Chapter 3. Continuous Distribution

Math 3215 Spring 2024

Georgia Institute of Technology

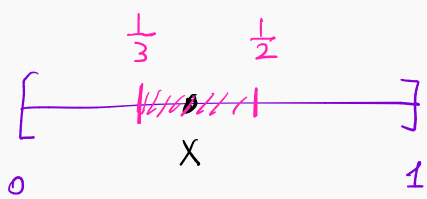
Section 1.
Random Variables of the Continuous Type

Continuous Random Variables

Let the random variable X denote the outcome when a point is selected at random from an interval $[0,1]$.

If the experiment is performed in a fair manner, it is reasonable to assume that the probability that the point is selected from an interval $[\frac{1}{3}, \frac{1}{2}]$ is

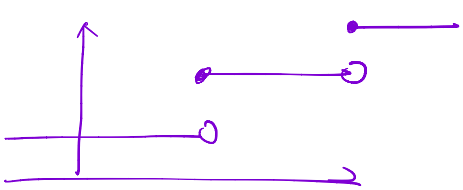
The CDF of X is

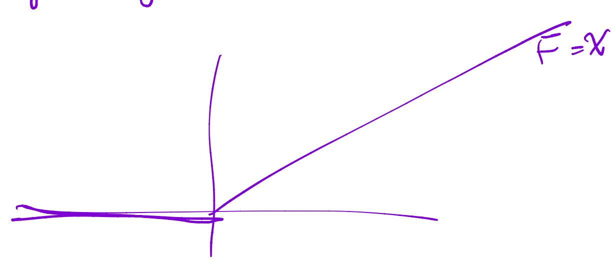


$$\mathbb{P}(X \in [\tfrac{1}{3}, \tfrac{1}{2}]) = \frac{\text{length of } [\frac{1}{3}, \frac{1}{2}]}{\text{length of } [0,1]} = \frac{1}{6}.$$

$$F_X(x) = \mathbb{P}(X \leq x) = \frac{\text{length of } [0, x]}{\text{length of } [0,1]} = x.$$

Discrete Case





Continuous Random Variables

Definition

We say a random variable X on a sample space S is a continuous random variable if there exists a function $f(x)$ such that

- $f(x) \geq 0$ for all x,
- $\int_{S(X)} f(x)\,dx = 1$, and
- For any interval $(a,b) \subset \mathbb{R}$, $\mathbb{P}(a \leq X < b) = \mathbb{P}(a < X \leq b)$

$$\mathbb{P}(a \leq X \leq b) = \mathbb{P}(a < X < b) = \int_a^b f(x)\,dx.$$

The function $f(x)$ is called **the probability density function (PDF)** of X.

density

Note For a continuous RV, $\mathbb{P}(X = a) = 0$.

$$\mathbb{P}(X = a) = \lim_{\varepsilon \to 0} \mathbb{P}(a - \varepsilon < X < a + \varepsilon)$$

$$= \lim_{\varepsilon \to \infty} \int_{a-\varepsilon}^{a+\varepsilon} f(x)\,dx = 0.$$

Note There many RVs $\begin{cases} \text{not discrete and.} \\ \text{not continuous} \end{cases}$

Discrete : $\mathbb{E}[X] = \sum x f(x)$

Continuous Random Variables

The CDF of X is $F(x) = \mathbb{P}(X \leq x) = \mathbb{P}(X \in (-\infty, x]) = \int_{-\infty}^{x} f(t)\, dt$

The expectation (mean) of X is $\mu = \mathbb{E}[X] = \int_{-\infty}^{\infty} x f(x)\, dx$

The variance of X is $Var(X) = \mathbb{E}[(X-\mu)^2] = \int_{-\infty}^{\infty} (x-\mu)^2 f(x) dx = \mathbb{E}[X^2] - \mathbb{E}[X]^2$

$= \int_{-\infty}^{\infty} x^2 f(x)\, dx - \left(\int_{-\infty}^{\infty} x f(x) dx\right)^2$

The standard deviation of X is $Std(X) = \sqrt{Var(X)}$

The moment generating function of X is

$$M(t) = \mathbb{E}[e^{tX}] = \int_{-\infty}^{\infty} e^{tx} f(x)\, dx.$$

Practice (or Recall) basic Integration.

e^x, $\sin x$, $\cos x$, $\ln x$, x^n, $\frac{1}{x+c}$, ...

Integration by parts

⋮

Continuous Random Variables

PMF = $\mathbb{P}(X = x) \leq 1$

Properties

The PMF of a discrete random variable is bounded by 1. But for PDF, $f(x)$ can be greater than 1.

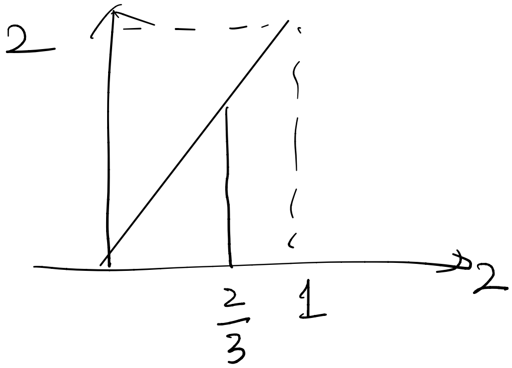
For CDF F, we have $F'(x) = f(x)$ where F is differentiable at x.

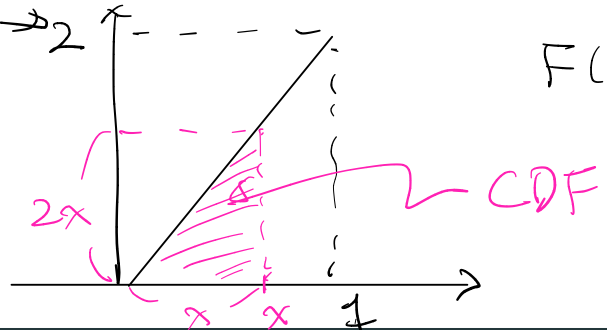
Derivative of CDF.

$$F(x) = \int_{-\infty}^{x} f(t)\,dt$$

$$F'(x) = \frac{d}{dx}\int_{-\infty}^{x} f(t)\,dt = f(x)$$

Fundamental Thm of Calculus.





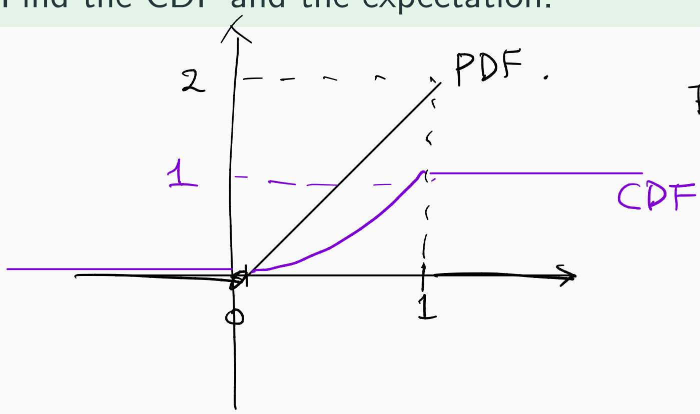
$$F(x) = \int_{-\infty}^{x} f(t)\,dt$$

Continuous Random Variables

Example

Let X be a continuous random variable with a PDF $f(x) = 2x$ for $0 < x < 1$.

Find the CDF and the expectation.



$$F(x) = P(X \le x)$$

$$= \int_{-\infty}^{x} f(t)\,dt$$

$$= \begin{cases} 0, & x < 0 \\ 1, & x > 1 \\ x^2 & 0 \le x \le 1 \end{cases}$$

$$1 = \int_{-\infty}^{\infty} f(x)\,dx$$

$$\int_0^x 2t \cdot dt = \left[t^2\right]_0^x = x^2$$

$$E[X] = \int_{-\infty}^{\infty} x \cdot f(x)\,dx = \int_0^1 \underbrace{x \cdot (2x)}_{2x^2}\,dx = \left[\tfrac{2}{3}x^3\right]_0^1$$

$$= \frac{2}{3}.$$

f to be a PDF $\begin{cases} \bullet \quad f(x) \geq 0 \\ \bullet \quad \int_{-\infty}^{\infty} f(x)\,dx = 1 \end{cases}$

Continuous Random Variables

Example

Let X have the PDF $f(x) = xe^{-x}$. Find the MGF.

