Section 1.1 : Systems of Linear Equations

Chapter 1 : Linear Equations

Math 1554 Linear Algebra

Section 1.1 Systems of Linear Equations

Topics

We will cover these topics in this section.

1. Systems of Linear Equations
2. Matrix Notation
3. Elementary Row Operations
4. Questions of Existence and Uniqueness of Solutions

Objectives

For the topics covered in this section, students are expected to be able to do the following.

1. Characterize a linear system in terms of the number of solutions, and whether the system is consistent or inconsistent.
2. Apply elementary row operations to solve linear systems of equations.
3. Express a set of linear equations as an augmented matrix.

A Single Linear Equation

A linear equation has the form

$$a_1x_1^{1} + a_2x_2^{1} + \cdots + a_nx_n^{1} = b$$

$a_1, \ldots, a_n$ and b are the **coefficients**, $x_1, \ldots, x_n$ are the **variables** or **unknowns**, and n is the **dimension**, or number of variables.

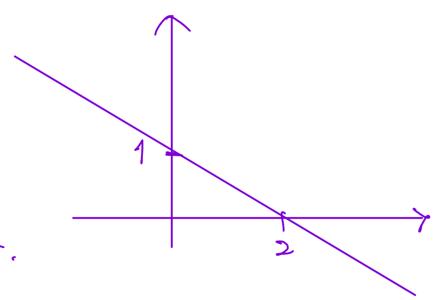
For example,

- $2x_1 + 4x_2 = 4$ is a line in two dimensions
- $3x_1 + 2x_2 + x_3 = 6$ is a plane in three dimensions

$$2x_1 + 4x_2 = 4$$

$$4x_2 = 4 - 2x_1$$

$$x_2 = \underbrace{-\frac{1}{2}}_{\text{slope}} x_1 + \underbrace{1}_{\text{y-int.}}$$



Systems of Linear Equations

When we have more than one linear equation, we have a **linear system** of equations. For example, a linear system with two equations is

$$\begin{cases} x_1 + 1.5x_2 + \pi x_3 = 4 \\ 5x_1 + 0 \cdot x_2 + 7x_3 = 5 \end{cases}$$

dimension = 3

Definition: Solution to a Linear System

The set of all possible values of $x_1, x_2, \ldots x_n$ that satisfy all equations is the **solution** to the system.

A linear system can have a unique solution, no solution, or an infinite number of solutions.

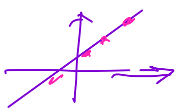
Theorem, Later

Solution of System = Intersection of lines

2 lines ⇑ 2 Eqns.   lines ⇑ 2 variables

Two Variables

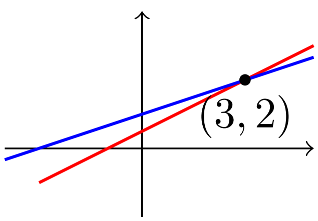
Consider the following systems. How are they different from each other?



$$\begin{cases} x_1 - 2x_2 = -1 \\ -x_1 + 3x_2 = 3 \end{cases}$$

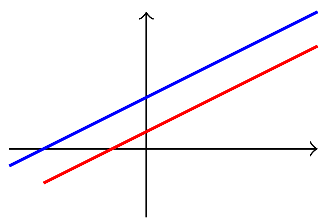
$$\begin{aligned} x_1 - 2x_2 &= -1 \\ -x_1 + 2x_2 &= 3 \end{aligned}$$

$$\begin{aligned} x_1 - 2x_2 &= -1 \\ -x_1 + 2x_2 &= 1 \end{aligned}$$



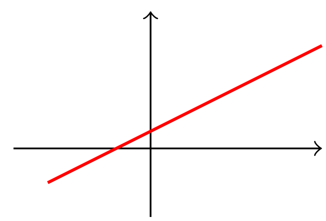
non-parallel lines

1 solution.



parallel lines

0 Solution



identical lines

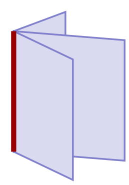
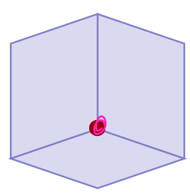
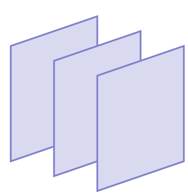
Infinitely many solutions.

3 variables $\Rightarrow$ planes

3 Equations

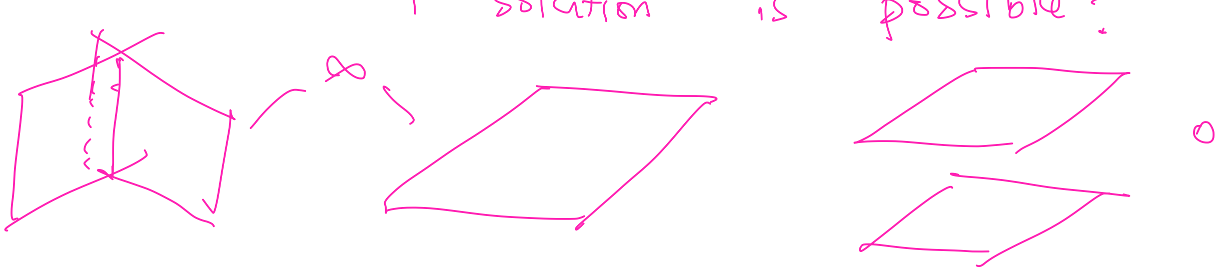
Three-Dimensional Case

An equation $a_1x_1 + a_2x_2 + a_3x_3 = b$ defines a plane in $\mathbb{R}^3$. The **solution** to a system of **three equations** is the set of intersections of the planes.

solution set	sketch	number of solutions
line		Infinity
point		1
empty		0

Q: 2 Eqns, 3 variables,

1 solution is possible?



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$$\text{linear system} \rightarrow \begin{cases} a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = b & \leftarrow \text{linear eqn.} \\ \vdots \\ \vdots \end{cases}$$

# of variables = dimension

Row Reduction by Elementary Row Operations

How can we find the solution set to a set of linear equations?
We can manipulate equations in a linear system using **row operations**.

1. (Replacement/Addition) Add a multiple of one row to another.
2. (Interchange) Interchange two rows.
3. (Scaling) Multiply a row by a non-zero scalar.

Let's use these operations to solve a system of equations.

Example 1

Identify the solution to the linear system.

$$\begin{cases} x_1 & -2x_2 & +x_3 & = 0 \\ & 2x_2 & -8x_3 & = 8 \\ 5x_1 & & -5x_3 & = 10 \end{cases}$$

$(-5)\times 1 : -5x_1 + 10x_2 - 5x_3 = 0$

$\xrightarrow{\quad}$ $3 \to 3 + (-5)\,1$

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 & \text{--- } 1 \\ 2x_2 - 8x_3 = 8 & \text{--- } 2 \\ 0 + 10x_2 - 10x_3 = 10 & \text{--- } 3 \end{cases}$$

: Replacement.

$\xrightarrow{\quad}$ $2 \to \frac{1}{2}\times 2$, $3 \to \frac{1}{10}\times 3$

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ \boxed{\begin{matrix} x_2 - 4x_3 = 4 \\ x_2 - x_3 = 1 \end{matrix}} \end{cases}$$

: scaling

$\xrightarrow{\quad}$ $2 \to 2 - 3$

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ \boxed{-3x_3 = 3} \\ x_2 - x_3 = 1 \end{cases}$$

$x_1 - 2\cdot 0 + (-1) = 0$

$\therefore x_1 = 1$

$\rightarrow$ $x_3 = -1$

$x_2 - (-1) = 1$

$\therefore x_2 = 0$

Augmented Matrices

It is redundant to write x_1, x_2, x_3 again and again, so we rewrite systems using matrices. For example,

$$\begin{array}{lllr} x_1 & -2x_2 & +x_3 & =0 \\ 0 \cdot x_1 & +2x_2 & -8x_3 & =8 \\ 5x_1 & +0 \cdot x_2 & -5x_3 & =10 \end{array}$$

can be written as the **augmented matrix**,

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 5 & 0 & -5 & 10 \end{array}\right]$$

The vertical line reminds us that the first three columns are the coefficients to our variables x_1, x_2, and x_3.

Linear Systems $\rightsquigarrow$ Augmented Matrices

$\downarrow$ Row operations $\downarrow$ Row operations

Simpler form Another matrices

Consistent Systems and Row Equivalence

Definition (Consistent)

A linear system is **consistent** if it has at least one <u>solution</u>.

Definition (Row Equivalence)

Two matrices are **row equivalent** if a sequence of <u>row operations</u> transforms one matrix into the other.

Note: if the augmented matrices of two linear systems are row equivalent, then they have the same solution set.

$$A = \begin{bmatrix} - & - & - & - \\ - & - & & - \\ - & - & \vdots & - \end{bmatrix} \xrightarrow{\text{row op.}} \square \longrightarrow \square \longrightarrow \cdots \longrightarrow B = \begin{bmatrix} \cdot & \cdot \end{bmatrix}$$

Fundamental Questions

Two questions that we will revisit many times throughout our course.

1. Does a given linear system have a solution? In other words, is it consistent?
2. If it is consistent, is the solution unique?

Section 1.2 : Row Reduction and Echelon Forms

Chapter 1 : Linear Equations

Math 1554 Linear Algebra

Section 1.2 : Row Reductions and Echelon Forms

Topics

We will cover these topics in this section.

1. Row reduction algorithm
2. Pivots, and basic and free variables
3. Echelon forms, existence and uniqueness

Objectives

For the topics covered in this section, students are expected to be able to do the following.

1. Characterize a linear system in terms of the number of leading entries, free variables, pivots, pivot columns, pivot positions.
2. Apply the row reduction algorithm to reduce a linear system to echelon form, or reduced echelon form.
3. Apply the row reduction algorithm to compute the coefficients of a polynomial.

Definition: Echelon Form and RREF

A rectangular matrix is in **echelon form** if

1. All zero rows (if any are present) are at the bottom.
2. The first non-zero entry (or **leading entry**) of a row is to the right of any leading entries in the row above it (if any).
3. All elements below a leading entry (if any) are zero.

A matrix in echelon form is in **reduced row echelon form** (RREF) if

1. All leading entries, if any, are equal to 1.
2. Leading entries are the only nonzero entry in their respective column.

Example of a Matrix in Echelon Form

$\blacksquare$ = non-zero number, $\quad *$ = any number

leading entries

$$\begin{array}{r}\\ \\ \\ \\ \text{zero row} \rightarrow\end{array}\begin{bmatrix} 0 & \blacksquare & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}, \begin{bmatrix}1 & 1\\0 & 0\end{bmatrix}, \begin{bmatrix}1 & 100\\0 & 0\end{bmatrix}$$

Example 1

Which of the following are in RREF?

$a)$ $\underline{\begin{bmatrix}1 & 0\\0 & \boxed{2}\end{bmatrix}} \to \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$

$d)$ $\begin{bmatrix}0 & 6 & 3 & 0\end{bmatrix} \to \begin{bmatrix}0 & 1 & \frac{1}{2} & 0\end{bmatrix}$

$b)$ $\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$

$e)$ $\begin{bmatrix}1 & 17 & 0\\0 & 0 & 1\end{bmatrix}$

$c)$ $\begin{bmatrix}0\\1\\0\\0\end{bmatrix} \to \begin{bmatrix}1\\0\\0\\0\end{bmatrix}$

Definition: Pivot Position, Pivot Column

A **pivot position** in a matrix A is a location in A that corresponds to a leading 1 in the reduced echelon form of A.

A **pivot column** is a column of A that contains a pivot position.

Example 2: Express the matrix in reduced row echelon form and identify the pivot columns.

pivot positions

$$A = \begin{bmatrix} 0 & -3 & -6 & 4 \\ -1 & -2 & -1 & 3 \\ -2 & -3 & 0 & 3 \end{bmatrix}$$

pivot columns

Row oper.

$$\begin{bmatrix} \blacksquare & * & * & * \\ 0 & 0 & \blacksquare & * \\ 0 & 0 & 0 & \blacksquare \end{bmatrix} : \text{Echelon form}$$

$$\begin{bmatrix} 1 & * & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} : \text{RREF}$$

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{matrix} 0 \\ 0 \\ \underline{1} \\ 0 \\ 0 \end{matrix}$$

Row Reduction Algorithm

The algorithm we used in the previous example produces a matrix in RREF. Its steps can be stated as follows.

- Step 1a Swap the 1st row with a lower one so the leftmost nonzero entry is in the 1st row
- Step 1b Scale the 1st row so that its leading entry is equal to 1
- Step 1c Use row replacement so all entries below this 1 are 0
- Step 2a Swap the 2nd row with a lower one so that the leftmost nonzero entry below 1st row is in the 2nd row
- etc. ...

  Now the matrix is in echelon form, with leading entries equal to 1.
- Last step Use row replacement so all entries above each leading entry are 0, starting from the right.

$$\begin{cases} x_1 + 3\underline{x_2} \quad + 7\underline{x_4} = 4 \\ \underline{x_3} \quad + 4\cdot\underline{x_4} = 5 \\ \boxed{x_5} = \underline{6} \end{cases}$$

Basic And Free Variables

- Choose any number for x_4
  → x_3 determined
- Choose x_2, x_4
  → x_1 determined.

RREF.

Consider the augmented matrix

$$\begin{array}{c} \\ \left[A \mid \vec{b}\right] = \end{array} \begin{array}{c} \begin{array}{ccccc|c} x_1 & & x_3 & & x_5 & \end{array} \\ \left[\begin{array}{ccccc|c} 1 & 3 & 0 & 7 & 0 & 4 \\ 0 & 0 & 1 & 4 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 6 \end{array}\right] \\ \begin{array}{ccccc} & x_2 & & x_4 & \to \text{free} \end{array} \end{array}$$

The leading one's are in first, third, and fifth columns. So:

- the pivot variables of the system $A\vec{x} = \vec{b}$ are x_1, x_3, and x_5.
- The free variables are x_2 and x_4. **Any choice** of the free variables leads to a solution of the system.

Note that A does not have basic variables or free variables. Systems have variables.

Linear system → Augmented matrix $\xrightarrow{R.O}$ RREF.

① $\left[\begin{array}{ccc|c} \underbrace{0 & \cdots & 0 & 1}_{\to 0} \end{array}\right]$ leading entry in the last column

$0 = 1 \Rightarrow$ No solution

$\Rightarrow$ Inconsistent.

② If free variable $\Rightarrow$ infinitely many solution.

Existence and Uniqueness

Theorem

A linear system is consistent if and only if (exactly when) the last column of the **augmented** matrix does not have a pivot. This is the same as saying that the RREF of the augmented matrix does **not** have a row of the form

$$\left(\begin{array}{ccccc|c} 0 & 0 & 0 & \cdots & 0 & 1 \end{array}\right)$$

Moreover, if a linear system is consistent, then it has

1. a unique solution if and only if there are no free variables.
2. infinitely many solutions that are parameterized by free variables.

