdot 9 \cdot 8}{3 \cdot 2 \cdot 1} \cdot \frac{15 \cdot 14 \cdot 13}{3 \cdot 2 \cdot 1}}{\frac{25 \cdot 24 \cdot 23 \cdot 22 \cdot 21 \cdot 20}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}} = \text{Simplify it}$$

Please do this by 2:35

Exercise

(2.3-34) Computer keyboard failures can be attributed to electrical defects or mechanical defects.

A repair facility currently has 25 failed keyboards, 6 of which have electrical defects and 19 of which have mechanical defects.

1. How many ways are there to randomly select 5 of these keyboards for a thorough inspection (without regard to order)?
2. In how many ways can a sample of 5 keyboards be selected so that exactly two have an electrical defect?
3. If a sample of 5 keyboards is randomly selected, what is the probability that at least 4 of these will have a mechanical defect?

① $\binom{25}{5} = \frac{25!}{5!\,20!} = \frac{\overset{5}{\cancel{25}} \cdot \overset{6}{\cancel{24}} \cdot 23 \cdot 22 \cdot 21}{\cancel{5} \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot 1} = 5 \cdot 23 \cdot 22 \cdot 21$

② $\binom{6}{2} \cdot \binom{19}{3}$

2 ED 3 MD.

③ P (At least 4 MD) = P(4 MD) + P(5 MD)

$= \frac{\binom{19}{4} \cdot \binom{6}{1}}{\binom{25}{5}} + \frac{\binom{19}{5} \cdot \binom{6}{0}}{\binom{25}{5}}$

$$\binom{6}{0} = 1 = \frac{6!}{\underbrace{0!}_{=1}(6-0)!} = \frac{6!}{6!} = 1.$$

$$0! = 1$$

Section 4.
Conditional Probability

"Bayes' Rule"

Conditional Probability

Example

Complex components are assembled in a plant that uses two different assembly lines, A and A'.

Line A uses older equipment than A', so it is somewhat slower and less reliable.

Suppose on a given day line A has assembled 8 components, of which 2 have been identified as defective (B) and 6 as nondefective (B'),

whereas A' has produced 1 defective and 9 nondefective components.

If we randomly choose one component,

1. what is the probability that it is from the line A? $P(A) = \frac{8}{18}$
2. what is the probability that it is from the line A given that it is defective?

$A \longrightarrow$ 8 comp.

$A' \longrightarrow$ 10 comp.

	B	B'
A	2	6
A'	1	9

$$P(A \mid B) = \frac{2}{3}$$

Conditional Probability

B is a new Sample space

Conditional probability

For any two events A and B with $\mathbb{P}(B) \neq 0$,

the conditional probability of A given that B has occurred is defined by

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}.$$

Conditional Probability

Example

Suppose that of all individuals buying a certain digital camera, 60% include an optional memory card in their purchase, 40% include an extra battery, and 30% include both a card and battery.

Given that the selected individual purchased an extra battery, what is the probability that an optional card was also purchased?

A = { memory card } $P(A) = 0.6$

B = { battery } $P(B) = 0.4$

$P(A \cap B) = 0.3$

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{0.3}{0.4} = 0.75$$

$$\mathbb{P}(A \cap B) = \mathbb{P}(B) \cdot \mathbb{P}(A|B) = \mathbb{P}(A) \cdot \mathbb{P}(B|A)$$

$$\mathbb{P}(B) \cdot \mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\cancel{\mathbb{P}(B)}} \cdot \cancel{\mathbb{P}(B)}$$

The Multiplication Rule

$$\mathbb{P}(B|A) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)}$$

The Multiplication Rule

$$\mathbb{P}(A \cap B) = \mathbb{P}(A|B)\mathbb{P}(B) = \mathbb{P}(B|A)\mathbb{P}(A).$$

Note

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$$

$P(A_1) = 0.5$, $P(A_2) = 0.3$, $P(A_3) = 0.2$

$P(B|A_1) = 0.25$, $P(B|A_2) = 0.2$, $P(B|A_3) = 0.1$

$A_1 = \{\text{Brand 1}\}$, $A_2 = \{\text{Brand 2}\}$, $A_3 = \cdots$

$B = \{\text{Need repair}\}$

The Multiplication Rule

Example

A chain of video stores sells three different brands of DVD players. Of its DVD player sales, 50% are brand 1 (the least expensive), 30% are brand 2, and 20% are brand 3. Each manufacturer offers a 1-year warranty on parts and labor. It is known that 25% of brand 1's DVD players require warranty repair work, whereas the corresponding percentages for brands 2 and 3 are 20% and 10%, respectively.