$x > 0$

Integration by parts.

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_0^{\infty} x e^{-x}\,dx =$$

$$(u \cdot v)' = u' \cdot v + \underline{u \cdot v'}$$

$$\int u'v = u \cdot v - \int u \cdot v'$$

$v = x \qquad v' = 1$

$u' = e^{-x} \qquad u = -e^{-x}$

$$\int_0^{\infty} x e^{-x}\,dx = \underbrace{\left[-x e^{-x}\right]_0^{\infty}} + \underbrace{\int_0^{\infty} e^{-x}\,dx}$$

$$= 0 + \left[-\underbrace{e^{-x}}\right]_0^{\infty} = 1.$$

Continuous Random Variables

Example

Let X have the PDF $f(x) = xe^{-x}$. Find the MGF.

$$M(t) = \mathbb{E}[e^{tX}] = \int_{-\infty}^{\infty} e^{tx} f(x)\, dx = \int_0^{\infty} x e^{-x} e^{tx}\, dx = \int_0^{\infty} x e^{-(1-t)x}\, dx$$

$$(u \cdot v)' = u' \cdot v + \underline{u \cdot v'}$$

$$\int u'v = u \cdot v - \int u \cdot v'$$

$$v = x \qquad v' = 1$$

$$u' = e^{-(1-t)x} \qquad u = -\frac{1}{(1-t)} \cdot e^{-(1-t)x}$$

$$\int_0^{\infty} x e^{-(1-t)x}\, dx = \left[-x e^{-(1-t)x}\right]_0^{\infty} + \int_0^{\infty} \frac{1}{(1-t)} e^{-(1-t)x}\, dx$$

$$= 0 - 0 + \left[-\frac{1}{(1-t)^2} e^{-(1-t)x}\right]_0^{\infty}$$

$$= \begin{cases} \frac{1}{(1-t)^2}, & (1-t) > 0 \quad t < 1 \\ \text{DNE}. & t \geq 1 \end{cases}$$

Uniform Random Variables

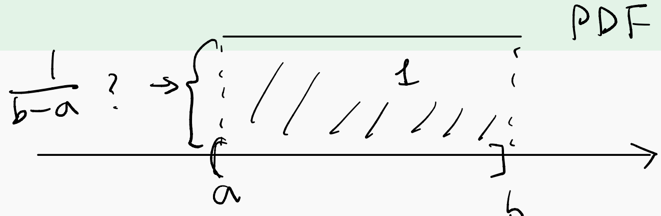
Definition

X is a uniform random variable if its PDF is constant on its support.

If its support is $[a, b]$, then the PDF is

We denote by $X \sim U(a, b)$.

PDF.



$$f(x) = \begin{cases} \frac{1}{b-a}, & a \leq x \leq b \\ 0, & \text{o.w.} \end{cases}$$

Ex : $X \sim U(1,3) = \mathrm{Unif}(1,3)$.

Uniform Random Variables

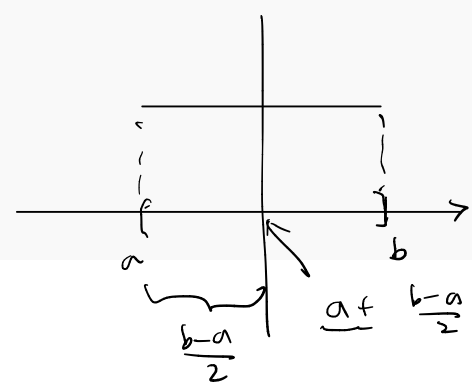
Theorem

If $X \sim U(a,b)$, then

$$\mathbb{E}[X] = \frac{a+b}{2} = \int_{-\infty}^{\infty} x f(x)\,dx = \int_a^b x \cdot \frac{1}{b-a}\,dx = \mu$$

$$\mathrm{Var}[X] = \frac{(b-a)^2}{12}$$

$$M(t) = \begin{cases} \dfrac{e^{tb} - e^{ta}}{t(b-a)} & \text{if } t \neq 0 \\ 1. & \text{if } t = 0 \end{cases}$$



$$a + \frac{b-a}{2} = \frac{b+a}{2}$$

$$\mathrm{Var}(X) = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx$$

$$= \int_a^b (x-\mu)^2 \frac{1}{(b-a)}\,dx = \frac{1}{b-a}\left[\frac{1}{3}(x-\mu)^3\right]_a^b$$

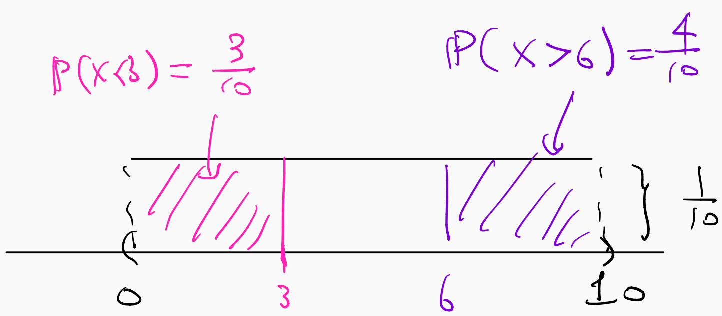
$$= \frac{1}{3(b-a)}\left[(b-\mu)^3 - (a-\mu)^3\right]$$

$$= \frac{1}{3(b-a)}\left[\left(\frac{b-a}{2}\right)^3 - \left(\frac{a-b}{2}\right)^3\right] = \frac{2}{3(b-a)} \cdot \frac{(b-a)^3}{8} = \frac{(b-a)^2}{12}$$

Uniform Random Variables

Example

If X is uniformly distributed over $(0, 10)$, calculate $\mathbb{P}(X < 3)$, $\mathbb{P}(X > 6)$, and $\mathbb{P}(3 < X < 8)$.



Recall

- X is a Conti. RV if there exists a PDF.
- $f(x)$ is a PDF if $\begin{cases} f(x) \geq 0 & \text{for all } x \\ \int_{\mathbb{R}} f(x)\,dx = 1 \\ P(a < X < b) = \int_a^b f(x)\,dx. \end{cases}$
- $X \sim \text{Unif}(a, b)$ if $f(x) = \begin{cases} \frac{1}{b-a} & a < x < b \\ 0 & \text{o.w.} \end{cases}$

Uniform Random Variables

Example

A bus travels between the two cities A and B, which are 100 miles apart.

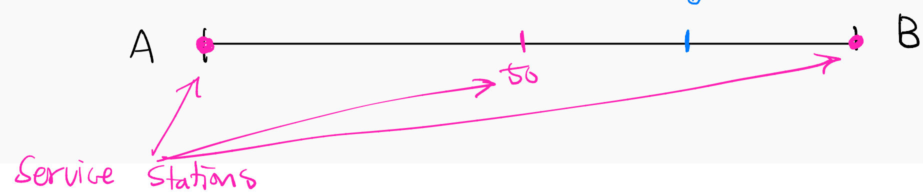
If the bus has a breakdown, the distance from the breakdown to city A has a $U(0, 100)$ distribution.

There are bus service stations in city A, in B, and in the center of the route between A and B.

It is suggested that it would be more efficient to have the three stations located 25, 50, and 75 miles, respectively, from A.

Do you agree? Why?