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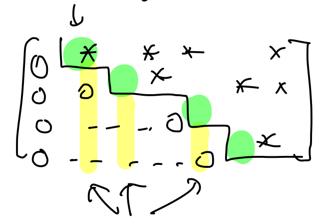
Linear Systems → Augmented matrix $\left[\begin{array}{ccc|c} \vdots & \cdots & & \vdots \\ & \ddots & & \vdots \end{array}\right]$

↓ Row operation $\begin{cases} \text{Replace} \\ \text{Swap} \\ \text{Scaling} \end{cases}$

RREF.

leading entries



pivot → basic variable

non pivot → free variable

① Consistent $\Leftrightarrow$ No $\left[\begin{array}{ccc|c} 0 & \cdots & 0 & 1 \end{array}\right]$ in RREF

last column $\neq$ pivot

② If consistent

$\begin{cases} \text{non pivot} \to \text{free var.} \to \text{Infinitely many solution} \\ \text{No nonpivot} \to \text{no free} \to 1 \text{ solution.} \end{cases}$

Section 1.3 : Vector Equations

Chapter 1 : Linear Equations

Math 1554 Linear Algebra

1.3: Vector Equations

Topics

We will cover these topics in this section.

1. Vectors in $\mathbb{R}^n$, and their basic properties
2. Linear combinations of vectors

Objectives

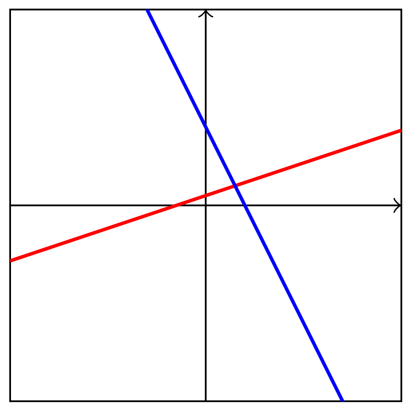
For the topics covered in this section, students are expected to be able to do the following.

1. Apply geometric and algebraic properties of vectors in $\mathbb{R}^n$ to compute vector additions and scalar multiplications.
2. Characterize a set of vectors in terms of **linear combinations**, their **span**, and how they are related to each other geometrically.

Motivation

We want to think about the **algebra** in linear algebra (systems of equations and their solution sets) in terms of **geometry** (points, lines, planes, etc).

$$x - 3y = -3$$
$$2x + y = 8$$



- This will give us better insight into the properties of systems of equations and their solution sets.
- To do this, we need to introduce n-dimensional space $\mathbb{R}^n$, and **vectors** inside it.

$\mathbb{R}^n$

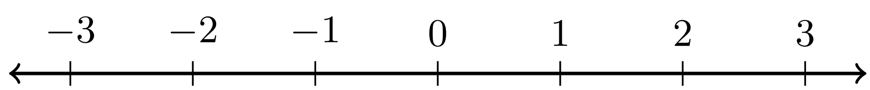
Recall that $\mathbb{R}$ denotes the collection of all real numbers.

Let n be a positive whole number. We define

$$\mathbb{R}^n = \text{all ordered } n\text{-tuples of real numbers } (x_1, x_2, x_3, \ldots, x_n).$$

$= \{ (x_1, \ldots, x_n) : x_1, \ldots, x_n \in \mathbb{R} \}$

When $n = 1$, we get $\mathbb{R}$ back: $\mathbb{R}^1 = \mathbb{R}$. Geometrically, this is the **number line**.



$\mathbb{R}^2$

$\mathbb{R}^2 = \{ (x_1, x_2) : x_1, x_2 \in \mathbb{R} \}$

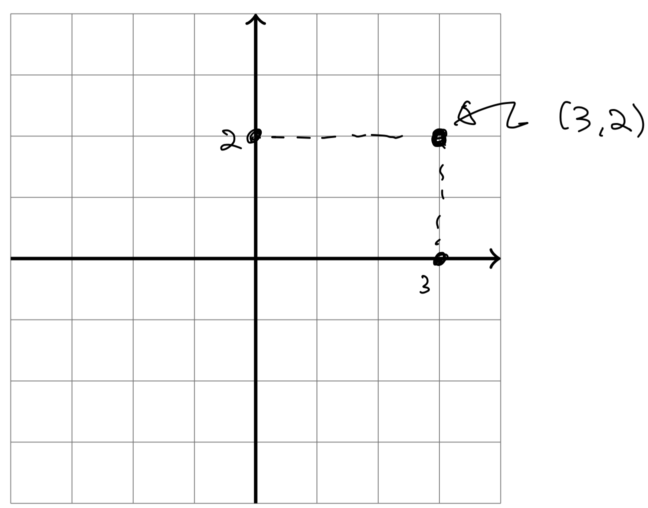
Note that:

- when $n = 2$, we can think of $\mathbb{R}^2$ as a **plane**
- every point in this plane can be represented by an ordered pair of real numbers, its x- and y-coordinates

$\in \mathbb{R}^2$

Example: Sketch the point $(3, 2)$ and the vector $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$.

$\mathbb{R}^2$ = { All points in 2-dim Space }

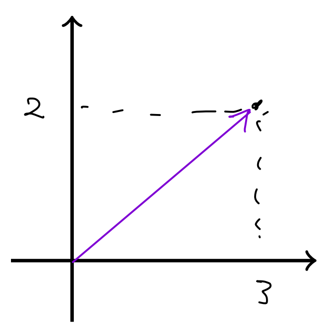


Vectors

In the previous slides, we were thinking of elements of $\mathbb{R}^n$ as **points**: in the line, plane, space, etc.

We can also think of them as **vectors**: arrows with a given length and direction.

$(3, 2)$ $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$



$\mathbb{R}^2 = \{$ All arrows from the origin in plane $\}$

For example, the vector $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ points **horizontally** in the amount of its x-coordinate, and **vertically** in the amount of its y-coordinate.

Addition , Scalar multiplication

Vector Algebra

When we think of an element of $\mathbb{R}^n$ as a vector, we write it as a matrix with n rows and one column:

$$\vec{v} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

Suppose

$$\vec{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad \vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}. \quad \in \mathbb{R}^2$$

Vectors have the following properties.

1. **Scalar Multiple**: $c \in \mathbb{R}$

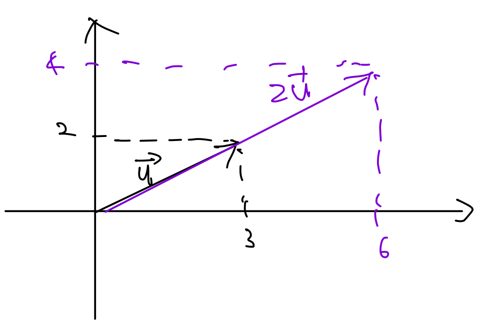
$$c\vec{u} = c \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} cu_1 \\ cu_2 \end{pmatrix}$$

2. **Vector Addition**:

$$\vec{u} + \vec{v} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} u_1 + v_1 \\ u_2 + v_2 \end{pmatrix}$$

Component-wisely!

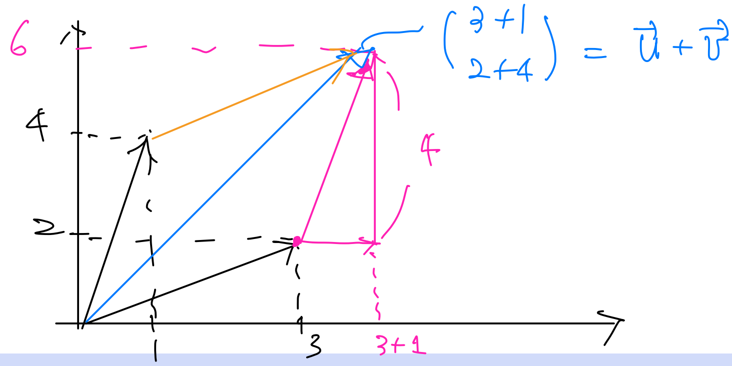
Note that vectors in higher dimensions have the same properties.



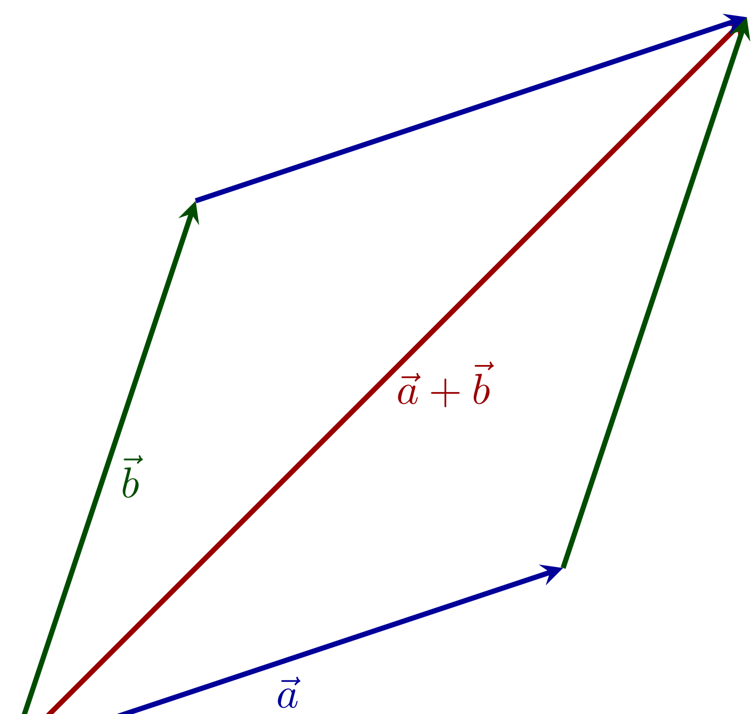
$$\vec{u} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

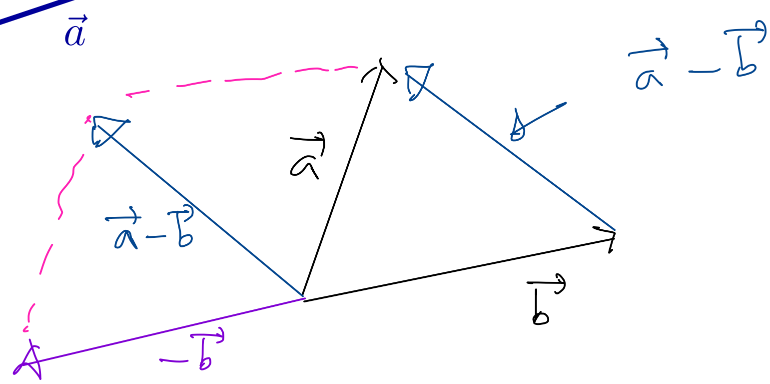
$$2 \cdot \vec{u} = 2 \cdot \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \cdot 3 \\ 2 \cdot 2 \end{pmatrix} = \begin{pmatrix} 6 \\ 4 \end{pmatrix}$$

$\vec{u} = \begin{pmatrix}3\\2\end{pmatrix}$ $\vec{v} = \begin{pmatrix}1\\4\end{pmatrix}$ $\vec{u} + \vec{v} = \begin{pmatrix}3\\2\end{pmatrix} + \begin{pmatrix}1\\4\end{pmatrix} = \begin{pmatrix}3+1\\2+4\end{pmatrix}$



Parallelogram Rule for Vector Addition





$$\vec{a} - \vec{b} = \vec{a} + (-\vec{b})$$

Linear Combinations and Span

Definition

1. Given vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p \in \mathbb{R}^n$, and scalars $c_1, c_2, \ldots, c_p$, the vector below

$$\vec{y} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_p\vec{v}_p$$

is called a **linear combination of** $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$ **with weights** $c_1, c_2, \ldots, c_p$.

2. The set of all linear combinations of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$ is called the **Span** of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$.

$\vec{u} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$

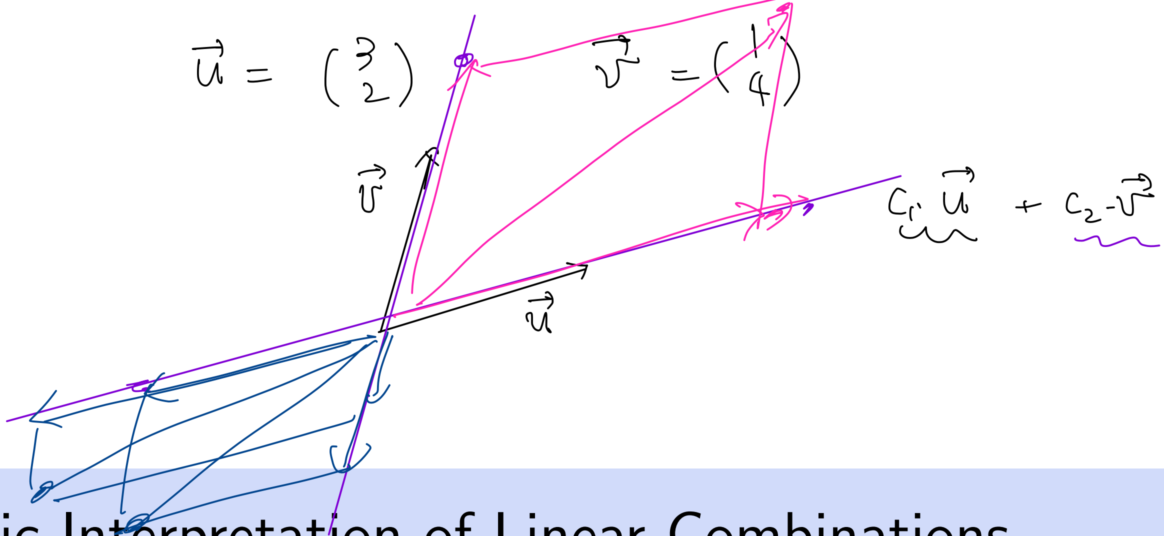
$\vec{u} + \vec{v} = \begin{pmatrix} 4 \\ 6 \end{pmatrix}$ linear combination of $\vec{u}, \vec{v}$

$\vec{u} - \vec{v} = 1 \cdot \vec{u} + (-1) \cdot v$, $\vec{u} = 1 \cdot \vec{u} + 0 \cdot \vec{v}$, $\vec{v}$

$\vec{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0 \cdot \vec{u} + 0 \cdot \vec{v}$

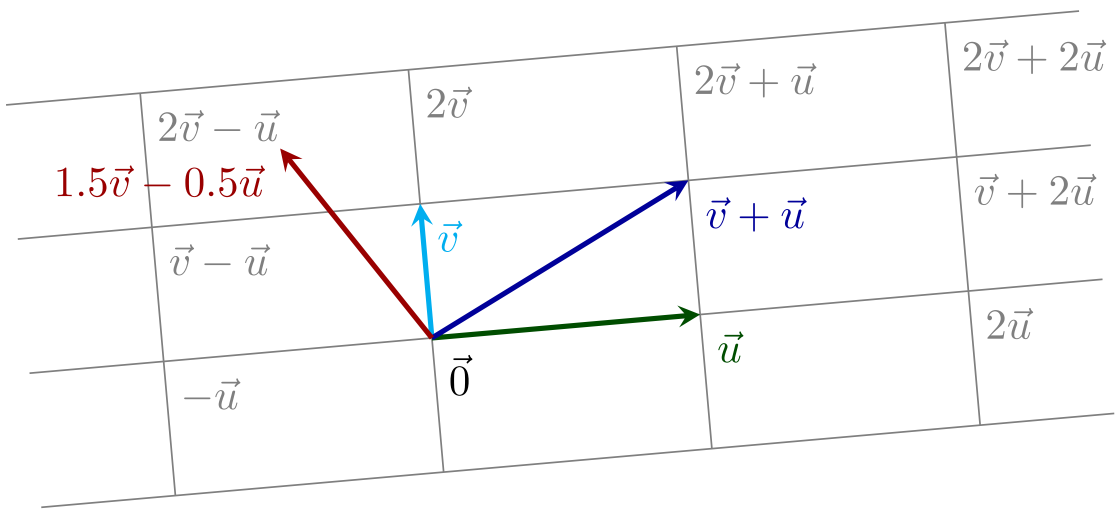
$\text{Span}(\{v_1, v_2, \ldots, v_p\}) = \{ c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_p\vec{v}_p : c_1, \ldots, c_p \in \mathbb{R} \}$

Span of $\vec{u}, \vec{v}$ = Plane containing $\vec{u}, \vec{v}$



Geometric Interpretation of Linear Combinations

Note that any two vectors in $\mathbb{R}^2$ that are not scalar multiples of each other, span $\mathbb{R}^2$. In other words, any vector in $\mathbb{R}^2$ can be represented as a linear combination of two vectors that are not multiples of each other.



