Solution To Midterm 2 Practice Problems

Q1) The random variable X ha sthe density

$$f(x) = \begin{cases} ax + bx^2, & 0 < x < 1 \\ 0, & \text{otherwise.} \end{cases}$$

If $\mathbb{E}[X] = 0.6$, find $\mathbb{P}(X < \frac{1}{2})$ and $\text{Var}(X)$.

$$\int_{\mathbb{R}} f(x)\, dx = \int_0^1 (ax + bx^2)\, dx = \frac{1}{2}a + \frac{1}{3}b = 1$$

$$\mathbb{E}[X] = \int_{\mathbb{R}} x \cdot f(x)\, dx = \int_0^1 x(ax+bx^2)\, dx = \frac{1}{3}a + \frac{1}{4}b = \frac{3}{5}$$

$$a = 2 - \frac{2}{3}b = \frac{9}{5} - \frac{3}{4}b \quad , \quad \frac{1}{5} = -\frac{1}{12}b \quad \therefore\ b = -\frac{12}{5} \ \ \& \ \ a = \frac{18}{5}$$

$$\therefore\ f(x) = \begin{cases} \frac{6}{5}(3x - 2x^2) & \text{if } 0 < x < 1 \\ 0 & \text{o.w.} \end{cases}$$

$$\mathbb{P}(X < \tfrac{1}{2}) = \int_0^{\frac{1}{2}} \frac{6}{5}(3x - 2x^2)\, dx = \frac{6}{5}\left[\frac{3}{2}x^2 - \frac{2}{3}x^3\right]_0^{\frac{1}{2}}$$

$$= \frac{6}{5}\left(\frac{3}{8} - \frac{2}{3\cdot 8}\right) = \frac{7}{20}.$$

$$\mathbb{E}[X^2] = \frac{6}{5}\left(\frac{3}{4} - \frac{2}{5}\right) = \frac{6}{5}\cdot\frac{7}{20} = \frac{21}{50}$$

$$\text{Var}(X) = \frac{21}{50} - \left(\frac{3}{5}\right)^2 = \frac{3}{50}.$$

Q2) Your company must make a sealed bid for a construction project. If you succeed in winning the contract (by having the lowest bid), then you plan to pay another firm \$100,000 to do the work. If you believe that the minimum bid (in thousands of dollars) of the other participating companies can be modeled as the value of a random variable that is uniformly distributed on $(70, 140)$, how much should you bid to maximize your expected profit?

X = the minimum bid of other companies

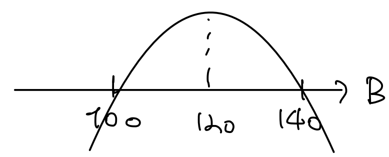
$\sim \text{Unif}(70, 140)$

B : my bid. P : profit

$$\Rightarrow P = \begin{cases} B - 100, & \text{if } X \geqslant B \\ 0, & \text{o.w.} \end{cases}$$

$$\mathbb{E}[P] = \int_B^{140} (B - 100)\, dx$$

$$= (B - 100) \cdot (140 - B)$$



$E[P]$ is maximized when $B = 120$.

Q3) The life of a certain type of automobile tire $=: X$ is normally distributed with mean 34,000 miles and standard deviation 4000 miles.

(a) What is the probability that such a tire lasts more than 40,000 miles?

(b) What is the probability that it lasts between 30,000 and 35,000 miles?

(c) Given that it has survived 30,000 miles, what is the conditional probability that the tire survives another 10,000 miles?

$X \sim N(34, 4^2)$ (in 1000 miles) $Z \sim N(0,1)$

(a) $$\mathbb{P}(X > 40) = \mathbb{P}(4Z + 34 > 40) = \mathbb{P}(Z > \tfrac{3}{2}) = 1 - \Phi(\tfrac{3}{2})$$

(b) $$\mathbb{P}(30 < X < 35) = \mathbb{P}(-1 < Z < \tfrac{1}{4}) = \Phi(\tfrac{1}{4}) - \Phi(-1) = \Phi(\tfrac{1}{4}) + \Phi(1) - 1.$$

(c) $$\mathbb{P}(X > 40 \mid X > 30) = \frac{\mathbb{P}(Z > \frac{3}{2})}{\mathbb{P}(Z > -1)} = \frac{1 - \Phi(\frac{3}{2})}{\Phi(1)}$$

Q4) At a certain bank, the amount of time that a customer spends being served by a teller $=: X$ is an exponential random variable with mean 5 minutes. If there is a customer in service when you enter the bank, what is the probability that he or she will still be with the teller after an additional 4 minutes?

$X \sim \mathrm{Exp}(\frac{1}{5})$, t = time spent for the customer when you enter

$$P(X > t+4 \mid X > t) = P(X > 4) = e^{-\frac{1}{5} \cdot 4} = e^{-\frac{4}{5}}.$$

Q5) Let X be a random variable with density

$$f(x) = \frac{1}{\pi(1+x^2)}, \quad -\infty < x < \infty.$$

Find the density of $1/X$.

$Y = \frac{1}{X}$

Let $t > 0$.