X breakdown $\sim U(0, 100)$

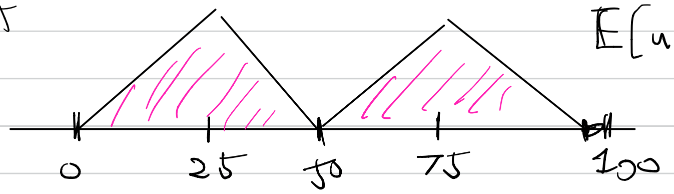


Idea: Optimize $\underbrace{\text{the distance}}_{= u(X)}$ from X to the nearest station.

$$u(X) = \begin{cases} X, & X \leq 25 \\ |X - 50|, & 25 < X < 75 \\ 100 - X & X \geq 75 \end{cases}$$

$$E[u(X)] = \int u(x) \cdot f(x)\,dx = \int_0^{\infty} u(x) \cdot \frac{1}{100}\,dx = \frac{1}{100} \cdot \int_0^{\infty} u(x)\,dx.$$

Plan A



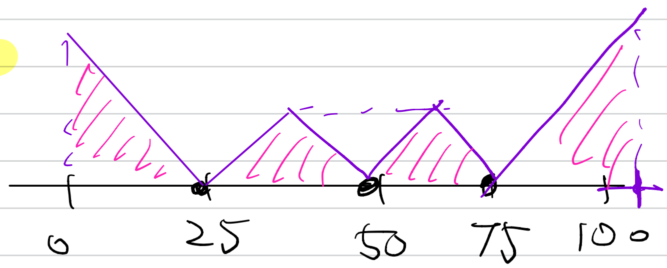
$u(x)$

$$\mathbb{E}[u(x)] = \frac{1}{100}\int_0^\infty u\,dx = \frac{25^2\cdot 2}{100} = \frac{25}{2} = 12.5$$

Plan B

↑

Better than Plan A.



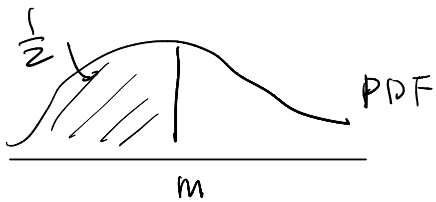
$$\text{Area} = 25^2 + 25\cdot\frac{25}{2} = 25^2\cdot\left(1+\frac{1}{2}\right)$$

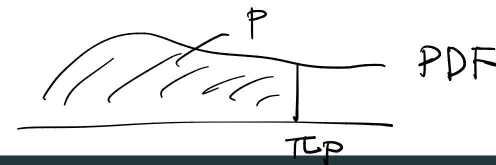
$$\mathbb{E}[u(x)] = \frac{1}{100}\cdot\text{Area} = \frac{25\cdot\cancel{25}\cdot\frac{3}{2}}{\cancel{25}\cdot 4}$$

$$= 25\cdot\frac{3}{8} < 12.5$$

m is median ($m = \pi_{\frac{1}{2}}$) if $F(m) = \frac{1}{2}$.

In general, π_p is $(100p)^{th}$ percentile





Percentile

Ex 50^{th} percentile $\underbrace{}_{}$ = median $(p = \frac{1}{2})$, 75^{th} percentile $(p = \frac{3}{4})$.

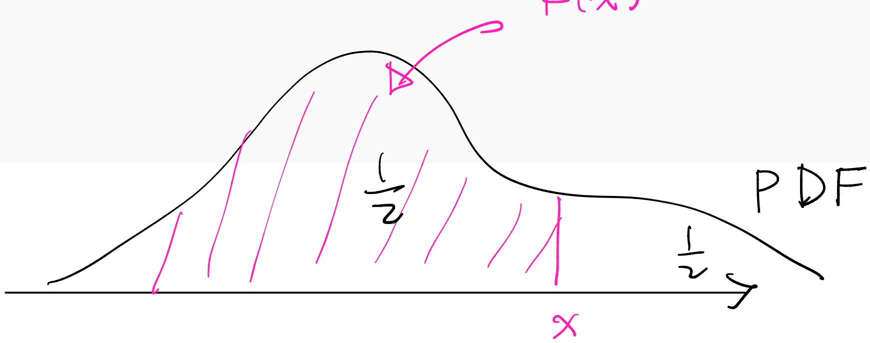
The $(100p)$-th percentile is a number π_p such that $F(\pi_p) = p$.

CDF.

For example, the 50th percentile is the number $\pi_{\frac{1}{2}} = q_2$ such that $F(\pi_{\frac{1}{2}}) = \frac{1}{2}$ and this is called the median.

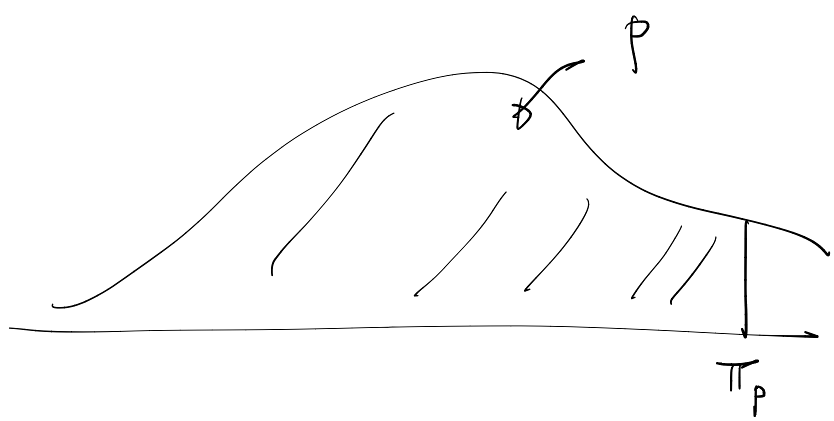
The 25th and 75th percentiles are called the first and third quartiles, respectively, and are denoted by $q_1 = \pi_{0.25}$ and $q_3 = \pi_{0.75}$.

$F(x) = P(X \leq x)$



$F(x) = \frac{1}{2}$

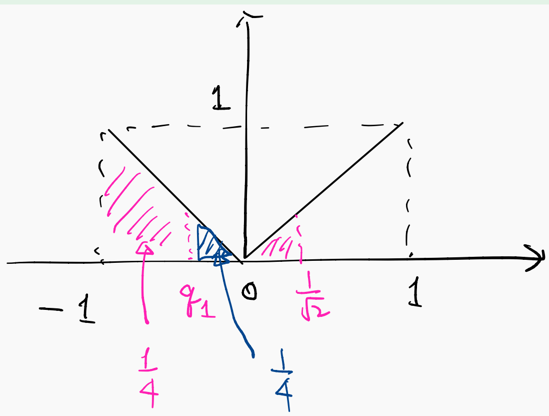
$\pi_{\frac{1}{2}}$



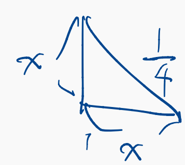
Percentile

Example

Let X be a continuous random variable with PDF $f(x) = |x|$ for $-1 < x < 1$. Find q_1, q_2, q_3.



q_1 ?



$$\frac{1}{2} \cdot x^2 = \frac{1}{4}$$

$$x^2 = \frac{1}{2}$$

$$x = \frac{1}{\sqrt{2}}.$$

$$\Rightarrow \quad q_1 = -\frac{1}{\sqrt{2}}$$

$q_2 = m = 0$ by symm.

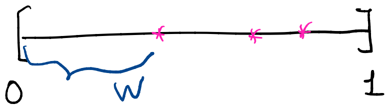
$q_3 = \frac{1}{\sqrt{2}}$ "

Exercise

Let $f(x) = c\sqrt{x}$ for $0 \leq x \leq 4$ be the PDF of a random variable X.

Find c, the CDF of X, and $\mathbb{E}[X]$.

Section 2.
The Exponential, Gamma, and Chi-Square Distributions



$X = \#$ of customers $\sim \text{Pois}(\lambda)$

$\lambda =$ average # in 1 hr.

$W =$ waiting time until 1st customer $\quad (F'(t) = f(t))$

Find CDF. $\quad \mathbb{P}(W \le t) = 1 - \mathbb{P}(W > t)$

Exponential random variables

Consider a Poisson random variable X with parameter λ.