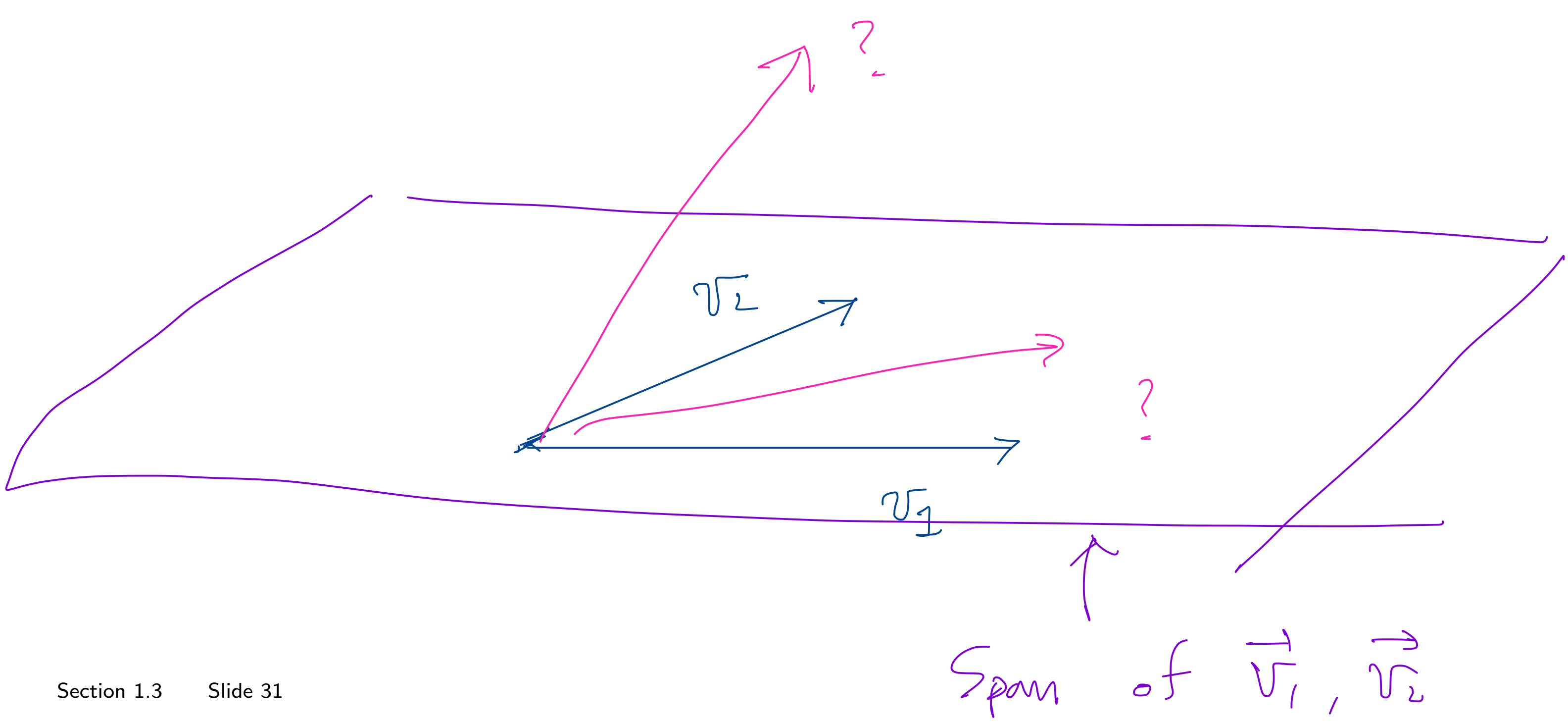
In $\mathbb{R}^2$, if $\vec{u}, \vec{v}$ are not on the same line

$\Rightarrow$ Span of $\vec{u}, \vec{v} = \mathbb{R}^2$

Example

Is $\vec{y}$ in the span of vectors $\vec{v}_1$ and $\vec{v}_2$?

$$\vec{v}_1 = \begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} 2 \\ 5 \\ 6 \end{pmatrix}, \text{ and } \vec{y} = \begin{pmatrix} 7 \\ 4 \\ 15 \end{pmatrix}.$$



Q: y belongs to plane

$\Leftrightarrow$ $y \in \text{Span}(\{\vec{v_1}, \vec{v_2}\})$ $\Leftrightarrow$ y is a linear combi. of v_1, v_2

$\Leftrightarrow$ $x_1, x_2 \in \mathbb{R}$ s.t.

$$\vec{y} = x_1 \cdot \vec{v_1} + x_2 \cdot \vec{v_2}$$

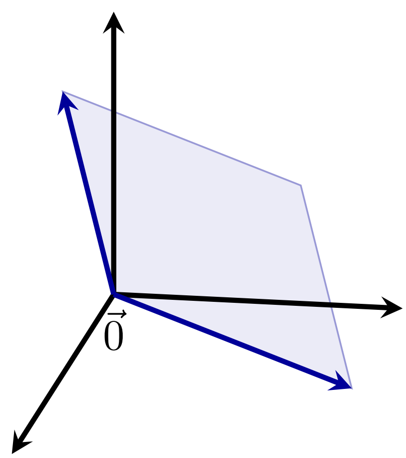
$$\begin{pmatrix} 7 \\ 4 \\ 15 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix} + x_2 \begin{pmatrix} 2 \\ 5 \\ 6 \end{pmatrix} = \begin{pmatrix} x_1 + 2x_2 \\ -2x_1 + 5x_2 \\ -3x_1 + 6x_2 \end{pmatrix}$$

$$\begin{cases} x_1 + 2x_2 = 7 \\ -2x_1 + 5x_2 = 4 \\ -3x_1 + 6x_2 = 15 \end{cases} \quad \Leftarrow \text{ Linear system.}$$

The Span of Two Vectors in $\mathbb{R}^3$

In the previous example, did we find that $\vec{y}$ is in the span of $\vec{v}_1$ and $\vec{v}_2$?

In general: Any two non-parallel vectors in $\mathbb{R}^3$ span a plane that passes through the origin. Any vector in that plane is also in the span of the two vectors.



Section 1.4 : The Matrix Equation

Chapter 1 : Linear Equations

Math 1554 Linear Algebra

"Mathematics is the art of giving the same name to different things."
- H. Poincaré

In this section we introduce another way of expressing a linear system that we will use throughout this course.

8/28/23

$\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_p \in \mathbb{R}^n$ $\qquad v_1 = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \in \mathbb{R}^n$

$c_1, c_2, \cdots, c_p \in \mathbb{R}$

$c_1 \cdot \vec{v}_1 + c_2\vec{v}_2 + \cdots + c_p\vec{v}_p \in \mathbb{R}^n$: a linear combi. of $\vec{v}_1, \cdots, \vec{v}_p$

$\mathrm{Span}(\{\vec{v}_1, \cdots, \vec{v}_p\}) = \{$ All linear combinations $\} = \{ c_1\vec{v}_1 + \cdots + c_p\vec{v}_p : c_1, \cdots, c_p \in \mathbb{R} \}$

1.4 : Matrix Equation $A\vec{x} = \vec{b}$

Topics

We will cover these topics in this section.

1. Matrix notation for systems of equations.
2. The matrix product $A\vec{x}$.

Objectives

For the topics covered in this section, students are expected to be able to do the following.

1. Compute matrix-vector products.
2. Express linear systems as vector equations and matrix equations.
3. Characterize linear systems and sets of vectors using the concepts of span, linear combinations, and pivots.

Notation

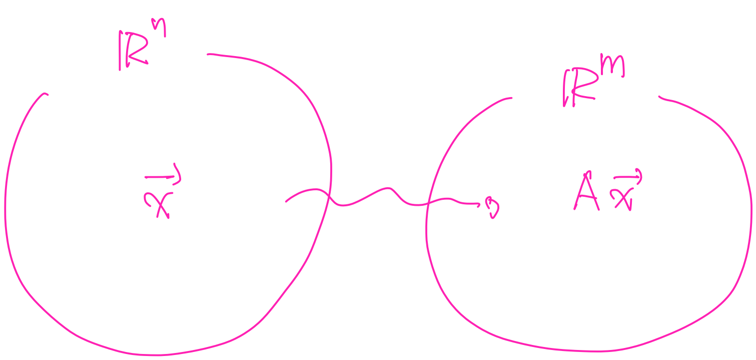
symbol	meaning
elements $\in$ sets	belongs to
$\mathbb{R}^n$	the set of vectors with n real-valued elements
$\mathbb{R}^{m\times n}$	the set of real-valued matrices with m rows and n columns

All entries are real

Example: the notation $\vec{x} \in \mathbb{R}^5$ means that $\vec{x}$ is a vector with five real-valued elements.

$$\mathbb{R}^{2\times 3} = \left\{ \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} : a_{ij} \in \mathbb{R} \quad \begin{matrix} i=1,2 \\ j=1,2,3 \end{matrix} \right\}$$

$$\mathbb{R}^n = \left\{ \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} : a_1, \dots, a_n \in \mathbb{R} \right\} = \mathbb{R}^{n\times 1}$$



$\underline{A \in \mathbb{R}^{m\times n}}$

collection of n $\mathbb{R}^m$ vectors

$$A = \underbrace{\begin{bmatrix} & & & \\ & & & \\ \end{bmatrix}}_{n \text{ columns}} \Big\} \; m \text{ rows}$$

$$= \begin{bmatrix} \underset{\mathbb{R}^m}{\vec{a}_1} & \underset{\mathbb{R}^m}{\vec{a}_2} & \cdots & \underset{\mathbb{R}^m}{\vec{a}_n} \end{bmatrix}$$

Linear Combinations

$$\begin{bmatrix} \vec{a}_1 & \cdots & \vec{a}_n \end{bmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$$

Definition

A is a $m \times n$ matrix with columns $\vec{a}_1, \ldots, \vec{a}_n$ and $x \in \mathbb{R}^n$, then the **matrix vector product** $A\vec{x}$ is a linear combination of the columns of A:

$\mathbb{R}^n$, $n \times 1$ matrix

$$A\vec{x} = \begin{bmatrix} | & | & \cdots & | \\ \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \\ | & | & \cdots & | \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \underbrace{x_1 \vec{a}_1 + x_2 \vec{a}_2 + \cdots + x_n \vec{a}_n}_{} \in \mathbb{R}^m$$

$m \times n$ matrix

$\vec{a}_1 \in \mathbb{R}^m$, $\vec{a}_2 \in \mathbb{R}^m$, $\vec{a}_n \in \mathbb{R}^m$

$m \times 1$ matrix.

Note that $A\vec{x}$ is in the span of the columns of A.

$(m \times n) \cdot \mathbb{R}^n = \mathbb{R}^m$

Example

The following product can be written as a linear combination of vectors:

$$\underset{2\times 3}{\begin{bmatrix} 1 & 0 & -1 \\ 0 & -3 & 3 \end{bmatrix}} \underset{\mathbb{R}^3}{\begin{bmatrix} 4 \\ 3 \\ 7 \end{bmatrix}} = 4\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 0 \\ -3 \end{bmatrix} + 7\begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

$\mathbb{R}^2$

$$A\vec{x} = \vec{b}$$

$$\begin{bmatrix} \vec{a_1} & \vec{a_2} & \cdots & \vec{a_n} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} = \vec{b}$$

$\vec{b}$ belongs to Span$\{\vec{a_1}, \cdots, \vec{a_n}\}$, is a linear combi.

$$x_1 \cdot \vec{a_1} + x_2 \cdot \vec{a_2} + \cdots + x_n \vec{a_n} = \vec{b}$$

$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

Solution Sets

$$A = \begin{bmatrix} \underset{\in \mathbb{R}^m}{\vec{a_1}} & \cdots & \underset{\in \mathbb{R}^m}{\vec{a_n}} \end{bmatrix} \qquad \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

Theorem

If A is a $m \times n$ matrix with columns $\vec{a}_1, \ldots, \vec{a}_n$, and $\vec{x} \in \mathbb{R}^n$ and $\vec{b} \in \mathbb{R}^m$, then the solutions to

$$\underset{m \times n}{A}\underset{\mathbb{R}^n}{\vec{x}} = \underset{\mathbb{R}^m}{\vec{b}} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

has the same set of solutions as the vector equation

$$x_1\vec{a}_1 + \cdots + x_n\vec{a}_n = \vec{b}$$

which as the same set of solutions as the set of linear equations with the augmented matrix

$$\left[\begin{array}{cccc|c} \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n & \vec{b} \end{array}\right]$$

Existence of Solutions

$A, \vec{b}$ are given

all possible $\vec{x}$ s.t. $A\vec{x} = \vec{b}$

Theorem

The equation $A\vec{x} = \vec{b}$ has a solution if and only if $\vec{b}$ is a linear combination of the columns of A.