$F_Y(t) = \mathbb{P}(Y \leq t) = \mathbb{P}(X \geqslant \frac{1}{t}) + \mathbb{P}(X < 0)$

$= \frac{1}{2} + \int_{\frac{1}{t}}^{\infty} \frac{1}{\pi(1+x^2)}\, dx = \frac{1}{\pi}\left[\arctan(x)\right]_{\frac{1}{t}}^{\infty} + \frac{1}{2}$

$= \frac{1}{\pi}\left(\frac{\pi}{2} - \arctan(\frac{1}{t})\right) + \frac{1}{2}$

Let $t < 0$.

$F_Y(t) = \mathbb{P}(Y \leq t) = \mathbb{P}(\frac{1}{t} \leqslant X < 0)$

$= \int_{\frac{1}{t}}^{0} \frac{1}{\pi(1+x^2)}\, dx = \frac{1}{\pi}\left[\arctan(x)\right]_{\frac{1}{t}}^{0}$

$= -\frac{1}{\pi}\arctan(\frac{1}{t})$

$F_Y(0) = \mathbb{P}(Y \leq 0) = \mathbb{P}(X < 0) = \frac{1}{2}$

$f_Y(t) = -\frac{1}{\pi} \cdot \frac{1}{1 + (1/t)^2} \cdot \left(-\frac{1}{t^2}\right) = \frac{1}{\pi} \frac{1}{1+t^2}$.

Q6) The joint probability mass function of the random variables X, Y, Z is

$$p(1,2,3) = p(2,1,1) = p(2,2,1) = p(2,3,2) = \frac{1}{4}.$$

Find $\mathbb{E}[XYZ]$ and $\mathbb{E}[XY + YZ + ZX]$.

$$\mathbb{E}[XYZ] = (1\cdot 2\cdot 3 + 2\cdot 1\cdot 1 + 2\cdot 2\cdot 1 + 2\cdot 3\cdot 2)\cdot \frac{1}{4}$$

$$= (6 + 2 + 4 + 12)\cdot \frac{1}{4} = 6.$$

$$\mathbb{E}[XY] = (1\cdot 2 + 2\cdot 1 + 2\cdot 2 + 2\cdot 3)\frac{1}{4} = \frac{7}{2}$$

$$\mathbb{E}[YZ] = (2\cdot 3 + 1\cdot 1 + 2\cdot 1 + 3\cdot 2)\frac{1}{4} = \frac{15}{4}$$

$$\mathbb{E}[ZX] = (1\cdot 3 + 2\cdot 1 + 2\cdot 1 + 2\cdot 2)\frac{1}{4} = \frac{11}{4}.$$

Q7) Let X and Y be continuous random variables with joint density function

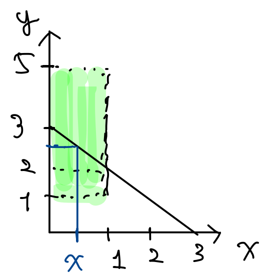
$$f(x,y) = \begin{cases} \frac{x}{5} + cy, & 0 < x < 1, 1 < y < 5, \\ 0, & \text{otherwise.} \end{cases}$$

Find c and $\mathbb{P}(X+Y>3)$. Are X, Y independent?

$$\int_1^5 \int_0^1 \left(\frac{x}{5} + cy\right) dx\, dy = \frac{4}{10} + 12c = 1 \quad \therefore\ c = \frac{1}{20}.$$

$$f(x,y) = \frac{1}{20}(4x+y)$$

$$\mathbb{P}(X+Y>3) = \int_0^1 \int_{3-x}^{5} \frac{1}{20}(4x+y)\, dy\, dx$$



$$= \frac{1}{20}\int_0^1 \left[4xy + \frac{1}{2}y^2\right]_{3-x}^{5} dx$$

$$= \frac{1}{20}\int_0^1 \left(4x(x+2) + \frac{25}{2} - \frac{1}{2}(3-x)^2\right) dx$$

$$= \frac{1}{20}\left[\frac{4}{3} + 4 + \frac{25}{2} + \frac{8}{6} - \frac{9}{2}\right]$$

$$= \frac{1}{20}\left(\frac{8}{3} + 12\right)$$

$$= \frac{1}{20} \cdot \frac{44}{3} = \frac{11}{15}$$

Dependent.

Q8) Consider two components and three types of shocks. A type 1 shock causes component 1 to fail, a type 2 shock causes component 2 to fail, and a type 3 shock causes both components 1 and 2 to fail. The times until shocks 1, 2, and 3 occur are independent exponential random variables with respective rates λ_1, λ_2, and λ_3. Let X_i denote the time at which component i fails, $i = 1, 2$. Find $\mathbb{P}(X_1 > s, X_2 > t)$.

T_i = time until shock i occurs $i = 1, 2, 3$.

$\sim \operatorname{Exp}(\lambda_i)$

$X_1 = \min\{T_1, T_3\}$ $\qquad X_2 = \min\{T_2, T_3\}$

$\mathbb{P}(X_1 > s, X_2 > t)$

$= \mathbb{P}(T_1 > s, T_2 > t, T_3 > \max\{t, s\})$

$= e^{-\lambda_1 s} \cdot e^{-\lambda_2 t} \cdot e^{-\lambda_3 \max\{t, s\}}$.