This represents the number of occurrances in a given interval, say $[0, 1]$.

If $\lambda = 5$, that means the expected number of occurrances in $[0, 1]$ is 5.

Let W be the waiting time for the first occurrence. Then,

$$\mathbb{P}(W > t) = \mathbb{P}(\text{no occurrences in } [0, t]) = e^{-\lambda t} \cdot \frac{(\lambda t)^0}{0!}$$

for $t > 0$.

(# of customers in $[0, t]$ $\sim \text{Pois}(\lambda t)$)

$= e^{-\lambda t}$

$F(t) = 1 - e^{-\lambda t}$

$f(t) = F'(t) = \lambda \cdot e^{-\lambda t}$. $\quad$ Exponential RV.

Exponential random variables

Definition

We say X is **an exponential random variable** with parameter λ (or mean θ where $\lambda = \frac{1}{\theta}$) if its pdf is

$$f(x) = \lambda e^{-\lambda x} \quad = \frac{1}{\theta} e^{-\frac{x}{\theta}}$$

for $x \geq 0$ and otherwise 0. Here, λ is the parameter and θ is the mean.

$$\mathbb{E}[X] = \theta = \frac{1}{\lambda}.$$

Exponential random variables

Theorem

Suppose that X is an exponential random variable with parameter $\lambda = \frac{1}{\theta}$.

$\mathbb{E}[X] = \frac{1}{\lambda} = \theta$

$\mathrm{Var}[X] = \frac{1}{\lambda^2} = \theta^2$

$M(t) = \frac{\lambda}{\lambda - t} = \frac{1}{1-\theta t}$

Exercise.

$$\mathbb{E}[X] = \int_{\mathbb{R}} x \cdot f(x)\, dx = \int_0^\infty \lambda x \cdot e^{-\lambda x}\, dx$$

$u = \lambda x$

$du = \lambda\, dx$

$dx = \frac{1}{\lambda} \cdot du$

$$= \frac{1}{\lambda} \int_0^\infty u \cdot e^{-u}\, du = \frac{1}{\lambda}.$$

$$\int_0^\infty \underbrace{u}_{1} \cdot \overbrace{e^{-u}}^{-e^{-u}}\, du = \left[u \cdot (-e^{-u}) \right]_0^\infty - \int_0^\infty 1 \cdot (-e^{-u})\, du = 1.$$

Integration by Parts.

Exercise.

Exponential random variables

$$f(x) = \frac{1}{\theta} e^{-\frac{x}{\theta}} = \lambda e^{-\lambda x}$$

$$F(x) = 1 - e^{-\frac{x}{\theta}} = 1 - e^{-\lambda t}$$

Example

Let X have an exponential distribution with a mean $\theta = 20$.

Find $\mathbb{P}(X < 18)$.

$$F(18) = \mathbb{P}(X < 18) = \int_0^{18} \frac{1}{20} e^{-\frac{x}{20}} dx = \left[\frac{1}{20}\left(-20 e^{-\frac{x}{20}}\right)\right]_0^{18}$$

$$= -e^{-\frac{18}{20}} + 1 = 1 - e^{-\frac{18}{20}}$$

Note : $\mathbb{P}(X > t) = e^{-\lambda t} = e^{-\frac{t}{\theta}}$

Exponential random variables

Example

Customers arrive in a certain shop according to an approximate Poison process at a mean rate of 20 per hour.

What is the probability that the shopkeeper will have to wait more than five minutes for the arrival of the first customer?

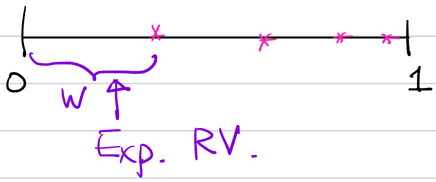
X = # of customers $\sim$ Pois (20)

W = waiting time for 1st customer in hr

$\sim$ Exp with $\lambda = 20$ $\quad \theta = \frac{1}{20}$

$$P\left(W > \frac{5}{60}\right) = e^{-\lambda \cdot \frac{5}{60}} = e^{-\frac{5}{3}}.$$

Exp. Random Variable. = the waiting time for 1st occurance for Poisson.



Poisson with λ

① $f_W(t) = \lambda e^{-\lambda t} = \frac{1}{\theta} e^{-\frac{t}{\theta}}$, $t \geq 0$

② $\mathbb{E}[W] = \frac{1}{\lambda} = \theta$

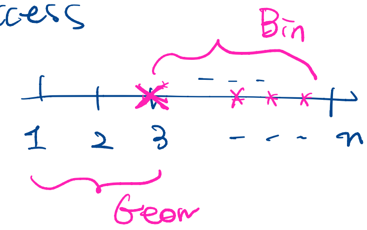
③ $Var(W) = \frac{1}{\lambda^2} = \theta^2$

④ $\mathbb{P}(W > t) = e^{-\lambda t}$

Bin : # of success in n trials

Geom : # of trials until 1^{st} success

NegBin : # of trials until r^{th} success



Poisson : # of customer

Exp : Waiting Time for 1^{st} Customer.

??

Gamma random variables

Consider a Poisson random variable X with λ.

Let W be the waiting time until α-th occurrences, then its CDF is

$$F(t) = \mathbb{P}(W \leq t) = 1 - \mathbb{P}(W > t) = 1 - \sum_{k}^{\alpha-1} \frac{(\lambda t)^k e^{-\lambda t}}{k!}.$$

Thus, the PDF is

$$f(x) = \frac{\lambda(\lambda x)^{\alpha-1}}{(\alpha-1)!} e^{-\lambda x}.$$

This random variable is called **a gamma random variable** with λ and α where $\lambda = \frac{1}{\theta} > 0$.

This can be extended to non-integer $\alpha > 0$.

Gamma functions

The gamma function is defined by

$$\Gamma(t) = \int_0^\infty \underbrace{y^{t-1}}_{(t-1)\cdot y^{t-2}} \overbrace{e^{-y}}^{-e^{-y}} \, dy$$

for $t > 0$.

By integration by parts, we have

$$\Gamma(t) = \left[y^{t-1} \cdot (-e^{-y}) \right]_0^{\infty \leftarrow N} - \int_0^\infty (t-1)\, y^{t-2} \cdot (-e^{-y})\, dy$$

$$= \lim_{N\to\infty} \left(\underbrace{N^{t-1} \cdot (-e^{-N})}_{\to 0} - 0 \right) + (t-1)\int_0^\infty y^{t-2} e^{-y}\, dy$$

$$= (t-1) \cdot \underbrace{\int_0^\infty y^{(t-1)-1} \cdot e^{-y}\, dy}_{= \Gamma(t-1)}$$

$$\boxed{\Gamma(t) = (t-1)\,\Gamma(t-1)}$$

$$\Gamma(t) = \int_0^\infty y^{t-1} e^{-y} dy$$
$$\Gamma(t) = (t-1)\,\Gamma(t-1)$$

Gamma functions

In particular, $\Gamma(1) = \int_0^\infty y^{1-1} e^{-y} dy = \int_0^\infty e^{-y} dy = \left[-e^{-y}\right]_0^\infty = 1.$

$\Gamma(2) = 1 \cdot \Gamma(1) = 1.$

$\Gamma(3) = 2 \cdot \Gamma(2) = 2 \cdot 1 = 2$ $\qquad \Gamma(4) = 3 \cdot \Gamma(3) = 3 \cdot 2 \cdot 1 = 3!$

$\Gamma(n) = (n-1)\,\underbrace{\Gamma(n-1)} = (n-1)(n-2)\,\Gamma(n-2) = \cdots = (n-1)(n-2)\cdots 2 \cdot 1$
$= (n-1)!$

for integers n.

Gamma Function = Generalized Factorial.

$\Gamma(\frac{1}{2}) = \cdots$ or $\Gamma(\frac{3}{2}) = \cdots$

$$\int y^{\frac{1}{2}-1} e^{-y} dy$$

any positive #.