$\Leftrightarrow$ Linear system is consistent

$\Leftrightarrow$ Aug. Matrix $[\,\vec{a}_1 \;\cdots\; \vec{a}_n \mid \vec{b}\,] = [A \mid \vec{b}]$

Last column is not pivot

Example

For what vectors $\vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ does the equation have a solution?

$$\underbrace{\begin{pmatrix} 1 & 3 & 4 \\ 2 & 8 & 4 \\ 0 & 1 & -2 \end{pmatrix}}_{= A \in \mathbb{R}^{3\times 3}} \underset{\mathbb{R}^3}{\vec{x}} = \vec{b} \in \mathbb{R}^3$$

Goal: last column = pivot?

$$\left[\begin{array}{ccc|c} 1 & 3 & 4 & b_1 \\ 2 & 8 & 4 & b_2 \\ 0 & 1 & -2 & b_3 \end{array}\right] \xrightarrow[② \to ② - 2\times①]{} \left[\begin{array}{ccc|c} 1 & 3 & 4 & b_1 \\ 0 & 2 & -4 & b_2 - 2b_1 \\ 0 & 1 & -2 & b_3 \end{array}\right] \xrightarrow[② \times \frac{1}{2}]{} \left[\begin{array}{ccc|c} 1 & 3 & 4 & b_1 \\ 0 & 1 & -2 & \frac{b_2 - 2b_1}{2} \\ 0 & 1 & -2 & b_3 \end{array}\right]$$

$$\xrightarrow[③ \to ③ - ②]{} \left[\begin{array}{ccc|c} 1 & 3 & 4 & b_1 \\ 0 & 1 & -2 & \frac{b_2 - 2b_1}{2} \\ 0 & 0 & 0 & \boxed{b_3 - \frac{b_2 - 2b_1}{2}} \end{array}\right]$$

has a solution
Consistent $\leftarrow$ $b_3 - \frac{b_2 - 2b_1}{2} = 0$ " $\neq 0$ $\rightarrow$ no solution

$A\vec{x} = \vec{b}$ has a solution

$\Leftrightarrow$ $\vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ satisfies $b_3 - \frac{b_2 - 2b_1}{2} = 0$

$\Leftrightarrow$ $\vec{b} \in \text{Span}\left(\left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 8 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ 4 \\ -2 \end{pmatrix} \right\} \right)$

Example

For what vectors $\vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ does the equation have a solution?

$$\begin{pmatrix} 1 & 3 & 4 \\ 2 & 8 & 4 \\ 0 & 1 & -2 \end{pmatrix} \vec{x} = \vec{b}$$

The Row Vector Rule for Computing $A\vec{x}$

$$\begin{bmatrix} 1 & 0 & 2 & 0 & 3 \\ 0 & 1 & 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \quad \\ \quad \end{bmatrix}$$

Summary

We now have four **equivalent** ways of expressing linear systems.

1. A system of equations:

$$\begin{aligned} 2x_1 + 3x_2 &= 7 \\ x_1 - x_2 &= 5 \end{aligned}$$

2. An augmented matrix:

$$\left[\begin{array}{cc|c} 2 & 3 & 7 \\ 1 & -1 & 5 \end{array}\right]$$

3. A vector equation: linear combi.

$$x_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \end{pmatrix}$$

4. As a matrix equation:

$$\begin{pmatrix} 2 & 3 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \end{pmatrix}$$

Each representation gives us a different way to think about linear systems.

Section 1.5 : Solution Sets of Linear Systems

Chapter 1 : Linear Equations

Math 1554 Linear Algebra

1.5 : Solution Sets of Linear Systems

Topics
We will cover these topics in this section.

1. Homogeneous systems
2. Parametric **vector** forms of solutions to linear systems

Objectives
For the topics covered in this section, students are expected to be able to do the following.

1. Express the solution set of a linear system in parametric vector form.
2. Provide a geometric interpretation to the solution set of a linear system.
3. Characterize homogeneous linear systems using the concepts of free variables, span, pivots, linear combinations, and echelon forms.

Linear system : $A \cdot \vec{x} = \vec{b}$

Homogeneous Systems

Definition

Linear systems of the form $A \cdot \vec{x} = \vec{0}$ are **homogeneous**.

Linear systems of the form $A\vec{x} = \vec{b}$, $\vec{b} \neq \vec{0}$ are **inhomogeneous**.

Because homogeneous systems always have the **trivial solution**, $\vec{x} = \vec{0}$, the interesting question is whether they have non-trivial solutions.

Observation

$\underline{A\vec{x} = \vec{0}}$ has a nontrivial solution $\iff$ Infinitely many solutions

$\iff$ there is a free variable

$\iff$ A has a column with no pivot.

$\swarrow$ always non-pivot.

$\left[\begin{array}{c|c} A & \vec{0} \end{array}\right]$

Example: a Homogeneous System

Identify the free variables, and the solution set, of the system.

$$\begin{aligned} x_1 + 3x_2 + x_3 &= 0 \\ 2x_1 - x_2 - 5x_3 &= 0 \\ x_1 - 2x_3 &= 0 \end{aligned}$$

$$\begin{bmatrix} 1 & 3 & 1 \\ 2 & -1 & -5 \\ 1 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & 1 \\ 2 & -1 & -5 \\ 1 & 0 & -2 \end{bmatrix} \xrightarrow[\substack{R_2 \to R_2 - 2\cdot R_1 \\ R_3 \to R_3 - R_1}]{} \begin{bmatrix} 1 & 3 & 1 \\ 0 & -7 & -7 \\ 0 & -3 & -3 \end{bmatrix}$$

$$\xrightarrow[\substack{(-\frac{1}{7})\cdot R_2 \\ (-\frac{1}{3})\cdot R_3}]{} \begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow[\substack{R_1 \to R_1 - 3R_2 \\ R_3 \to R_3 - R_2}]{} \underbrace{\begin{bmatrix} \underline{1} & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}}_{}$$

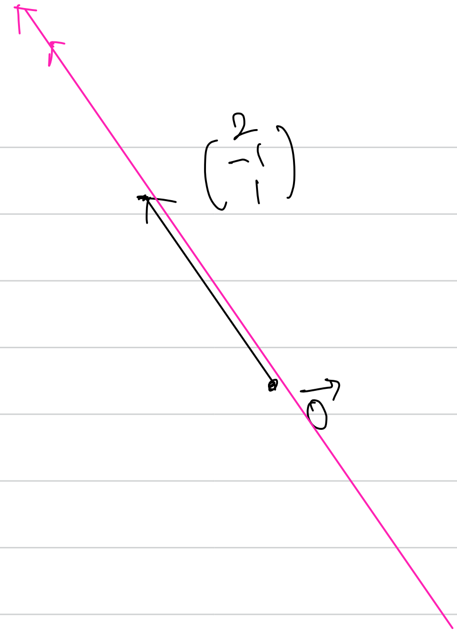
RREF.

$$\begin{cases} x_1 - 2x_3 = 0 \\ x_2 + x_3 = 0 \end{cases} \qquad \begin{aligned} x_1 &= 2x_3 \\ x_2 &= -x_3 \end{aligned}$$

Solution Set $= \{ (2x_3, -x_3, x_3) : \quad x_3 \in \mathbb{R} \}$

$= \mathrm{Span}\left(\left\{ \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \right\} \right)$

$$\begin{pmatrix} 2x_3 \\ -x_3 \\ x_3 \end{pmatrix} = x_3 \cdot \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$



Ex

$$\xrightarrow[\text{operation}]{\text{row}} \begin{bmatrix} 1 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$x_1 - x_2 - x_3 = 0$$

$$x_1 = x_2 + x_3$$

Solution set $= \{ (x_2 + x_3, x_2, x_3) : x_2 \in \mathbb{R}, \ x_3 \in \mathbb{R} \}$

$$\left(\begin{pmatrix} x_2 + x_3 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_2 \\ 0 \end{pmatrix} + \begin{pmatrix} x_3 \\ 0 \\ x_3 \end{pmatrix} = x_2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right)$$

$= \{$ All linear Combi. of $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \}$

$= \text{Span}\left(\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\} \right)$

Parametric Forms, Homogeneous Case

In the example on the previous slide we expressed the solution to a system using a vector equation. This is a **parametric form** of the solution.

In general, suppose the free variables for $A\vec{x} = \vec{0}$ are $x_k, \ldots, x_n$. Then all solutions to $A\vec{x} = \vec{0}$ can be written as

$$\vec{x} = x_k\vec{v}_k + x_{k+1}\vec{v}_{k+1} + \cdots + x_n\vec{v}_n$$

for some $\vec{v}_k, \ldots, \vec{v}_n$. This is the **parametric form** of the solution.

Example 2 (non-homogeneous system)

Write the parametric vector form of the solution, and give a geometric interpretation of the solution.

$$\begin{aligned} x_1 + 3x_2 + x_3 &= 9 \\ 2x_1 - x_2 - 5x_3 &= 11 \\ x_1 - 2x_3 &= 6 \end{aligned}$$

(Note that the left-hand side is the same as Example 1).

$$\left[\begin{array}{ccc|c} 1 & 3 & 1 & 9 \\ 2 & -1 & -5 & 11 \\ 1 & 0 & -2 & 6 \end{array}\right] \xrightarrow[\substack{R_2 \to R_2 - 2\cdot R_1 \\ R_3 \to R_3 - R_1}]{} \left[\begin{array}{ccc|c} 1 & 3 & 1 & 9 \\ 0 & -7 & -7 & -7 \\ 0 & -3 & -3 & -3 \end{array}\right]$$

$$\xrightarrow[\substack{(-\frac{1}{7})\cdot R_2 \\ (-\frac{1}{3})\cdot R_3}]{} \left[\begin{array}{ccc|c} 1 & 3 & 1 & 9 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{array}\right] \xrightarrow[\substack{R_1 \to R_1 - 3R_2 \\ R_3 \to R_3 - R_2}]{} \underbrace{\left[\begin{array}{ccc|c} 1 & 0 & -2 & 6 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]}_{\text{RREF.}}$$

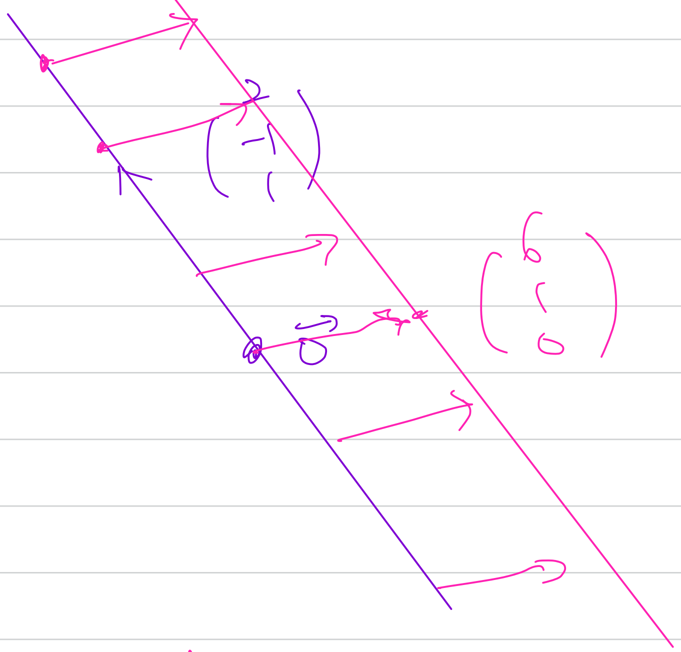
$x_1 - 2x_3 = 6$ $\qquad$ $x_1 = 2x_3 + 6$

$x_2 + x_3 = 1$ $\qquad$ $x_2 = -x_3 + 1$

Solution Set $= \{ (2x_3 + 6, -x_3 + 1, x_3) : x_3 \in \mathbb{R} \}$

$$\begin{pmatrix} 2x_3 + 6 \\ -x_3 + 1 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2x_3 \\ -x_3 \\ x_3 \end{pmatrix} + \begin{pmatrix} 6 \\ 1 \\ 0 \end{pmatrix} = x_3 \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} + \begin{pmatrix} 6 \\ 1 \\ 0 \end{pmatrix}$$

Hom.



InHom.

Section 1.7 : Linear Independence

Chapter 1 : Linear Equations

Math 1554 Linear Algebra

1.7 : Linear Independence

Topics
We will cover these topics in this section.

- Linear independence
- Geometric interpretation of linearly independent vectors

Objectives
For the topics covered in this section, students are expected to be able to do the following.

1. Characterize a set of vectors and linear systems using the concept of linear independence.
2. Construct dependence relations between linearly dependent vectors.

Motivating Question
What is the smallest number of vectors needed in a parametric solution to a linear system?

$\{\vec{v}_1, \cdots \vec{v}_k\}$ is linearly indep. if

$$\begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_k \end{bmatrix} \cdot \underbrace{\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{\text{homogeneous}} \quad \text{implies} \quad \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Linear Independence

homogeneous has always a solution
trivial solution
unique solution

A set of vectors $\{\vec{v}_1, \ldots, \vec{v}_k\}$ in $\mathbb{R}^n$ are **linearly independent** if

$$\underbrace{c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k}_{\text{linear combi.}} = \vec{0} \quad A \leftarrow \text{vector equation}$$

has $\boxed{\text{only}}$ the **trivial** solution. It is **linearly dependent** otherwise.

$c_1 = c_2 = \cdots = c_k = 0$

In other words, $\{\vec{v}_1, \ldots, \vec{v}_k\}$ are linearly dependent if there are real numbers $c_1, c_2, \ldots, c_k$ **not all zero** so that

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k = \vec{0}$$

$\{\vec{v}_1, \cdots, \vec{v}_k\}$ lin. indep. $\Leftrightarrow$ no free variable

$\Leftrightarrow \quad A = \begin{bmatrix} \vec{v}_1 & \cdots & \vec{v}_k \end{bmatrix} \xrightarrow[\text{row operations}]{}$ Echlon form.

has no non-pivot columns

Or, Every column is pivot.

Consider the vectors:

$$\vec{v}_1, \vec{v}_2, \dots \vec{v}_k$$

To determine whether the vectors are linearly independent, we can set the linear combination to the zero vector:

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_k \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = V\vec{c} \stackrel{??}{=} \vec{0}$$

Linear independence: There is NO non-zero solution $\vec{c}$

Linear dependence: There is a non-zero solution $\vec{c}$.

Example 1

For what values of h are the vectors linearly independent?

$$\vec{v}_1 = \begin{bmatrix}1\\1\\h\end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix}1\\h\\1\end{bmatrix}, \quad \begin{bmatrix}h\\1\\1\end{bmatrix} = \vec{v}_3$$

Definition: $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$ implies $c_1 = c_2 = c_3 = 0$

$\Leftrightarrow \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ lin. indep.

$\Leftrightarrow \begin{bmatrix}\vec{v}_1 & \vec{v}_2 & \vec{v}_3\end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$ 　 $\begin{bmatrix}1&1&h\\1&h&1\\h&1&1\end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$ has the only trivial solution $\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$

$\Leftrightarrow \begin{bmatrix}1&1&h\\1&h&1\\h&1&1\end{bmatrix}$ has 3 pivots.

Assume ($h \neq 1$: $h-1 \neq 0$ or $1-h \neq 0$)

$$\begin{bmatrix}1&1&h\\1&h&1\\h&1&1\end{bmatrix} \xrightarrow[R_3 \to R_3 - hR_1]{R_2 \to R_2 - R_1} \begin{bmatrix}1&1&h\\0&h-1&1-h\\0&1-h&1-h^2\end{bmatrix} \xrightarrow[\left(\frac{1}{1-h}\right)R_3]{\left(\frac{1}{h-1}\right)R_2} \begin{bmatrix}1&1&h\\0&1&-1\\0&1&1+h\end{bmatrix}$$

$1-h = -(h-1)$

$1-h^2 = 1^2 - h^2 = (1+h)(1-h)$

$$\xrightarrow{R_3 \to R_3 - R_2} \begin{bmatrix}1&1&h\\0&1&-1\\0&0&h+2\end{bmatrix} \quad h+2 \neq 0$$

$\begin{cases} h \neq 1 \\ h \neq -2 \end{cases} \Leftrightarrow \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ lin. indep.