Q9) A model proposed for NBA basketball supposes that when two teams with roughly the same record play each other, the number of points scored in a quarter by the home team minus the number scored by the visiting team is approximately a normal random variable with mean 1.5 and variance 6. In addition, the model supposes that the point differentials for the four quarters are independent. Assume that this model is correct.

$X_i :=$, $i = 1, 2, 3, 4$

(a) What is the probability that the home team wins?

(b) What is the conditional probability that the home team wins, given that it is behind by 5 points at halftime?

(c) What is the conditional probability that the home team wins, given that it is ahead by 5 points at the end of the first quarter?

$$X_i \sim N(1.5, 6) \qquad i = 1, 2, 3, 4 \qquad \text{indep.}$$

(a) $$\mathbb{P}(\underbrace{X_1 + X_2 + X_3 + X_4}_{=: S \sim N(6, 24)} > 0)$$

$$= \mathbb{P}(2\sqrt{6} \cdot Z + 6 > 0)$$

$$= \mathbb{P}\left(Z > -\frac{\sqrt{6}}{2}\right) = \Phi\left(\frac{\sqrt{6}}{2}\right)$$

$$X_1 + X_2 = T_1 \sim N(3, 12)$$

$$X_3 + X_4 = T_2 \sim N(3, 12)$$

(b) $$\mathbb{P}(S > 0 \mid X_1 + X_2 = -5)$$

$$= \mathbb{P}(T_1 + T_2 > 0 \mid T_1 = -5)$$

$$f_{T_1, S}(z, w) = f_{T_1, T_2}(z, w - z)$$

$$T_1 = T_1 = g_1(T_1, T_2) \qquad z = x \qquad x = z$$

$$S = T_1 + T_2 = g_2(T_1, T_2) \qquad w = x + y \qquad y = w - z$$

$$f_{S \mid T_1}(w \mid z) = f_{T_2}(w - z) = \frac{1}{\sqrt{24\pi}} e^{-\frac{1}{24}(w - z - 3)^2} \sim N(3 + z, 12)$$

$$\mathbb{P}(S > 0 \mid T_1 = -5) = \int_0^\infty \frac{1}{\sqrt{24\pi}} e^{-\frac{1}{24}(w + 2)^2} \, dw$$

$$= \mathbb{P}(\sqrt{12} Z - 2 > 0) = \mathbb{P}\left(Z > \frac{1}{\sqrt{3}}\right).$$

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(c) $\mathbb{P}(S' > 0 \mid X_1 = 5)$

$$X_1 = X_1 \qquad\qquad X_1 = X_1$$

$$S' = X_1 + Y \qquad\qquad Y = S' - X_1$$

$$f_{X_1, S'}(z, w) = f_{X_1, Y}(z, w - z)$$

$$f_{S' \mid X_1}(w \mid z) = f_Y(w - z) \sim N(4.5 + z, 18)$$

$$\mathbb{P}(S' > 0 \mid X_1 = 5) = \mathbb{P}(\sqrt{18}\, Z + 9.5 > 0)$$

$$= \mathbb{P}\left(Z > -\frac{9.5}{\sqrt{18}}\right).$$

Q10) Let X and Y be independent uniform $(0,1)$ random variables.

(a) Find the joint density of $U = X$,$V = X + Y$.

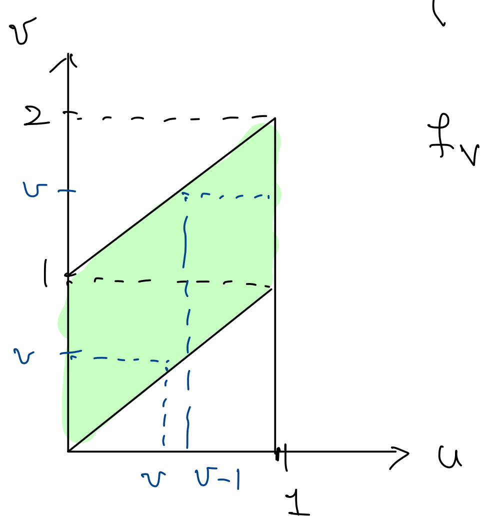
(b) Compute the density function of V.

$U = X \qquad X = U \qquad |J| = 1.$

$V = X + Y \qquad Y = V - U$

$$f_{U,V}(u,v) = f_{X,Y}(u, v-u)$$

$$= \begin{cases} 1 , & 0 < u < 1 \text{ and } 0 < v-u < 1 \\ & \quad u < v < u+1 \\ 0 , & \text{o.w.} \end{cases}$$



$$f_V(v) = \int_{\mathbb{R}} f_{U,V}(u,v)\, du$$

$$= \begin{cases} \int_0^v du , & 0 < v < 1 \\ \int_{v-1}^1 du , & 1 \leq v < 2 \end{cases}$$

$$= \begin{cases} v , & 0 < v < 1 \\ 2 - v , & 1 \leq v < 2 . \end{cases}$$