Def $\quad X \sim \text{Gamma}(\lambda, \alpha)$

$$f_X(x) = \frac{\lambda^\alpha x^{\alpha-1}}{\Gamma(\alpha)} e^{-\lambda x}$$

Ex. $\quad X \sim$ Normal $\qquad x^2$

Gamma random variables

Theorem

$\mathbb{E}[X] = \frac{\alpha}{\lambda}$ = $\alpha \cdot$ (Expectation of Exp.)

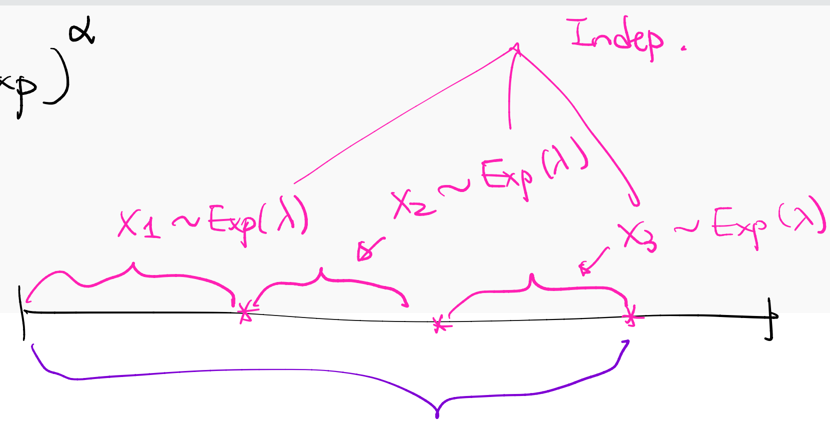
$\mathrm{Var}[X] = \frac{\alpha}{\lambda^2}$ = $\alpha \cdot$ (Variance of Exp)

$M(t) = \frac{1}{(1-\theta t)^\alpha}$ for $t \leq \frac{1}{\theta}$.

= (MGF of Exp $)^\alpha$

Heuristic Reason

$\alpha = 3$.



$X = X_1 + X_2 + X_3$ $\qquad X \sim \mathrm{Gamma}(\lambda, 3)$

Gamma = Indep Sum of Exp.

$W \sim \text{Gamma}(\lambda, n)$ $\leftarrow$ positive integer

$X_t \sim \text{Pois}(\lambda t)$

$$\Rightarrow \quad \underbrace{\mathbb{P}(W > t)}_{\text{wait more than } t \text{ for } n^{th} \text{ customer}} = \mathbb{P}(\underbrace{X_t}_{\#\text{ of customers in } [0,t]} < n)$$

Gamma random variables

Example

Suppose the number of customers per hour arriving at a shop follows a Poisson random variable with mean 20.

That is, if a minute is our unit, then $\lambda = \frac{1}{3}$.

What is the probability that the second customer arrives more than five minutes after the shop opens for the day?

$$W \boxed{\sim \text{Gamma}(\overset{\lambda}{\tfrac{1}{3}}, \overset{\alpha}{2})}$$

means

$$f_W(t) = \frac{\lambda^\alpha t^{\alpha-1}}{\Gamma(\alpha)} e^{-\lambda t}$$

$X \sim \text{Exp}(1)$, $Y \sim \text{Exp}(1)$

$X = Y \Leftrightarrow \mathbb{P}(X=Y)=1$

$X \sim Y$

different meaning.

$$\mathbb{P}(W > 5) = \int_5^\infty \left(\frac{1}{3}\right)^2 \frac{t^{2-1}}{1!} e^{-\frac{t}{3}}\, dt$$

$$= \frac{1}{9}\int_5^\infty t\, e^{-\frac{t}{3}}\, dt = \frac{1}{9}\left[\left[t\left(-3e^{-\frac{t}{3}}\right)\right]_5^\infty + 3\int_5^\infty e^{-\frac{t}{3}}\, dt\right]$$

$t \to 1$, $e^{-\frac{t}{3}} \to -3e^{-\frac{t}{3}}$

$$= \frac{1}{9}\left[5 \cdot 3 \cdot e^{-\frac{5}{3}} + 3 \cdot \left[-3e^{-\frac{t}{3}}\right]_5^\infty\right]$$

$$= \frac{1}{9}\left[15 e^{-\frac{5}{3}} + 9 e^{-\frac{5}{3}}\right] = \frac{24}{9} e^{-\frac{5}{3}}.$$

$[0 \qquad 5]$

Exp. # of customers $= \frac{5}{3}$

$X = \#$ of customesr $\sim \mathrm{Pois}\left(\frac{5}{3}\right)$

$$\mathbb{P}(W > 5) = \mathbb{P}(X = 0, 1) = \mathbb{P}(X=0) + \mathbb{P}(X=1) = e^{-\frac{5}{3}} + \frac{\left(\frac{5}{3}\right)^1}{1!} \cdot e^{-\frac{5}{3}} = \frac{8}{3} e^{-\frac{5}{3}}$$

Chi-square distribution

$\lambda = \frac{1}{2}$

Let X have a gamma distribution with $\theta = 2$ and $\alpha = r/2$, where r is a positive integer.

The pdf of X is

$$f(x) = \frac{1}{\Gamma(\frac{r}{2})2^{\frac{r}{2}}} x^{\frac{r}{2}-1} e^{-\frac{x}{2}}$$

for $x > 0$.

We say that X has **a chi-square distribution** with r degrees of freedom and we use the notation $X \sim \chi^2(r)$.

$X \sim \mathrm{Exp}(\lambda) \Rightarrow \mathbb{P}(X > t) = e^{-\lambda t} = e^{-\frac{t}{\theta}}$

Exercise

$f_X(t) = \frac{1}{\theta} e^{-\frac{t}{\theta}}, \quad t \geq 0.$

Let X have an exponential distribution with mean θ.

Compute $\mathbb{P}(X > 15 | X > 10)$ and $\mathbb{P}(X > 5)$.

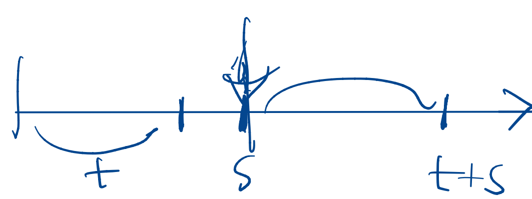
$= e^{-\frac{5}{\theta}}$

$$= \frac{\mathbb{P}(X > 15)}{\mathbb{P}(X > 10)} = \frac{e^{-\frac{15}{\theta}}}{e^{-\frac{10}{\theta}}} = e^{-\frac{5}{\theta}}$$

$\text{CDF} = F(t) = \mathbb{P}(X \leq t) = 1 - \mathbb{P}(X > t)$

$= 1 - e^{-\frac{t}{\theta}}$

Memorylessness Property



If $X \sim \mathrm{Exp}(\lambda)$ then

$$\mathbb{P}(X > t+s \mid X > s) = \mathbb{P}(X > t)$$

$Y \sim \mathrm{Geom}(p)$

$$\mathbb{P}(Y > t+s \mid Y > s) = \mathbb{P}(Y > t)$$

Section 3.
The Normal Distribution

Gaussian random variables

Definition

We say X is **a Gaussian random variable** or has **a normal distribution** if its PDF is given by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right).$$

Here μ is the mean and σ is the standard deviation. We use the notation $X \sim N(\mu, \sigma^2)$.

mean
exp.

variance

$X \sim N(\mu, \sigma^2) \qquad f(x) = \frac{1}{\sqrt{2\pi \cdot \sigma^2}} \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}} \qquad x \in \mathbb{R} = (-\infty, \infty)$