1. What is the probability that a randomly selected purchaser has bought a brand 1 DVD player that will need repair while under warranty?
2. What is the probability that a randomly selected purchaser has a DVD player that will need repair while under warranty?
3. If a customer returns to the store with a DVD player that needs warranty repair work, what is the probability that it is a brand 1 DVD player? A ~~brand 2 DVD~~ player? A brand ~~3 DVD~~ player?

① $P(\text{Brand 1 \& Needs Repair})$

$= P(A_1 \cap B) = P(A_1) \cdot P(B|A_1)$

$= 0.5 \cdot 0.25$

② $P(B) = P(B \cap A_1) + P(B \cap A_2) + P(B \cap A_3)$

$= P(A_1) \cdot P(B|A_1) + P(A_2) \cdot P(B|A_2) + P(A_3) \cdot P(B|A_3)$.

why?

A_1, A_2, A_3 : disjoint (mutually exclusive) exhaustive

$$\Rightarrow \quad B = \underline{(B \cap A_1)} \cup \underline{(B \cap A_2)} \cup \underline{(B \cap A_3)}$$

disjoint

why?

$$\Rightarrow \quad P(B) = P(B \cap A_1) + P(B \cap A_2) + P(B \cap A_3)$$

③ $$P(A_1 \mid B) \underset{\text{def.}}{=} \frac{P(A_1 \cap B) \leftarrow ①}{P(B) \leftarrow ②}$$

$$= \frac{P(B \mid A_1) \cdot P(A_1)}{P(B \mid A_1) \cdot P(A_1) + P(B \mid A_2) \cdot P(A_2) + P(B \mid A_3)\, P(A_3)}$$

Bayes' Rule

The Multiplication Rule

Example

There are 4 green balls and 6 red balls in a box. Choose 2 balls at random without replacement.

1. What is the probability that the first ball is green and the second ball is red?
2. What is the probability that the second ball is red?
3. What is the conditional probability that the first ball is green given that the second ball is red?

① $\underbrace{\frac{4}{10}}_{=P(A)} \cdot \underbrace{\frac{6}{9}}_{=P(B|A)} = P(A \cap B)$

$A = \{ 1^{st} = G \}$ $\quad B = \{ 2^{nd} = R \}$

② $P(B) = P(1^{st} = G \;\&\; 2^{nd} = R) + P(1^{st} = R, 2^{nd} = R)$

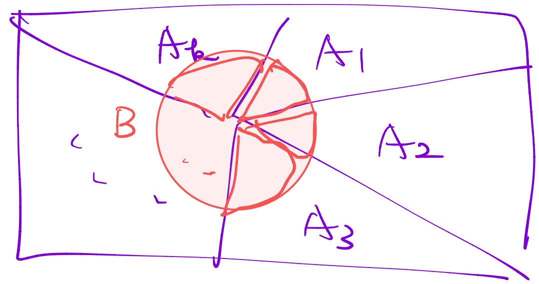
$= P(B \cap A) + P(B \cap A^c)$

$= P(B|A) \cdot \underline{P(A)} + P(B|A^c) \cdot \underline{P(A^c)}$

$= \frac{6}{9} \cdot \frac{4}{10} + \frac{5}{9} \cdot \frac{6}{10} = \frac{24+30}{90} = \frac{6}{10}.$

③ $\mathbb{P}(A \mid B) = \dfrac{\mathbb{P}(B \cap A)}{\mathbb{P}(B)}$

$$= \frac{\dfrac{4 \cdot \cancel{6}}{\cancel{10} \cdot 9}}{\dfrac{\cancel{6}}{\cancel{10}}} = \frac{4}{9}.$$



Bayes’ Theorem

$$B = (B \cap A_1) \cup (B \cap A_2) \cup \cdots \cup (B \cap A_k)$$

The Law of Total Probability (LoTP)

Let $A_1, \cdots, A_k$ be mutually exclusive and exhaustive events. Partition

Then for any other event B,

$$\mathbb{P}(B) = \mathbb{P}(A_1 \cap B) + \cdots + \mathbb{P}(A_k \cap B) = \mathbb{P}(B|A_1)\mathbb{P}(A_1) + \cdots + \mathbb{P}(B|A_k)\mathbb{P}(A_k).$$

Bayes' Theorem

Example

An individual has 3 different email accounts. Most of her messages, in fact 70%, come into account #1, whereas 20% come into account #2 and the remaining 10% into account #3. Of the messages into account #1, only 1% are spam, whereas the corresponding percentages for accounts #2 and #3 are 2% and 5%, respectively.