$h = 1$

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ h \end{bmatrix}, \begin{bmatrix} 1 \\ h \\ 1 \end{bmatrix}, \begin{bmatrix} h \\ 1 \\ 1 \end{bmatrix} \right\} = \left\{ \underset{=\vec{v}_1}{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}, \underset{=\vec{v}_2}{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}, \underset{=\vec{v}_3}{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}} \right\}$$

$$\overset{1 \,=}{\underset{\neq\, 0}{c_1}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \overset{-1 \,=}{\underset{\neq\, 1}{c_2}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \overset{0 \,=}{\underset{= -1}{c_3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 0$$

lin. dep.

$\Rightarrow c_1 = c_2 = c_3 = 0 \Rightarrow$ indep.

Ex

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix}, \vec{v}_3, \vec{v}_4, \cdots, \vec{v}_{200} \right\}$$

$c_1 = 2 \quad c_2 = -1, \quad c_3 = \cdots = c_{200} = 0.$

$h = -2$

$$c_1 \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} = 0$$

if $c_1 = c_2 = c_3 = 1.$

Example 2 (One Vector)

Suppose $\vec{v} \in \mathbb{R}^n$. When is the set $\{\vec{v}\}$ linearly dependent?

$$c \cdot \vec{v} = \vec{0} \Rightarrow \begin{cases} c = 0 \Rightarrow \text{lin. indep} \\ \text{Can find } c \neq 0 \text{ and } c \cdot \vec{v} = 0 \Rightarrow \text{lin. dep.} \end{cases}$$

Possible only when $\vec{v} = \vec{0}$

Example 3 (Two Vectors)

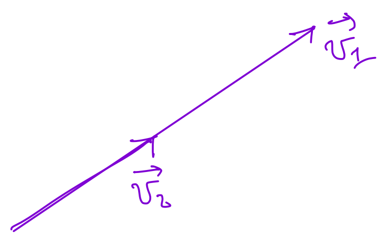
Suppose $\vec{v}_1, \vec{v}_2 \in \mathbb{R}^n$. When is the set $\{\vec{v}_1, \vec{v}_2\}$ linearly dependent? Provide a geometric interpretation.

$$\begin{cases} c_1 \vec{v}_1 + c_2 \vec{v}_2 = \vec{0} \\ c_1 \neq 0 \quad \text{or} \quad c_2 \neq 0 \end{cases} \iff \{\vec{v}_1, \vec{v}_2\} \text{ lin. dep}$$

$c_1 \neq 0$

$$c_1 \vec{v}_1 = -c_2 \vec{v}_2$$

$$\vec{v}_1 = \left(-\frac{c_2}{c_1}\right) \vec{v}_2 = \text{a scalar multiple of } \vec{v}_2$$



$\Rightarrow$ $\vec{v}_1, \vec{v}_2$ are on the same line.

$\Rightarrow$ $\text{Span}(\{\vec{v}_1, \vec{v}_2\}) =$ a line or $\{\vec{0}\}$

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$\{\vec{v}_1, \dots, \vec{v}_p\}$ lin. indep.

$\Leftrightarrow$ $c_1\vec{v}_1 + \dots + c_p\vec{v}_p = \vec{0}$ implies $c_1 = \dots = c_p = 0$

$\Leftrightarrow$ $A\vec{x} = [\vec{v}_1, \dots, \vec{v}_p] \cdot \begin{bmatrix} c_1 \\ \vdots \\ c_p \end{bmatrix} = \vec{0}$ $\Big\langle$ has the only trivial solution / has no free variable

$\Leftrightarrow$ Every column of A is a pivot column.

Two Theorems

Fact 1. Suppose $\vec{v}_1, \dots, \vec{v}_k$ are vectors in $\mathbb{R}^n$. If $k > n$, then $\{\vec{v}_1, \dots, \vec{v}_k\}$ is linearly dependent.

maximum pivot columns

$[\vec{v}_1, \dots, \vec{v}_k] \in \mathbb{R}^{n \times k}$ $= \min\{n, k\}$

lin. indep $\Leftrightarrow$ k pivot columns $\leftarrow$ impossible.

Fact 2. If any one or more of $\vec{v}_1, \dots, \vec{v}_k$ is $\vec{0}$, then $\{\vec{v}_1, \dots, \vec{v}_k\}$ is linearly dependent.

$\vec{v}_1 = \vec{0}$

$\underset{1}{c_1}\vec{v}_1 + \underset{0}{c_2}\vec{v}_2 + \underset{0}{c_k}\vec{v}_k = 0 \Rightarrow$ lin. dep.

Section 1.8 : An Introduction to Linear Transforms

Chapter 1 : Linear Equations

Math 1554 Linear Algebra

1.8 : An Introduction to Linear Transforms

Topics

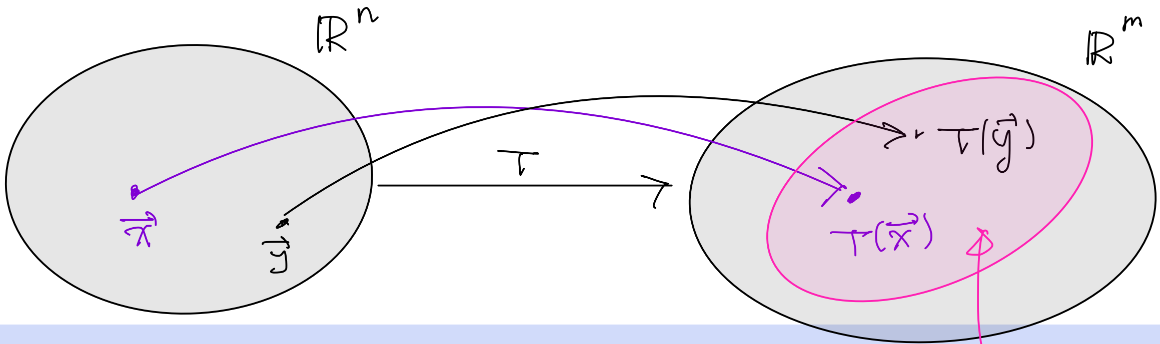
We will cover these topics in this section.

1. The definition of a linear transformation.
2. The interpretation of matrix multiplication as a linear transformation.

Objectives

For the topics covered in this section, students are expected to be able to do the following.

1. Construct and interpret linear transformations in $\mathbb{R}^n$ (for example, interpret a linear transform as a projection, or as a shear).
2. Characterize linear transforms using the concepts of
   - existence and uniqueness
   - domain, co-domain and range



From Matrices to Functions

Let A be an $m \times n$ matrix. We define a function

# of rows ↑ # of columns

$$T : \mathbb{R}^n \to \mathbb{R}^m, \quad T(\vec{v}) = A\vec{v}$$

# of columns # of rows

Input outputs

This is called a **matrix transformation**.

- The **domain** of T is $\mathbb{R}^n$.
- The **co-domain** or **target** of T is $\mathbb{R}^m$.
- The vector $T(\vec{x})$ is the **image** of $\vec{x}$ under T $\in \mathbb{R}^n$ (domain)
- The set of all possible images $T(\vec{x})$ is the **range**.

This gives us **another** interpretation of $A\vec{x} = \vec{b}$:

- set of equations
- augmented matrix
- matrix equation
- vector equation
- linear transformation equation

Ex $A \in \mathbb{R}^{1\times 1}$, $A = (a)$

$T(x) = a \cdot x$

Ex $A \in \mathbb{R}^{1\times 2}$, $A = [2 \;\; 3]$

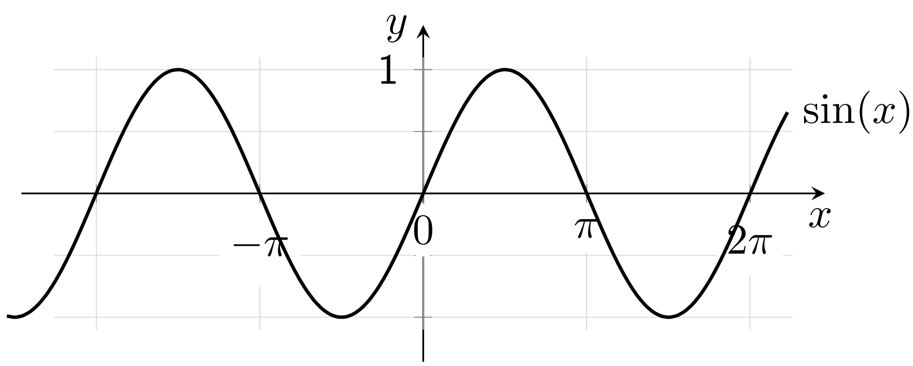
$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = [2 \;\; 3]\begin{bmatrix} x \\ y \end{bmatrix} = 2x + 3y$

Functions from Calculus

Many of the functions we know have **domain** and **codomain** $\mathbb{R}$.We can express the **rule** that defines the function $\sin$ this way:

$$f : \mathbb{R} \to \mathbb{R} \qquad f(x) = \sin(x)$$

In calculus we often think of a function in terms of its graph, whose horizontal axis is the **domain**, and the vertical axis is the **codomain**.



This is ok when the domain and codomain are $\mathbb{R}$. It's hard to do when the domain is $\mathbb{R}^2$ and the codomain is $\mathbb{R}^3$. We would need five dimensions to draw that graph.

$T: \mathbb{R}^2 \longrightarrow \mathbb{R}^3$

$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \longmapsto T(\vec{x}) = A\vec{x}$

Example 1

$= \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$, $\vec{u} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 7 \\ 5 \\ 7 \end{bmatrix}$.

$\in \mathbb{R}^{3\times 2}$

a) Compute $T(\vec{u})$. $T(\vec{u}) = A \cdot \vec{u} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = 3 \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + 4 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$= \begin{bmatrix} 3+4 \\ 4 \\ 3+4 \end{bmatrix}$

b) Calculate $\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$ so that $T(\vec{v}) = \vec{b}$

$\begin{bmatrix} x+y \\ y \\ x+y \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 5 \\ 7 \end{bmatrix}$

$y = 5$

$x = 2$

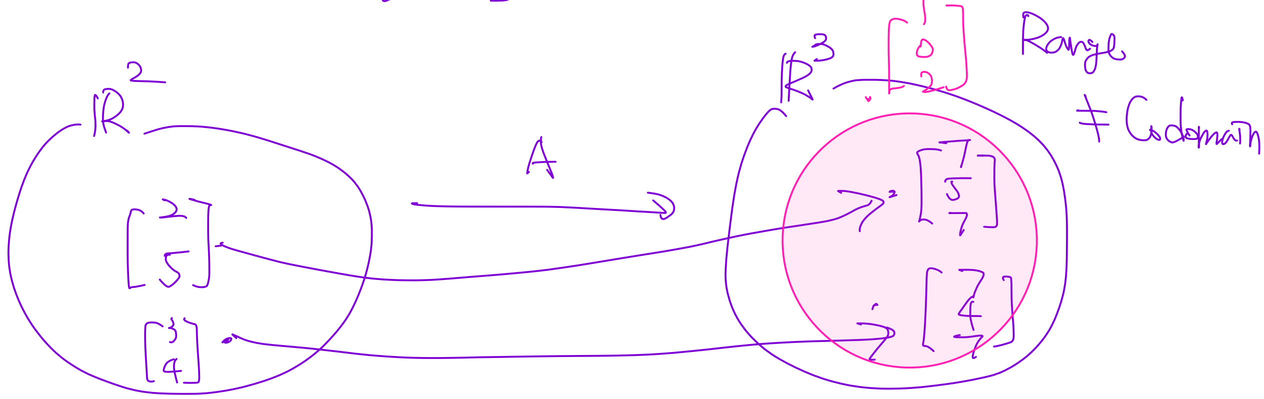
$\vec{v} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}.$

c) Give a $\vec{c} \in \mathbb{R}^3$ so there is no $\vec{v}$ with $T(\vec{v}) = \vec{c}$

or: Give a $\vec{c}$ that is not in the range of T.

or: Give a $\vec{c}$ that is not in the span of the columns of A.

$\vec{c} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \neq T(\vec{v})$ for any $\vec{v}$



Linear Transformations

A function $T : \mathbb{R}^n \to \mathbb{R}^m$ is **linear** if

Image of sum

- $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$ for all $\vec{u}, \vec{v}$ in $\mathbb{R}^n$. = Sum of Images.
- $T(c\vec{v}) = cT(\vec{v})$ for all $\vec{v} \in \mathbb{R}^n$, and c in $\mathbb{R}$. Image of scalar multiple

So if T is linear, then = scalar multiple of Image.

Image of lin. comb. = lin. combi. of Images.

$$T(c_1\vec{v}_1 + \cdots + c_k\vec{v}_k) = c_1T(\vec{v}_1) + \cdots + c_kT(\vec{v}_k)$$

This is called the **principle of superposition**. The idea is that if we know $T(\vec{e}_1), \ldots, T(\vec{e}_n)$, then we know every $T(\vec{v})$.

Fact: Every matrix transformation T_A is linear.

$$T_A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$$
$$\vec{x} \longmapsto A\vec{x}$$

Example 2

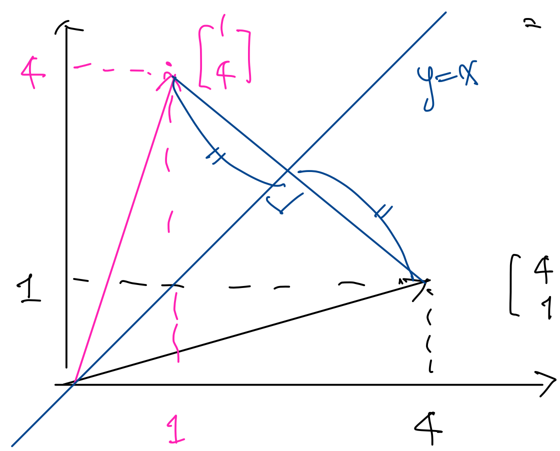
Suppose T is the linear transformation $T(\vec{x}) = A\vec{x}$. Give a short geometric interpretation of what $T(\vec{x})$ does to vectors in $\mathbb{R}^2$.

1) $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \in \mathbb{R}^{2\times 2}$

Reflection

$$T : \mathbb{R}^2 \to \mathbb{R}^2$$

$$\begin{bmatrix} x \\ y \end{bmatrix} \longmapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = x\begin{bmatrix} 0 \\ 1 \end{bmatrix} + y\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix}$$



$$\begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

2) $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

3) $A = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$ for $k \in \mathbb{R}$

Example 2

Suppose T is the linear transformation $T(\vec{x}) = A\vec{x}$. Give a short geometric interpretation of what $T(\vec{x})$ does to vectors in $\mathbb{R}^2$.