Gaussian random variables

Theorem

$\int_{\mathbb{R}} f(x)\, dx = 1$

$\mathbb{E}[X] = \mu$

$\mathrm{Var}[X] = \sigma^2$

$M(t) = \exp\left(\mu t + \frac{\sigma^2 t^2}{2}\right)$

$$1 = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\, dx \underset{\uparrow}{=} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}\, dz$$

repeat $\rightarrow$ $z = \frac{x-\mu}{\sigma}$, $dz = \frac{dx}{\sigma}$

$$= 2 \cdot \int_0^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}\, dz$$

$$\boxed{\int_0^{\infty} e^{-\frac{z^2}{2}}\, dz = \sqrt{\frac{\pi}{2}}}$$

$$A^2 = \int_0^{\infty} e^{-\frac{z^2}{2}}\, dz \cdot \int_0^{\infty} e^{-\frac{w^2}{2}}\, dw$$

$$= \iint_{\mathbb{R}^2} e^{-\frac{1}{2}(z^2+w^2)}\, dz\, dw$$

$\uparrow$ Polar coordinate

$\frac{z^2}{2} = y \qquad z = \sqrt{2y}$

$dz = \sqrt{2} \cdot \frac{1}{2\sqrt{y}}\, dy = \frac{1}{\sqrt{2}} \cdot y^{-\frac{1}{2}}\, dy$

$$\int_0^{\infty} e^{-\frac{z^2}{2}}\, dz = \int_0^{\infty} \frac{1}{\sqrt{2}} y^{\frac{1}{2}-1} e^{-y}\, dy = \frac{1}{\sqrt{2}} \cdot \Gamma\left(\frac{1}{2}\right)$$

X is a normal (Gaussian) random variable

$X \sim N(\mu, \sigma^2)$ μ : mean, σ^2 : variance

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad -\infty < x < \infty.$$

$\mathbb{E}[X] = \mu$, $Var(X) = \sigma^2$.

If $\mu = 0$, $\sigma^2 = 1$, X is the standard normal.

Fact $X \sim N(\mu, \sigma^2) \Rightarrow Z = \frac{X-\mu}{\sigma} \sim N(0,1)$

Standard normal distribution

Notation : The CDF of Standard Normal $\Phi(z) = \mathbb{P}(Z \leq z)$

In particular, if $\mu = 0$ and $\sigma = 1$, then $Z \sim N(0,1)$ is called **the standard normal random variable**.

Example

Let Z be $N(0,1)$.

Find $\mathbb{P}(Z \leq 1.24)$, $\mathbb{P}(1.24 \leq Z \leq 2.37)$, and $\mathbb{P}(-2.37 \leq Z \leq -1.24)$.

$$\mathbb{P}(Z \leq 1.24) = \Phi(1.24)$$

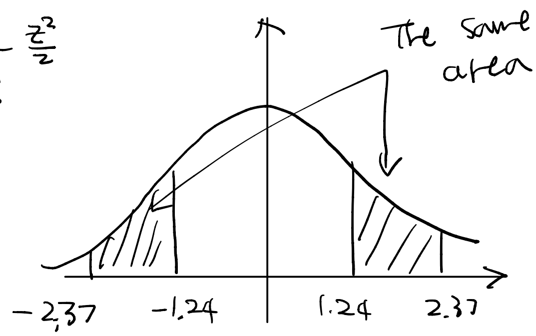
$$\mathbb{P}(1.24 \leq Z \leq 2.37) = \mathbb{P}(Z \leq 2.37) - \mathbb{P}(Z < 1.24)$$

$$= \Phi(2.37) - \Phi(1.24)$$

$\|$

$$\mathbb{P}(-2.37 \leq Z \leq -1.24) =$$

$$f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$$



$X \sim Bin(n, p)$

- If n large, p small, $\lambda = np$, then $X \approx Pois(\lambda)$
- If n large,

$$\underbrace{\frac{X - np}{\sqrt{np(1-p)}}}_{\text{Random variable}} \Rightarrow N(0,1)$$

"Central limit thm".

Standard normal distribution

$$Var(X+c) = \mathbb{E}\left[\left((X+\cancel{c}) - \mathbb{E}[X+\cancel{c}]\right)^2\right]$$
$$= \mathbb{E}\left[(X-\mathbb{E}[X])^2\right] = Var(X)$$

Theorem

If $X \sim N(\mu, \sigma^2)$, then $Z = \frac{X-\mu}{\sigma}$ is the standard normal.

① X is normal $\Rightarrow$ $aX+b$ is normal

② $Z = \frac{X-\mu}{\sigma} = \left(\frac{1}{\sigma}\right)\cdot X - \frac{\mu}{\sigma}$

$$\mathbb{E}[Z] = \mathbb{E}\left[\frac{X-\mu}{\sigma}\right] = \frac{1}{\sigma}\,\mathbb{E}[X-\mu] = \frac{1}{\sigma}\left(\mathbb{E}[X]-\mu\right) = 0$$
$$Var(Z) = Var\left(\frac{X-\mu}{\sigma}\right) = \frac{1}{\sigma^2}\cdot Var(X-\mu) = \frac{1}{\sigma^2}\cdot Var(X) = 1.$$

$$Var(2X) = 4\,Var(X)$$
$$Std(2X) = 2\,Std(X)$$

Standard normal distribution

$\Phi(z) = \mathbb{P}(Z \leqslant z)$

Example

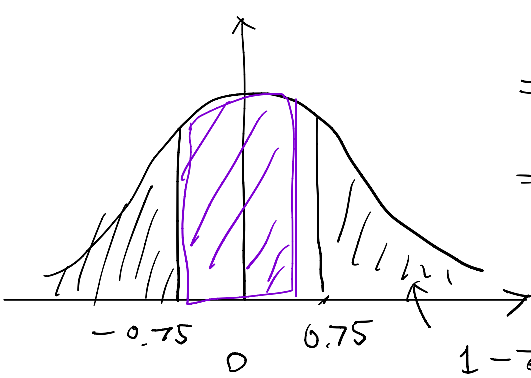
Let $X \sim N(3, 16)$.

Find $\mathbb{P}(4 \leq X \leq 8)$, $\mathbb{P}(0 \leq X \leq 5)$, and $\mathbb{P}(-2 \leq X \leq 1)$.

$\mu = 3, \quad \sigma^2 = 16, \quad \sigma = 4 \qquad Z = \frac{X-\mu}{\sigma} = \frac{X-3}{4} \sim N(0,1)$

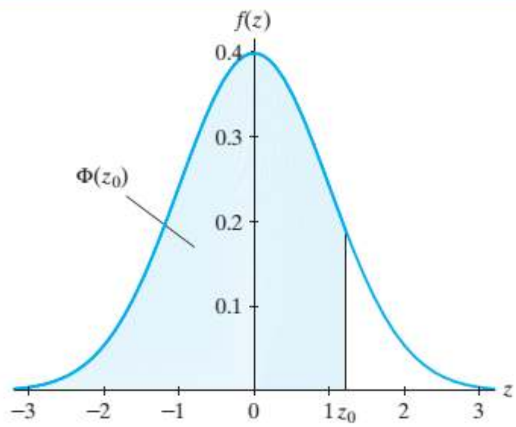
$$\begin{aligned}
\mathbb{P}(4 \leqslant X \leqslant 8) &= \mathbb{P}\left(\frac{4-3}{4} \leqslant \frac{X-3}{4} \leqslant \frac{8-3}{4}\right) \\
&= \mathbb{P}\left(\frac{1}{4} \leqslant Z \leqslant \frac{5}{4}\right) \\
&= \Phi(1.25) - \Phi(0.25). \\
&\approx 0.8944 - 0.5987
\end{aligned}$$

$$\begin{aligned}
\mathbb{P}(0 \leqslant X \leqslant 5) &= \mathbb{P}\left(\frac{0-3}{4} \leqslant Z \leqslant \frac{5-3}{4}\right) \\
&= \mathbb{P}\left(-\frac{3}{4} \leqslant Z \leqslant \frac{1}{2}\right) \\
&= \Phi(0.5) - \Phi(-0.75) \\
&= \Phi(0.5) - (1 - \Phi(0.75)) \\
&= \Phi(0.5) + \Phi(0.75) - 1.
\end{aligned}$$



Note : $\Phi(-z) = 1 - \Phi(z)$.