What is the probability that a randomly selected message is spam?

$A_1 = \{$ random message to #1 $\}$

$A_2 = \{$ " to #2 $\}$

$A_3 = \{$ " to #3 $\}$

$B = \{$ " is spam $\}$

$\mathbb{P}(A_1) = 0.7$, $\mathbb{P}(A_2) = 0.2$, $\mathbb{P}(A_3) = 0.1$

$\mathbb{P}(B \mid A_1) = 0.01$, $\mathbb{P}(B \mid A_2) = 0.02$, $\mathbb{P}(B \mid A_3) = 0.05$

$$\mathbb{P}(B) = \mathbb{P}(A_1 \cap B) + \mathbb{P}(A_2 \cap B) + \mathbb{P}(A_3 \cap B)$$

$$= \mathbb{P}(A_1) \cdot \mathbb{P}(B \mid A_1) + \mathbb{P}(A_2) \cdot \mathbb{P}(B \mid A_2) + \mathbb{P}(A_3) \cdot \mathbb{P}(B \mid A_3)$$

Bayes' Theorem

Bayes' Theorem

Let $A_1, \cdots, A_k$ be mutually exclusive and exhaustive events with prior probabilities $\mathbb{P}(A_i)$, $i = 1, 2, \cdots, k$.

Then for any other event B with $\mathbb{P}(B) \neq 0$, the posterior probability of A_j given that B has occurred is

$$\mathbb{P}(A_j|B) \underset{\text{def}}{=} \frac{\mathbb{P}(B \cap A_j)}{\mathbb{P}(B)}$$

$$= \frac{\mathbb{P}(B|A_j) \cdot \mathbb{P}(A_j)}{\mathbb{P}(B|A_1)\mathbb{P}(A_1) + \mathbb{P}(B|A_2)\mathbb{P}(A_2) + \cdots + \mathbb{P}(B|A_k)\mathbb{P}(A_k)}$$

Bayes' Theorem

Example

Only 1 in 1000 adults is afflicted with a rare disease for which a diagnostic test has been developed.

The test is such that when an individual actually has the disease, a positive result will occur 99% of the time, whereas an individual without the disease will show a positive test result only 2% of the time.

If a randomly selected individual is tested and the result is positive, what is the probability that the individual has the disease?

Note $P(B \mid C_1) + P(B^c \mid C_1) = 1$

Please do this by 2:49

Exercise

(2.4-60) Seventy percent of the light aircraft that disappear while in flight in a certain country are subsequently discovered.

Of the aircraft that are discovered, 60% have an emergency locator, whereas 90% of the aircraft not discovered do not have such a locator.

Suppose a light aircraft has disappeared.

1. If it has an emergency locator, what is the probability that it will not be discovered?
2. If it does not have an emergency locator, what is the probability that it will be discovered?

$A = \{ \text{discovered} \}$

$B = \{ \text{has locator} \}$

$P(A) = 0.7$, $P(B \mid A) = 0.6$

$P(B^c \mid A^c) = 0.9$

1. $P(A^c \mid B) = \dfrac{P(A^c \cap B)}{P(B)} = \dfrac{P(B \mid A^c) \cdot P(A^c)}{P(B \mid A) P(A) + P(B \mid A^c) \cdot P(A^c)}$

$= \dfrac{(1 - P(B^c \mid A^c)) \cdot (1 - P(A))}{\text{''}}$

2. $\mathbb{P}(A \mid B^c)$ = Similar computation.

Section 5. Independence

$$\mathbb{P}(B) = \underline{\mathbb{P}(B \mid A)} = \mathbb{P}(B \mid A^c)$$

$$\parallel$$

$$\frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)}$$

Independence

Independence

Two events A and B are independent if

$$\boxed{\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B),}$$

are dependent otherwise.