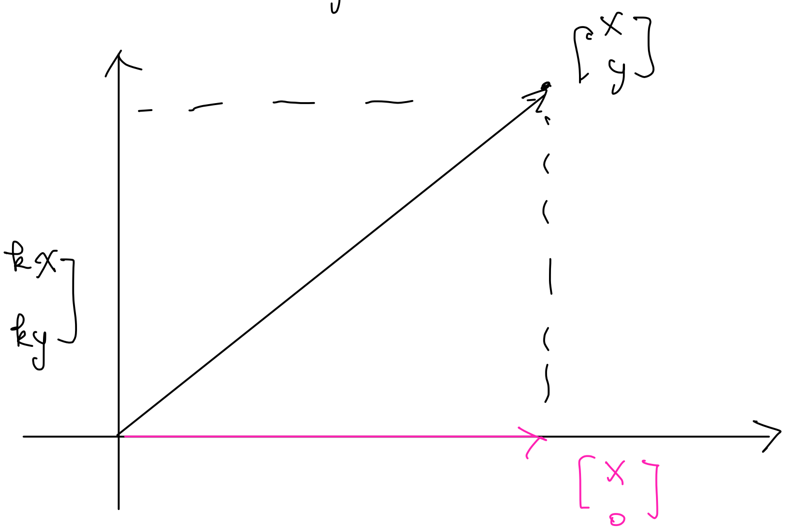
1) $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

2) $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

$T : \mathbb{R}^2 \to \mathbb{R}^2$

$\begin{bmatrix} x \\ y \end{bmatrix} \longmapsto \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 0 \end{bmatrix}$

Projection onto x-axis

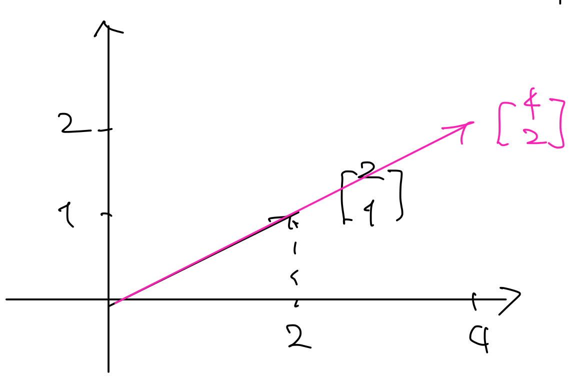


3) $A = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$ for $k \in \mathbb{R}$

$\begin{bmatrix} x \\ y \end{bmatrix} \longmapsto \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} kx \\ ky \end{bmatrix}$

Scaling by k

<u>Ex</u> $k = 2$



Example 3

What does T_A do to vectors in $\mathbb{R}^3$?

a) $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

b) $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Example 4

A linear transformation $T : \mathbb{R}^2 \mapsto \mathbb{R}^3$ satisfies

$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}5\\-7\\2\end{bmatrix}, \qquad T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}-3\\8\\0\end{bmatrix}$$

What is the matrix that represents T?

$\mathbb{R}^{3\times 2}$

Find A s.t. $\underline{T(\vec{x}) = A\vec{x}}$

$$1\cdot\vec{v}_1 + 0\cdot\vec{v}_2 = \underbrace{\begin{bmatrix}\vec{v}_1 & \vec{v}_2\end{bmatrix}\cdot\begin{bmatrix}1\\0\end{bmatrix}}_{} = \begin{bmatrix}5\\-7\\2\end{bmatrix}$$

$\mathbb{R}^3 \ni \vec{v}_1$, $\vec{v}_2 \in \mathbb{R}^3$

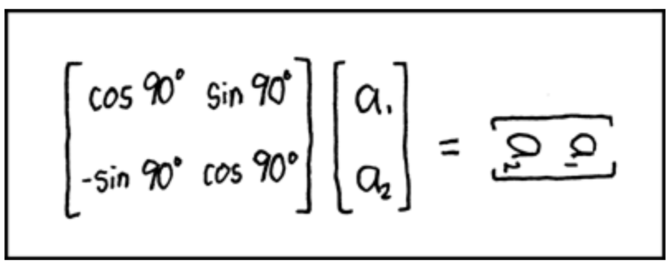
$$0\cdot\vec{v}_1 + 1\cdot\vec{v}_2 = \begin{bmatrix}\vec{v}_1 & \vec{v}_2\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}-3\\8\\0\end{bmatrix}$$

$$A = \begin{bmatrix}5 & -3\\-7 & 8\\2 & 0\end{bmatrix}$$

Section 1.9 : Linear Transforms

Chapter 1 : Linear Equations

Math 1554 Linear Algebra



https://xkcd.com/184

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$T : \mathbb{R}^n \to \mathbb{R}^m$

$\vec{x} \in \mathbb{R}^n$ (domain)

$A \in \mathbb{R}^{m \times n}$

Image of $\vec{x}$

$T(\vec{x}) = A \cdot \vec{x} \in \mathbb{R}^m$

$$\begin{cases} T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v}) \\ T(c\vec{u}) = c \cdot T(\vec{u}). \end{cases}$$

1.9 : Matrix of a Linear Transformation

Topics

We will cover these topics in this section.

1. The **standard vectors** and the **standard matrix**.
2. Two and three dimensional transformations in more detail.
3. **Onto** and **one-to-one** transformations.

Objectives

For the topics covered in this section, students are expected to be able to do the following.

1. Identify and construct linear transformations of a matrix.
2. Characterize linear transformations as onto and/or one-to-one.
3. Solve linear systems represented as linear transforms.
4. Express linear transforms in other forms, such as as matrix equations or as vector equations.

Definition: The Standard Vectors

The **standard vectors** in $\mathbb{R}^n$ are the vectors $\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n$, where:

$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad \vec{e}_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

For example, in $\mathbb{R}^3$,

$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad \vec{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$A = \begin{bmatrix} T(\vec{e}_1) & T(\vec{e}_2) & T(\vec{e}_3) \end{bmatrix}$$

$$T : \mathbb{R}^3 \to \mathbb{R}^4$$

$$\vec{x} \longmapsto A \cdot \vec{x}$$

$$A = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{bmatrix}$$

$$T(\vec{e}_1) = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 1 \cdot \vec{v}_1 + 0 \cdot \vec{v}_2 + 0 \cdot \vec{v}_3$$

$$= \vec{v}_1$$

$$T(\vec{e}_2) = \vec{v}_2 \quad , \quad T(\vec{e}_3) = \vec{v}_3$$

A Property of the Standard Vectors

Note: if A is an $m \times n$ matrix with columns $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$, then

$$A\vec{e}_i = \vec{v}_i, \text{ for } i = 1, 2, \ldots, n$$

So multiplying a matrix by $\vec{e}_i$ gives column i of A.

Example

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \vec{e}_2 = \begin{pmatrix} 2 \\ 5 \\ 8 \end{pmatrix}$$

The Standard Matrix

Theorem

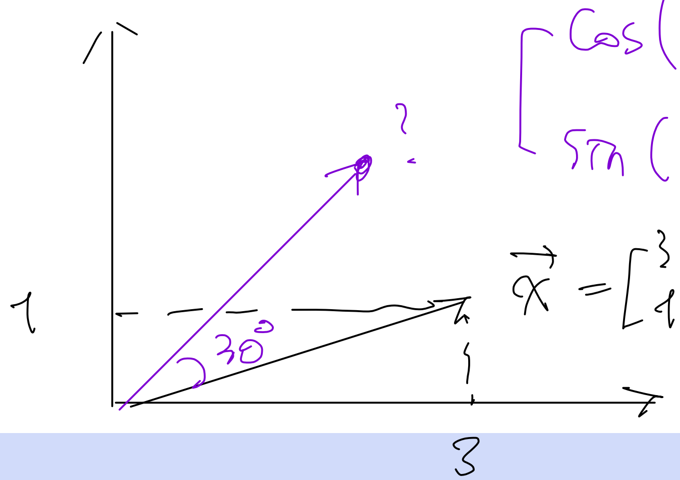
Let $T : \mathbb{R}^n \mapsto \mathbb{R}^m$ be a linear transformation. Then there is a unique matrix A such that

$$T(\vec{x}) = A\vec{x}, \qquad \vec{x} \in \mathbb{R}^n.$$

In fact, A is a $m \times n$, and its j^{th} column is the vector $T(\vec{e}_j)$.

$$A = \begin{bmatrix} T(\vec{e}_1) & T(\vec{e}_2) & \cdots & T(\vec{e}_n) \end{bmatrix}$$

The matrix A is the **standard matrix** for a linear transformation.



$$\begin{bmatrix}\cos\left(\frac{\pi}{6}\right) & -\sin\left(\frac{\pi}{6}\right)\\ \sin\left(\frac{\pi}{6}\right) & \cos\left(\frac{\pi}{6}\right)\end{bmatrix}\begin{bmatrix}3\\1\end{bmatrix}$$

$$\vec{x} = \begin{bmatrix}3\\1\end{bmatrix}$$

$$= \begin{bmatrix} \\ \end{bmatrix}$$

Rotations

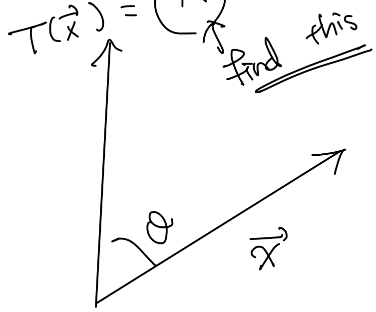
Example 1
What is the linear transform $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

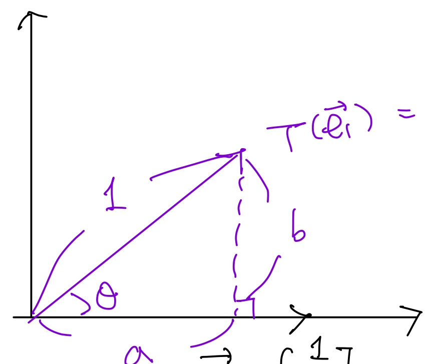
$$T(\vec{x}) = \vec{x} \text{ rotated counterclockwise by angle } \theta?$$

$$A = \begin{bmatrix} T(\vec{e}_1) & T(\vec{e}_2) \end{bmatrix}$$

$T(\vec{x}) = A\vec{x}$

find this



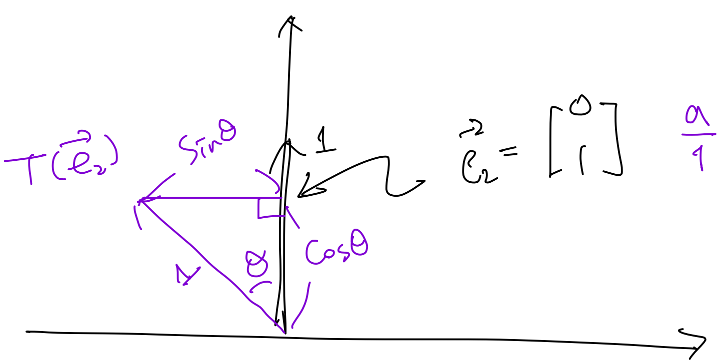


$$T(\vec{e}_1) = \begin{bmatrix}a\\b\end{bmatrix} = \begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix}$$

$$\vec{e}_1 = \begin{bmatrix}1\\0\end{bmatrix}$$

$$\frac{a}{1} = \cos\theta, \quad \frac{b}{1} = \sin\theta$$

$$\begin{bmatrix}-\sin\theta\\ \cos\theta\end{bmatrix} = T(\vec{e}_2)$$



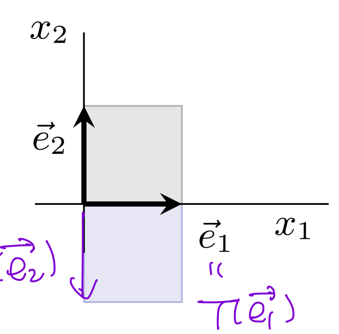
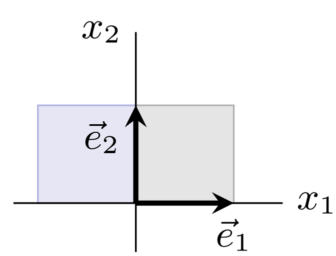
$$\vec{e}_2 = \begin{bmatrix}0\\1\end{bmatrix}$$

$$\Rightarrow \quad A = \begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$$

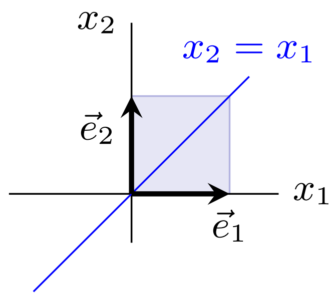
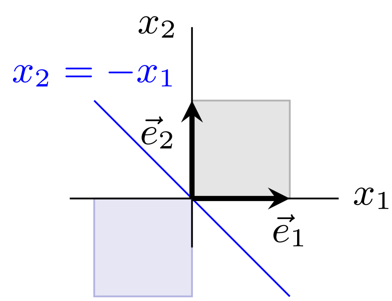
Standard Matrices in $\mathbb{R}^2$

- There is a long list of geometric transformations of $\mathbb{R}^2$ in our textbook, as well as on the next few slides (reflections, rotations, contractions and expansions, shears, projections, . . .)
- Please familiarize yourself with them: you are expected to memorize them (or be able to derive them)

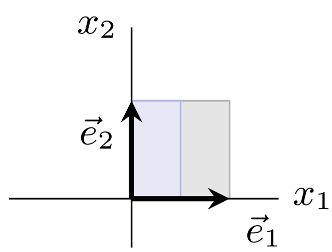
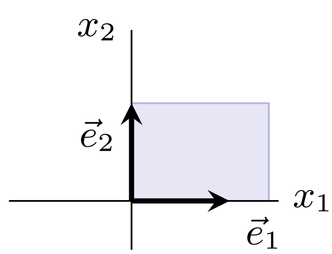
Two Dimensional Examples: Reflections

transformation	image of unit square	standard matrix
reflection through x_1−axis	 $-\vec{e}_2 = T(\vec{e}_2)$	$\begin{pmatrix} \boxed{\begin{matrix}1\\0\end{matrix}} & \boxed{\begin{matrix}0\\-1\end{matrix}} \end{pmatrix}$ $T(\vec{e}_1)$ $\quad -\vec{e}_2 = T(\vec{e}_2)$
reflection through x_2−axis		$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

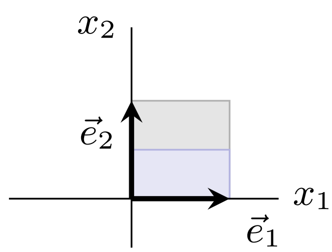
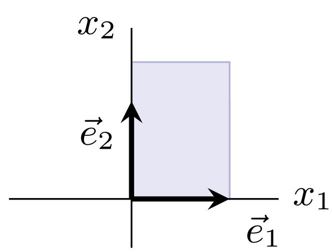
Two Dimensional Examples: Reflections

transformation	image of unit square	standard matrix
reflection through $x_2 = x_1$		$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
reflection through $x_2 = -x_1$		$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