Table Va The Standard Normal Distribution Function



$\Phi(1.24) \approx 0.8925$

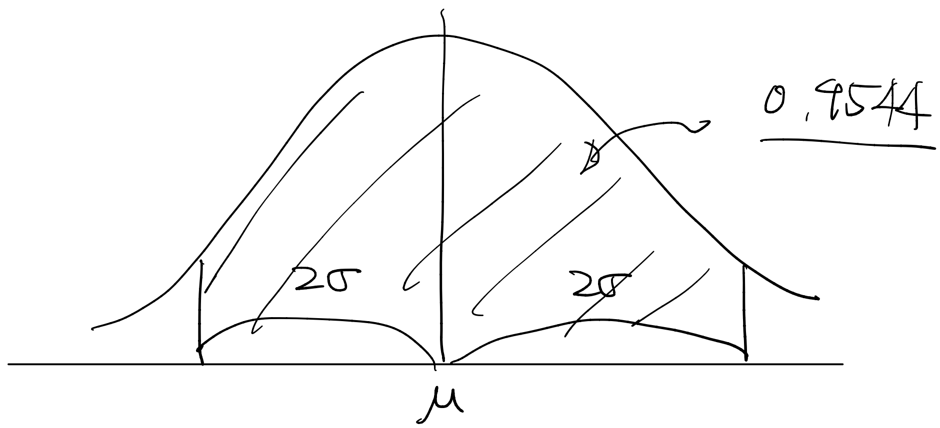
$\Phi(2.37) \approx 0.9911$

$$P(Z \le z) = \Phi(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-w^2/2}\, dw$$

$$\Phi(-z) = 1 - \Phi(z)$$

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7703	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990

α	0.400	0.300	0.200	0.100	0.050	0.025	0.020	0.010	0.005	0.001
z_α	0.253	0.524	0.842	1.282	1.645	1.960	2.054	2.326	2.576	3.090
$z_{\alpha/2}$	0.842	1.036	1.282	1.645	1.960	2.240	2.326	2.576	2.807	3.291



Standard normal distribution

Example

Let $X \sim N(25, 36)$.

Find a constant c such that $\mathbb{P}(|X - 25| \leq c) = 0.9544$.

$\mu = 25$, $\sigma^2 = 36$, $\sigma = 6$

$$Z = \frac{X-\mu}{\sigma} = \frac{X-25}{6} \sim N(0,1)$$

$$\mathbb{P}\left(\frac{|X-25|}{6} \leq \frac{c}{6} \right) = 0.9544$$

$$= \mathbb{P}\left(|Z| \leq \frac{c}{6} \right)$$

$$= \mathbb{P}\left(-\frac{c}{6} \leq Z \leq \frac{c}{6} \right)$$

$$= \Phi\left(\frac{c}{6}\right) - \Phi\left(-\frac{c}{6}\right) = \Phi\left(\frac{c}{6}\right) - \left(1 - \Phi\left(\frac{c}{6}\right)\right)$$

$$= 2 \cdot \Phi\left(\frac{c}{6}\right) - 1$$

$$2 \cdot \Phi\left(\frac{c}{6}\right) = 1 + 0.9544$$

$$\Phi\left(\frac{c}{6}\right) = 0.9772 = \Phi(2) \Rightarrow \frac{c}{6} = 2 \quad \therefore c = 12$$

Standard normal distribution

$$X = Z^2$$

Theorem

If Z is the standard normal, then Z^2 is $\chi^2(1)$. $= \text{Gamma}\left(\underset{\lambda}{\overset{\shortparallel}{\tfrac{1}{2}}}, \underset{\alpha}{\overset{\shortparallel}{\tfrac{1}{2}}}\right)$

How to find the distribution of Z^2 ?

"CDF"

CDF of Z^2

$$F_X(x) = \mathbb{P}(Z^2 \leq x) = \mathbb{P}(-\sqrt{x} \leq Z \leq \sqrt{x})$$

$x \geq 0$

$$= \Phi(\sqrt{x}) - \Phi(-\sqrt{x})$$

$$= \Phi(\sqrt{x}) - (1 - \Phi(\sqrt{x}))$$

$$= \underbrace{2\,\Phi(\sqrt{x}) - 1}.$$

$$f_X(x) = \frac{d}{dx} F_X(x) = \frac{d}{dx}\left(2\,\Phi(\sqrt{x}) - 1\right)$$

$$= 2 \cdot \Phi'(\sqrt{x}) \cdot \frac{1}{2\sqrt{x}} = \frac{1}{\sqrt{x}} \cdot \Phi'(\sqrt{x})$$

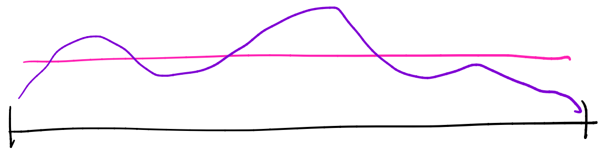
$$= \frac{1}{\sqrt{x}} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{(\sqrt{x})^2}{2}} = \frac{1}{\sqrt{2\pi x}} e^{-\frac{x}{2}} \sim \text{Gamma}\left(\tfrac{1}{2}, \tfrac{1}{2}\right)$$

Section 4.
Additional Models

Weibull distribution

Recall the postulates of an approximate Poisson:

- The numbers of occurrences in nonoverlapping subintervals are independent.
- The probability of two or more occurrences in a sufficiently short subinterval is essentially zero.
- The probability of exactly one occurrence in a sufficiently short subinterval of length h is approximately λh.



rate of incoming customers

Poisson

Constant λ $\downarrow$

function $\lambda(t)$

Weibull distribution

One can think the event occurrence as a failure and so λ can be understood as the failure rate.

Poisson distribution and its waiting time (exponential distribution) has a constant failure rate.

Sometimes, it is more natural to choose λ as a function of t in the last assumption.

Then the waiting time W for the first occurrence satisfies

$$\mathbb{P}(W > t) = \exp\left(-\int_0^t \lambda(w)\,dw\right).$$

Poisson Case (Exp.)

$$\mathbb{P}(W > t) = \exp\left(-\int_0^t \underset{\text{const.}}{\underline{\lambda}}\,dw\right) = e^{-\lambda t}.$$

Weibull distribution

$\lambda(t) = c \cdot t^{\alpha-1}$

Definition

If $\lambda(t) = \alpha \frac{t^{\alpha-1}}{\beta^\alpha}$, then the waiting time W for the first occurrence has the density

$$g(t) = \lambda(t) \exp\left(-\int_0^t \lambda(w)\, dw\right) = \alpha \frac{t^{\alpha-1}}{\beta^\alpha} \exp\left(-(\frac{t}{\beta})^\alpha\right).$$

W is called **the Weibull random variable**.

Weibull distribution

$$\lambda(t) = \frac{\alpha\, t^{\alpha-1}}{\beta^{\alpha}}$$

Example

If $\lambda(t) = 2t$, then the waiting time W has the density

and it is a Weibull random variable with $\alpha = 2$ and $\beta = 1$.

(Exp.) (maximum?)