Note that if A, B are independent and $\mathbb{P}(B) \neq 0$, then

$$\underline{\mathbb{P}(A|B)} = \mathbb{P}(A)$$

__Note__ A, B are disjoint if $A \cap B = \phi$

$$\mathbb{P}(A \cap B) = 0$$

Independence

$$\boxed{P(A \cap B) = P(A) \cdot P(B)}$$

Property

If two events A and B are independent, then so are

1. A and B',
2. A' and B,
3. A' and B'.

??

① WANT: $P(A \cap B^c) + P(A \cap B) = P(A)$

$$P(A \cap B^c) = P(A) - P(A \cap B)$$
$$= P(A) - P(A)P(B)$$
$$= P(A)(1 - P(B))$$
$$= P(A) \cdot P(B^c)$$

Recall Two events A, B are independent if $\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B)$

Note In general, (multiplication rule)

$$\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B \mid A) = \mathbb{P}(B) \cdot \mathbb{P}(A \mid B)$$

Independence

Example

Roll a die twice. Let A be the event that the sum is 7, and B be the event that the first outcome is even. Are they independent?

$\mathbb{P}(A) = \frac{6}{36} = \frac{1}{6}$ $\mathbb{P}(B) = \frac{1}{2}$

1, 6
$\boxed{2, 5}$
3, 4
$\boxed{4, 3}$
5, 2
$\boxed{6, 1}$

$$\mathbb{P}(A \cap B) = \frac{3}{36} = \frac{1}{12} = \frac{1}{6} \cdot \frac{1}{2} = \mathbb{P}(A) \cdot \mathbb{P}(B)$$

A, B are independent.

$C = \{\text{sum} = 6\}$ B, C ~~indep~~?

$= \{(1,5), (2,4), (3,3), (4,2), (5,1)\}$

$\mathbb{P}(C) = \frac{5}{36}$

$\mathbb{P}(B \cap C) = \frac{2}{36} = \frac{1}{18}$

More Than Two Events

Independence - More Than Two Events

Events $A_1, \cdots, A_n$ are mutually independent if every subset of indices $\{i_1, \cdots, i_k\} \subset \{1, 2, \cdots, n\}$,

$$\mathbb{P}(A_{i_1} \cap \cdots \cap A_{i_k}) = \mathbb{P}(A_{i_1}) \cdots \mathbb{P}(A_{i_k}).$$

For example, A, B, C are mutually independent if

$$\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B)$$

$$\mathbb{P}(B \cap C) = \mathbb{P}(B) \cdot \mathbb{P}(C)$$

$$\mathbb{P}(C \cap A) = \mathbb{P}(A) \cdot \mathbb{P}(C)$$

$$\mathbb{P}(A \cap B \cap C) = \mathbb{P}(A)\, \mathbb{P}(B)\, \mathbb{P}(C)$$

Four Events? A_1, A_2, A_3, A_4 mutually indep.

if $\mathbb{P}(A_1 \cap A_2) = \mathbb{P}(A_1) \cdot \mathbb{P}(A_2)$ --- $6 = \binom{4}{2}$

$\mathbb{P}(A_1 \cap A_2 \cap A_3) = \; ---$ --- $4 = \binom{4}{3}$

$\mathbb{P}(A_1 \cap A_2 \cap A_3 \cap A_4) = \; ---$ --- $1 = \binom{4}{4}$

Def $A_1, \dots, A_n$ pairwise indep.

if A_i, A_j are indep. for all $i \neq j \in \{1, \dots, n\}$

Please Do this by 2:25

Exercise

(2.5-82) Consider independently rolling two fair dice, one red and the other green. Let

$A = \{\text{the red die shows 3 dots}\}$

$B = \{\text{the green die shows 4 dots}\}$

$C = \{\text{the total number of dots showing on the two dice is 7}\}$

(1, 6) (2, 5) (3, 4) (4, 3) (5, 2), (6, 1)

1. Are these events pairwise independent? (i.e., are A and B independent events, are A and C independent, and are B and C independent) Yes
2. Are the three events mutually independent?

$\mathbb{P}(A) = \frac{1}{6} = \mathbb{P}(B) = \frac{1}{6}$, $\mathbb{P}(A \cap B) = \frac{1}{36}$

$\mathbb{P}(C) = \frac{1}{6}$, $\mathbb{P}(B \cap C) = \frac{1}{36}$

$\mathbb{P}(A \cap B \cap C) = \frac{1}{36} \neq \frac{1}{6} \cdot \frac{1}{6} \cdot \frac{1}{6}$

Exercise