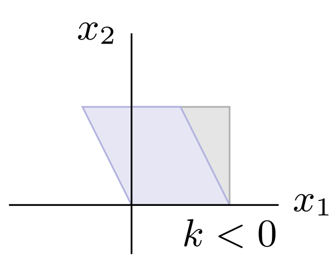
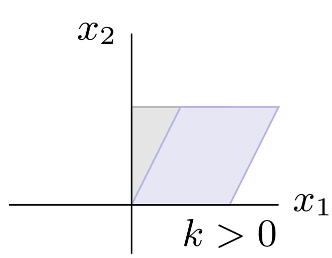
Two Dimensional Examples: Contractions and Expansions

transformation	**image of unit square**	**standard matrix**
Horizontal Contraction		$\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$. $\lvert k\rvert < 1$
Horizontal Expansion		$\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$, $k > 1$

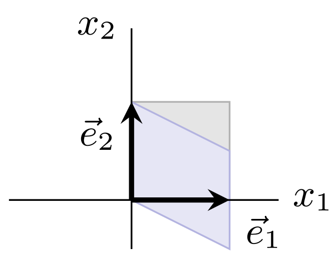
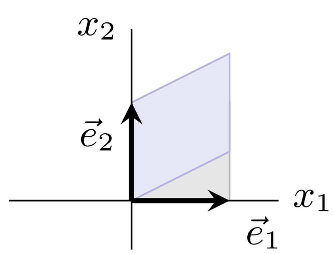
Two Dimensional Examples: Contractions and Expansions

transformation	**image of unit square**	**standard matrix**
Vertical Contraction		$\begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}$, $\lvert k \rvert < 1$
Vertical Expansion		$\begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}$, $k > 1$

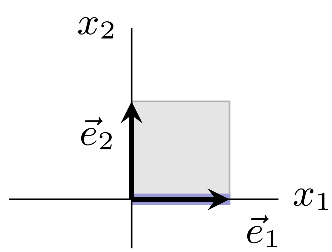
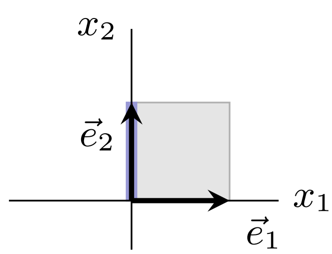
Two Dimensional Examples: Shears

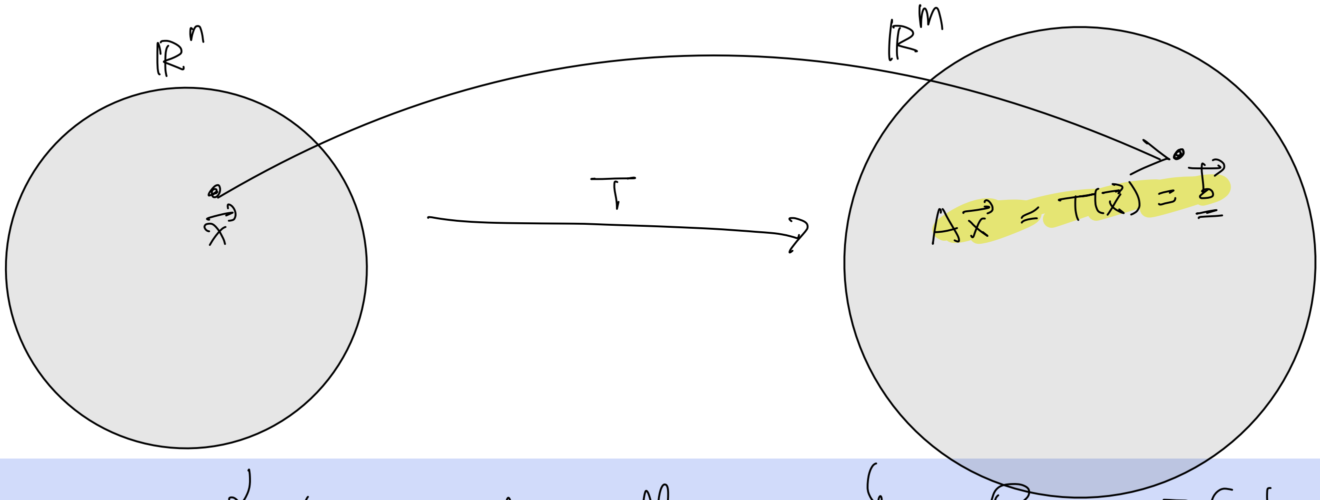
transformation	**image of unit square**	**standard matrix**
Horizontal Shear(left)		$\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$, $k < 0$
Horizontal Shear(right)		$\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$, $k > 0$

Two Dimensional Examples: Shears

transformation	**image of unit square**	**standard matrix**
Vertical Shear(down)		$\begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix}$, $k < 0$
Vertical Shear(up)		$\begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix}$, $k > 0$

Two Dimensional Examples: Projections

transformation	image of unit square	standard matrix
Projection onto the x_1-axis		$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$
Projection onto the x_2-axis		$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$



Onto

{ the set of all images } = Range = Codomain = $\mathbb{R}^m$

$A\vec{x} \leftarrow$ lin. combi. of columns in A

Definition

A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is **onto** if for all $\vec{b} \in \mathbb{R}^m$ there is a $\vec{x} \in \mathbb{R}^n$ so that $T(\vec{x}) = \vec{b}$.

target = codomain   domain

Onto is an **existence property:** for any $\vec{b} \in \mathbb{R}^m$, $A\vec{x} = \vec{b}$ has a solution.

Examples

- A rotation on the plane is an onto linear transformation.
- A projection in the plane is not onto.

Useful Fact

T is onto if and only if its standard matrix (A) has a pivot in every row.

$\Updownarrow$

$A\vec{x} = \vec{b}$ consistent for all $\vec{b} \in \mathbb{R}^m$

$\Updownarrow$

Range of T = $\mathbb{R}^m$ = Span of Columns in A

$\Updownarrow$

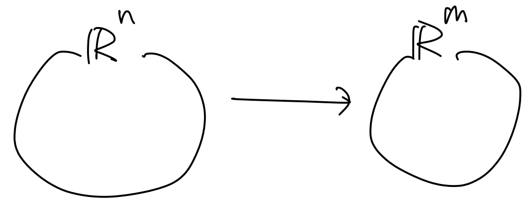
$[A \mid \vec{b}]$ last column is not pivotal for any $\vec{b}$

$\Updownarrow$

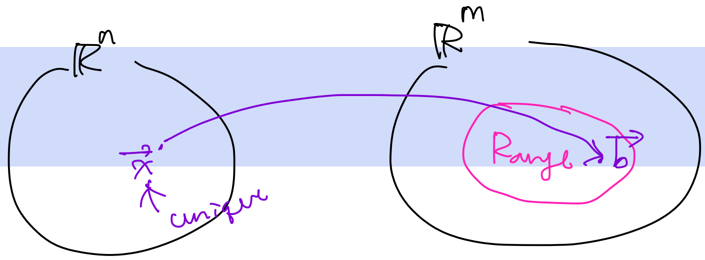
$\nearrow$

More rows than columns $\Leftrightarrow$ $T: \mathbb{R}^{\underline{n}} \to \mathbb{R}^{\underline{m}}$

$m > n$



One-to-One



Definition

A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is **one-to-one** if for all $\vec{b} \in \mathbb{R}^m$ there is at most one (possibly no) $\vec{x} \in \mathbb{R}^n$ so that $T(\vec{x}) = \vec{b}$.

① $\vec{x} \neq \vec{y} \Rightarrow T(\vec{x}) \neq T(\vec{y})$

② $T(\vec{x}) = T(\vec{y}) \Rightarrow \vec{x} = \vec{y}$

One-to-one is a uniqueness property, it does not assert existence for all $\vec{b}$.

Examples

- A rotation on the plane is a one-to-one linear transformation.
- A projection in the plane is not one-to-one.

Useful Facts

- T is one-to-one if and only if the only solution to $T(\vec{x}) = 0$ is the zero vector, $\vec{x} = \vec{0}$.
- T is one-to-one if and only if the standard matrix A of T has no free variables.

T is 1-1 $\Leftrightarrow$ $T(\vec{x}) = \vec{0}$ implies $\vec{x} = \vec{0}$

$\Leftrightarrow$ $A\vec{x} = \vec{0}$ has the only trivial solution

$\Leftrightarrow$ A has no free variable

$\Leftrightarrow$ Every column in A is pivotal

$\Leftrightarrow$ the columns in A are lin. indep.

Example

Complete the matrices below by entering numbers into the missing entries so that the properties are satisfied. **If it isn't possible to do so, state why**.

a) A is a 2×3 standard matrix for a one-to-one linear transform.

$$A = \begin{pmatrix} 1 & 0 & \\ 0 & & 1 \end{pmatrix}$$

↑ Every $\underline{\text{columns}}$ is pivotal

N. P

b) B is a 3×2 standard matrix for an onto linear transform.

$$B = \begin{pmatrix} 1 & \\ & \\ & \end{pmatrix}$$

Every row is pivotal

3 pivots

N. P.

c) C is a 3×3 standard matrix of a linear transform that is one-to-one and onto.

$$C = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix} \leftarrow \text{any numbers}$$

Q: 3×3, $\underbrace{1-1}$, not onto

Q: 3×3, $1-1$, onto $\Rightarrow$ RREF

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Def: For any $\vec{b} \in \mathbb{R}^m$, there exists $\vec{x} \in \mathbb{R}^n$ s.t. $T(\vec{x}) = \vec{b}$

$A\vec{x} = \vec{b}$

$[A \mid \vec{b}]$

Theorem

For a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ with standard matrix A these are equivalent statements.

1. T is onto.
2. The matrix A has columns which span $\mathbb{R}^m$.
3. The matrix A has m pivotal columns.

Theorem

For a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ with standard matrix A these are equivalent statements.

1. T is one-to-one.
2. The unique solution to $T(\vec{x}) = \vec{0}$ is the trivial one.
3. The matrix A linearly independent columns.
4. Each column of A is pivotal.

Def $T(\vec{x}) = T(\vec{y})$ implies $\vec{x} = \vec{y}$

$\Leftrightarrow$ (Linear case) $T(\vec{x}) = \vec{0}$ implies $\vec{x} = \vec{0}$

$A\vec{x} = \vec{0}$

lin. combi. of columns in A

Additional Examples

1. Construct a matrix $A \in \mathbb{R}^{2\times 2}$, such that $T(\vec{x}) = A\vec{x}$, where T is a linear transformation that rotates vectors in $\mathbb{R}^2$ counterclockwise by $\pi/2$ radians about the origin, then reflects them through the line $x_1 = x_2$.
2. Define a linear transformation by

$$T(x_1, x_2) = (3x_1 + x_2, 5x_1 + 7x_2, x_1 + 3x_2)$$

   Is T one-to-one? Is T onto?

Section 2.1 : Matrix Operations

Chapter 2 : Matrix Algebra

Math 1554 Linear Algebra

Topics and Objectives

Topics

We will cover these topics in this section.

1. Identity and zero matrices
2. Matrix algebra (sums and products, scalar multiplies, matrix powers)
3. Transpose of a matrix

Objectives

For the topics covered in this section, students are expected to be able to do the following.

1. **Apply** matrix algebra, the matrix transpose, and the zero and identity matrices, to **solve** and **analyze** matrix equations.

Definitions: Zero and Identity Matrices

1. A **zero matrix** is any matrix whose every entry is zero.

$$0_{2\times 3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad 0_{2\times 1} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

2. The $n \times n$ **identity matrix** has ones on the main diagonal, otherwise all zeros.

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Note: any matrix with dimensions $n \times n$ is **square**. Zero matrices need not be square, identity matrices must be square.

$$\underline{5} \cdot \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 5\cdot 1 & 5\cdot 2 \\ 5\cdot 3 & 5\cdot 4 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1+1 & 2+0 & 3+1 \\ 4+0 & 5+1 & 6+0 \end{pmatrix}$$

Sums and Scalar Multiples : Component-wise

Suppose $A \in \mathbb{R}^{m\times n}$, and $a_{i,j}$ is the element of A in row i and column j.

1. If A and B are $m \times n$ matrices, then the elements of $A + B$ are $a_{i,j} + b_{i,j}$.
2. If $c \in \mathbb{R}$, then the elements of cA are $ca_{i,j}$.

For example, if

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} + c \begin{bmatrix} 7 & 4 & 7 \\ 0 & 0 & k \end{bmatrix} = \begin{bmatrix} 15 & 10 & 17 \\ 4 & 5 & 16 \end{bmatrix}$$

What are the values of c and k?

Properties of Sums and Scalar Multiples

Scalar multiples and matrix addition have the expected properties.

If $r, s \in \mathbb{R}$ are scalars, and A, B, C are $m \times n$ matrices, then

1. $A + 0_{m \times n} = A$
2. $(A + B) + C = A + (B + C)$ ← Associativity
3. $r(A + B) = rA + rB$ ← Distributive property
4. $(r + s)A = rA + sA$
5. $r(sA) = (rs)A$

① $A \in \mathbb{R}^{m\times n}$ $\qquad\qquad A\cdot B \in \mathbb{R}^{m\times p}$

$B \in \mathbb{R}^{n\times p}$

$\| $

$[\vec b_1 \quad \vec b_2 \quad \cdots \quad \vec b_p]$

a linear combi. of columns of A

$A\cdot B = [\underbrace{A\cdot\vec b_1}_{\in\,\mathbb{R}^m} \quad A\cdot\vec b_2 \quad \cdots \quad A\vec b_p] \in \mathbb{R}^{m\times p}$.

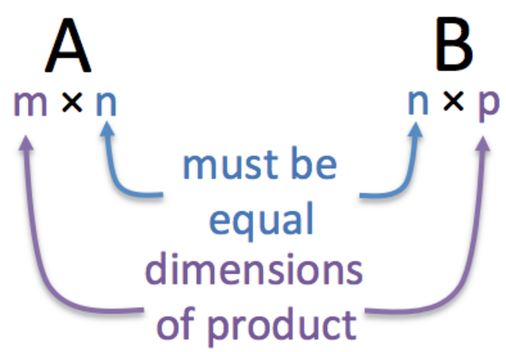
Matrix Multiplication

Definition

Let A be a $m\times n$ matrix, and B be a $n\times p$ matrix. The product is AB a $m\times p$ matrix, equal to

$$AB = A\begin{bmatrix}\vec b_1 & \cdots & \vec b_p\end{bmatrix} = \begin{bmatrix}A\vec b_1 & \cdots & A\vec b_p\end{bmatrix}$$

Note: the dimensions of A and B determine whether AB is defined, and what its dimensions will be.