If W_1, W_2 are independent Weibull with α and β above, is the minimum of W_1, W_2 Weibull?

$$\mathbb{P}(W > t) = e^{-\int_0^t 2w\, dw} = e^{-t^2}$$

$X = \min\{W_1, W_2\} \leftarrow$ dist.?

$$\mathbb{P}(X \le t) = 1 - \mathbb{P}(X > t)$$

$$\mathbb{P}(X > t) = \mathbb{P}(\min\{W_1, W_2\} > t)$$

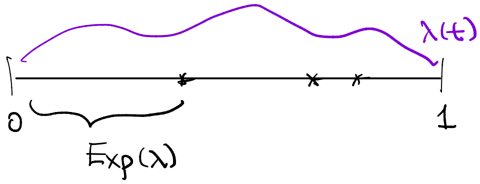
$$= \mathbb{P}(\{W_1 > t\} \cap \{W_2 > t\})$$

indep.

$$= \mathbb{P}(W_1 > t) \cdot \mathbb{P}(W_2 > t)$$

$$= e^{-t^2} \cdot e^{-t^2} = e^{-2 \cdot t^2} \leftarrow \text{Weibull}$$

$$\Rightarrow \quad X = \min\{W_1, W_2\} \quad \text{Weibull.}$$



Poiss(λ) : # of customers

Const λ

$\downarrow$

λ : function of time

W : the waiting time

$P(W > t) = e^{-\int_0^t \lambda(w)\,dw}$

Special Case : $\lambda(t) = \dfrac{\alpha t^{\alpha-1}}{\beta^\alpha} = \dfrac{d}{dw}\left(\dfrac{t}{\beta}\right)^\alpha$ $\leftarrow$ Weibull distribution.

Weibull distribution

Note $W_1, W_2 \sim \begin{cases} \text{Weibull} \\ \text{Exp.} \end{cases} \Rightarrow \min\{W_1, W_2\} \begin{cases} \text{Weibull} \\ \text{Exp.} \end{cases}$

indep

Theorem

The mean of W is $\mu = \beta\Gamma(1 + \frac{1}{\alpha})$.

The variance is $\sigma^2 = \beta^2 \left(\Gamma(1 + \frac{2}{\alpha}) - \Gamma(1 + \frac{1}{\alpha})^2\right)$.

Discrete RV ← when Ω is countable

Conti. RV ← RV having PDF

There are many others

Looking at CDF

$$F(x) = \mathbb{P}(X \le x)$$

Mixed type random variables

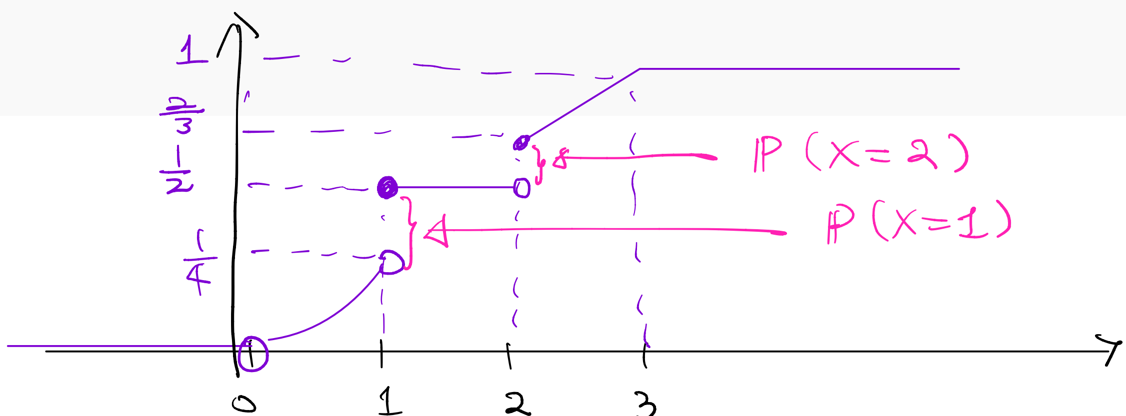
Example

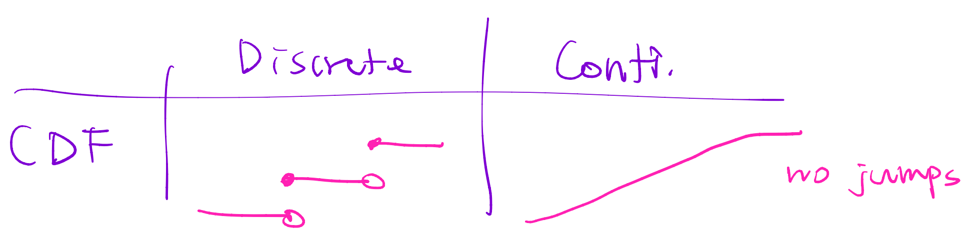
Suppose X has a CDF

$$F(x) = \begin{cases} 0, & x < 0 \\ \frac{x^2}{4}, & 0 \le x < 1 \\ \frac{1}{2}, & 1 \le x < 2 \\ \frac{x}{3}, & 2 \le x < 3 \\ 1, & x \ge 3. \end{cases}$$

Find $\mathbb{P}(0 < X < 1)$, $\mathbb{P}(0 < X \le 1)$, and $\mathbb{P}(X = 1)$.

$$\mathbb{P}(0 < X < 1)$$
$$= \mathbb{P}(X \le 1) - \mathbb{P}(X \le 0) - \mathbb{P}(X = 1)$$
$$= F(1) - F(0) - \mathbb{P}(X = 1)$$
$$= \frac{1}{2} - 0 - \underbrace{\mathbb{P}(X = 1)}_{= (\frac{1}{2} - \frac{1}{4})}$$





$$\mathbb{P}(X < 1)$$
$$= \lim_{\varepsilon \downarrow 0} \mathbb{P}(X \le 1 - \varepsilon)$$

Conti. RV :

$$\mathbb{P}(X=1) = \lim_{\varepsilon \downarrow 0} \mathbb{P}(1-\varepsilon \leq X \leq 1+\varepsilon)$$

$$= \underline{\lim_{\varepsilon \downarrow 0} \int_{1-\varepsilon}^{1+\varepsilon} f(x)\, dx} = 0$$

CDF

$$F(x) = \mathbb{P}(X \leq x) = \int_{-\infty}^{x} f(t)\, dt$$

Mixed type random variables

Example

Consider the following game: A fair coin is tossed.

If the outcome is heads, the player receives \$2.

If the outcome is tails, the player spins a balanced spinner that has a scale from 0 to 1.

The player then receives that fraction of a dollar associated with the point selected by the spinner.

Let X be the amount received. Draw the graph of the cdf $F(x)$.

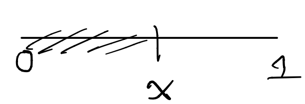
$$X = \begin{cases} 2 & , \text{ if } H \\ U & , \text{ if } T \end{cases} , \quad U \sim \text{Unif}(0,1)$$

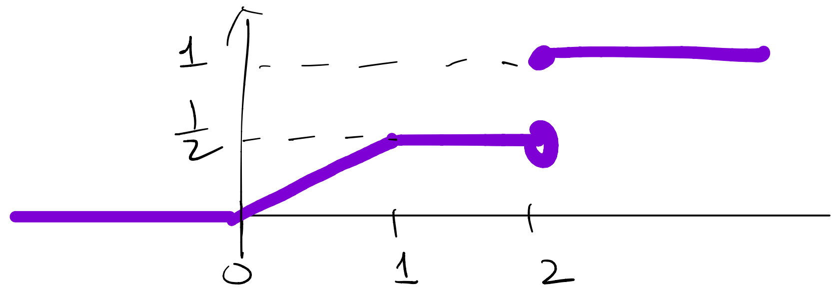
$$F(x) = \mathbb{P}(X \leq x) = \begin{cases} 0 & , \ x < 0 \\ \frac{1}{2} \cdot x & , \ 0 \leq x < 1 \\ \frac{1}{2} & , \ 1 \leq x < 2 \\ 1 & , \ x \geq 2 \end{cases}$$

$x = -1$

$x = \frac{1}{2}$

$x = \frac{3}{2}$





Exercise

The cdf of X is given by

$$F(x) = \begin{cases} 0, & x < -1 \\ \frac{x}{4} + \frac{1}{2}, & -1 \leq x < 1 \\ 1, & x \geq 1. \end{cases}$$

Find $\mathbb{P}(X < 0)$, $\mathbb{P}(X < -1)$, and $\mathbb{P}(-1 \leq X < \frac{1}{2})$.