$A \in \mathbb{R}^{m\times n}$ $\qquad$ $B \in \mathbb{R}^{n\times p}$ $\qquad$ dot product.

$$\begin{bmatrix} - & \vec{a}_1 \in \mathbb{R}^n & - \\ - & \vec{a}_2 & - \\ & \vdots & \\ - & \vec{a}_m & - \end{bmatrix} \cdot \begin{bmatrix} | & | & & | \\ \vec{b}_1 & \vec{b}_2 & \cdots & \vec{b}_p \\ | & | & & | \end{bmatrix} \in \mathbb{R}^n = \begin{bmatrix} \vec{a}_1\cdot\vec{b}_1 & \vec{a}_1\cdot\vec{b}_2 & \cdots & \vec{a}_1\cdot\vec{b}_p \\ \vec{a}_2\cdot\vec{b}_1 & \vec{a}_2\cdot\vec{b}_2 & \cdots & \\ \vdots & \vdots & \ddots & \\ \vec{a}_m\cdot\vec{b}_1 & \vec{a}_m\cdot\vec{b}_2 & \cdots & \vec{a}_m\cdot\vec{b}_p \end{bmatrix}$$

$\vec{x} = (x_1, \dots, x_n)$ $\qquad$ $\vec{y} = (y_1, \dots, y_n)$ $\qquad$ $\vec{x}\cdot\vec{y} = x_1y_1 + x_2y_2 + \cdots + x_ny_n$

Row Column Rule for Matrix Multiplication

The Row Column Rule is a convenient way to calculate the product AB that many students have encountered in pre-requisite courses.

Row Column Method

If $A \in \mathbb{R}^{m\times n}$ has rows $\vec{a}_i$, and $B \in \mathbb{R}^{n\times p}$ has columns $\vec{b}_j$, each element of the product $C = AB$ is $c_{ij} = \vec{a}_i \cdot \vec{b}_j$.

Example

Compute the following using the row-column method.

$$C = \underbrace{A}_{\in \mathbb{R}^{2\times 2}}\underbrace{B}_{\in \mathbb{R}^{2\times 3}} = \begin{pmatrix} 2 & 0 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 3 & 0 & 1 \\ 4 & 5 & 6 \end{pmatrix} \in \mathbb{R}^{2\times 3}$$

$$= \begin{pmatrix} 6 & * & * \\ * & * & -5 \end{pmatrix}$$

Ex Find $A, B \in \mathbb{R}^{2\times 2}$ $AB = 0$ $A \neq 0$, $B \neq 0$

$$\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \neq \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}$$

Properties of Matrix Multiplication

Let A, B, C be matrices of the sizes needed for the matrix multiplication to be defined, and A is a $m \times n$ matrix.

1. (Associative) $(AB)C = A(BC)$
2. (Left Distributive) $A(B + C) = AB + AC$
3. (Right Distributive) $\cdots$ $(A+B)\cdot C = AC + BC$
4. (Identity for matrix multiplication) $I_m A = AI_n = A$

Warnings:

1. (non-commutative) In general, $AB \neq BA$.
2. (non-cancellation) $AB = AC$ does not mean $B = C$.
3. (Zero divisors) $AB = 0$ does not mean that either $A = 0$ or $B = 0$.

① $A \in \mathbb{R}^{m\times n}$ $B \in \mathbb{R}^{n\times p}$ $A \cdot B \in \mathbb{R}^{m\times p}$

$B \cdot A$ Not defined unless $m = p$.
$(n \times p)$ $(m \times n)$

Suppose $m = p$ $A \cdot B \in \mathbb{R}^{m\times m}$

$B \cdot A \in \mathbb{R}^{n\times n}$

Suppose $m = n = p$ $AB, BA \in \mathbb{R}^{n\times n}$

$AB \neq BA$ in general

② $a, b \in \mathbb{R}$ $a \cdot b = 0 \Rightarrow a = 0$ or $b = 0$

$a \cdot b = 0$ & $a \neq 0 \Rightarrow b = 0$

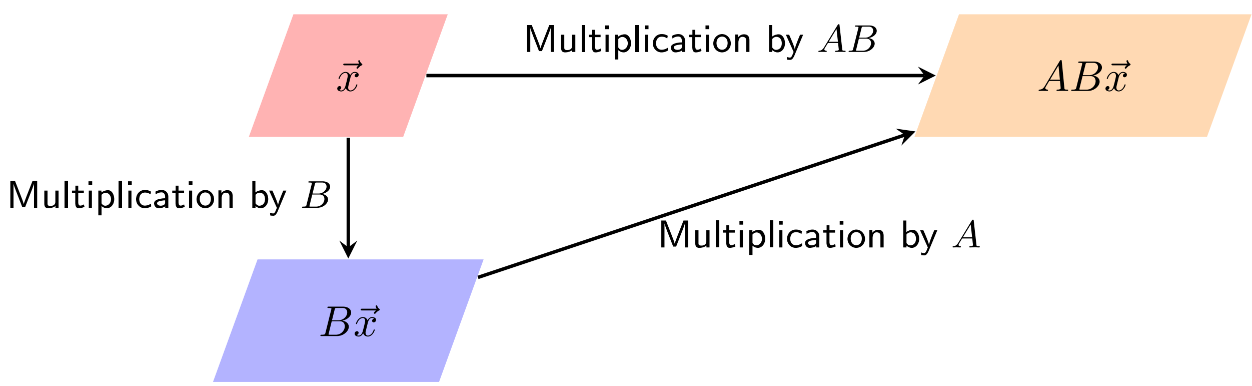
$ab = ac$, $a \neq 0 \Rightarrow a\cdot(b-c) = 0$, $a \neq 0 \Rightarrow b = c$

The Associative Property

The associative property is $(AB)C = A(BC)$. If $C = \vec{x}$, then

$$(AB)\vec{x} = A(B\vec{x})$$

Schematically:



The matrix product $AB\vec{x}$ can be obtained by either: multiplying by matrix AB, or by multiplying by B then by A. This means that matrix multiplication corresponds to **composition of the linear transformations**.

Example

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

Give an example of a 2×2 matrix B that is non-commutative with A.

Find $\underline{\underline{B}} \in \mathbb{R}^{2\times 2}$ s.t. $AB \neq BA$

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a+c & b+d \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a & a \\ c & c \end{bmatrix}$$

If $c \neq 0$

Additional Examples

True or false:

1. For any I_n and any $A \in \mathbb{R}^{n\times n}$, $(I_n + A)(I_n - A) = I_n - A^2$.

2. For any A and B in $\mathbb{R}^{n\times n}$, $(A+B)^2 = A^2 + B^2 + 2AB$.

The Transpose of a Matrix

A^T is the matrix whose columns are the rows of A.

Example

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 0 & 2 & 0 \end{bmatrix}^T = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 0 \\ 4 & 2 \\ 5 & 0 \end{bmatrix}$$

Properties of the Matrix Transpose

1. $(A^T)^T =$ A
2. $(A+B)^T =$ $A^T + B^T$
3. $(rA)^T =$ $r \cdot A^T$.
4. $(AB)^T =$ $B^T \cdot A^T$.

$\underset{m\times n}{A} \cdot \underset{n\times p}{B} \in \mathbb{R}^{m\times p}$

$\underset{n\times m}{A^T} \cdot \underset{p\times n}{B^T}$ Not defined in general

$(AB)^T = \underset{p\times n}{B^T} \cdot \underset{n\times m}{A^T} \in \mathbb{R}^{p\times m}$

$$A^3 = \underbrace{A \cdot A \cdot A}_{3 \text{ times}} = A^2 \cdot A = A \cdot A^2$$

Matrix Powers

For any $n \times n$ matrix and positive integer k, A^k is the product of k copies of A.

$$A^k = AA \dots A$$

Example: Compute C^8.

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$C^8 = \underbrace{C \cdot C \cdots C}_{8 \text{ times}}$$

$$C^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2^2 & 0 \\ 0 & 0 & 2^2 \end{bmatrix}$$

$$C^3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix}$$

$$\vdots$$

$$C^8 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2^8 & 0 \\ 0 & 0 & 2^8 \end{bmatrix}$$

$C^T \cdot (\underline{AB})^T =$

$\parallel$

$((A \cdot B) \cdot C)^T = C^T \cdot B^T \cdot A^T$

Example

Define

$$A = \underset{2\times 2}{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}, \quad B = \underset{2\times 3}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 8 \end{bmatrix}}, \quad C = \underset{3\times 3}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}}$$

Which of these operations are defined, and what are the dimensions of the result?

1. $A + 3C$ Not Defined

   2×2 3×3

   2×2 3×2 2×2

2. $A(AB)^T = A \cdot B^T \cdot A^T$

3. $A + \boxed{ABCB^T}$: Defined.

   2×2 2×2

$A \cdot B \in \mathbb{R}^{2\times 3}$

2×2 2×3

$(AB)^T \in \mathbb{R}^{3\times 2}$

$A \cdot \underline{(AB)^T} \notin$ Not defined.

2×2 3×2

$A \cdot B \cdot C \cdot \underline{B^T} \in \mathbb{R}^{2\times 2}$

$2\times\cancel{2}$ $\cancel{2}\times\cancel{3}$ $\cancel{3}\times\cancel{3}$ $\cancel{3}\times 2$

Section 2.2 : Inverse of a Matrix

Chapter 2 : Matrix Algebra

Math 1554 Linear Algebra

"Your scientists were so preoccupied with whether or not they could, they didn't stop to think if they should."

- Spielberg and Crichton, Jurassic Park, 1993 film

The algorithm we introduce in this section **could** be used to compute an inverse of an $n \times n$ matrix. At the end of the lecture we'll discuss some of the problems with our algorithm and why it can be difficult to compute a matrix inverse.

Topics and Objectives

Topics

We will cover these topics in this section.

1. Inverse of a matrix, its algebraic properties, and its relation to solving systems of linear equations.
2. Elementary matrices and their role in calculating the matrix inverse.

Objectives

For the topics covered in this section, students are expected to be able to do the following.

1. Apply the formal definition of an inverse, and its algebraic properties, to solve and analyze linear systems.
2. Compute the inverse of an $n \times n$ matrix, and use it to solve linear systems.
3. Construct elementary matrices.

Motivating Question

Is there a matrix, A, such that $\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} A = I_3$?

The Matrix Inverse

Definition

$A \in \mathbb{R}^{n\times n}$ is **invertible** (or **non-singular**) if there is a $C \in \mathbb{R}^{n\times n}$ so that

$$AC = CA = I_n. = \begin{bmatrix} 1 & & 0 \\ & 1 & \\ & & \ddots & \\ 0 & & & 1 \end{bmatrix}$$

If there is, we write $C = A^{-1}$.

both square matrices.

Facts ① $AC = I$ implies $CA = I$.

② C is unique.

Inverse of A, A^{-1}

$$\begin{bmatrix} ax+bz=1 & ay+bw=0 \\ \cdot & \cdot \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow 4 \text{ Eqn}$$

find

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

4 variables

The Inverse of a 2×2 Matrix

There's a formula for computing the inverse of a 2×2 matrix.

Theorem

The 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is non-singular (invertible) if and only if $ad - bc \neq 0$, and then

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Example

State the inverse of the matrix below.

$$A = \begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix}$$

① $ad - bc = 2 \cdot (-7) - 5 \cdot (-3) = 1 \neq 0$

A is invertible.

② $A^{-1} = \frac{1}{1} \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix}$

Suppose A^{-1} exists.

$A\vec{x} = \vec{b}$

$\vec{x} = \underbrace{A^{-1} \cdot A}_{= I_n}\vec{x} = A^{-1} \cdot \vec{b}$

The Matrix Inverse

$A \cdot A^{-1} = I = A^{-1} \cdot A$

$\Leftarrow$

$\Rightarrow$

Theorem

$A \in \mathbb{R}^{n\times n}$ has an inverse if and only if for all $\vec{b} \in \mathbb{R}^n$, $\boxed{A\vec{x} = \vec{b}}$ has a unique solution. And, in this case, $\vec{x} = A^{-1}\vec{b}$.

Example

Solve the linear system.

$$3x_1 + 4x_2 = 7$$
$$5x_1 + 6x_2 = 7$$

$$\underbrace{\begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 7 \\ 7 \end{bmatrix}}_{\vec{b}}$$

① A is Invertible? Yes.

$ad - bc = 3 \cdot 6 - 4 \cdot 5 = -2 \neq 0.$

② $A^{-1} = \frac{1}{-2} \cdot \begin{bmatrix} 6 & -4 \\ -5 & 3 \end{bmatrix}$

③ $\vec{x} = A^{-1} \cdot \vec{b} = -\frac{1}{2} \begin{bmatrix} 6 & -4 \\ -5 & 3 \end{bmatrix} \cdot \begin{bmatrix} 7 \\ 7 \end{bmatrix}.$

Properties of the Matrix Inverse

A and B are invertible $n \times n$ matrices.

$A \cdot A^{-1} = I$

1. $(A^{-1})^{-1} = A$
2. $(AB)^{-1} = B^{-1}A^{-1}$ (Non-commutative!)
3. $(A^T)^{-1} = (A^{-1})^T$

$(AB)\cdot(B^{-1}A^{-1}) = I$

$I^T = I$

Example

True or false: $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$.

True.

Find

$[A\vec{x}_1 \; A\vec{x}_2 \; \cdots \; A\vec{x}_n] = A\underbrace{[\vec{x}_1 \; \vec{x}_2 \; \cdots \; \vec{x}_n]} = I = [\vec{e}_1 \; \vec{e}_2 \; \cdots \; \vec{e}_n]$

Solve
$$\begin{cases} A\vec{x}_1 = \vec{e}_1 & \longrightarrow [A \mid \vec{e}_1] \xrightarrow{\text{row operations}} \\ A\vec{x}_2 = \vec{e}_2 & [A \mid \vec{e}_2] \\ \vdots & \vdots \\ A\vec{x}_n = \vec{e}_n & [A \mid \vec{e}_n] \end{cases}$$

$[A \mid \underbrace{\vec{e}_1 \; \vec{e}_2 \cdots \vec{e}_n}] \longrightarrow \cdots \quad [\underset{\substack{\| \\ I_n}}{\text{RREF}} \mid \underset{= A^{-1}}{\vec{x}_1 \; \vec{x}_2 \cdots \vec{x}_n}]$

$[A \mid I] \longrightarrow [I \mid A^{-1}]$

9/18/23

$T(\vec{x}) = A\vec{x}$

$A \in \mathbb{R}^{n\times n}$ has inverse

$\Leftrightarrow$ there exists $C \in \mathbb{R}^{n\times n}$ s.t. $AC = CA = I$

$\Leftrightarrow$ $A \cdot [\vec{x_1} \ \vec{x_2} \ \cdots \ \vec{x_n}] = [\vec{e_1} \ \vec{e_2} \ \cdots \ \vec{e_n}]$

$$\left\{\begin{array}{l} A\vec{x_1} = \vec{e_1} \\ A\vec{x_2} = \vec{e_2} \\ \vdots \\ A\vec{x_n} = \vec{e_n} \end{array}\right.$$ are all consistent.

T is onto $\Leftarrow$

$\Leftrightarrow$ For any $\vec{b} \in \mathbb{R}^n$, $A\vec{x} = \vec{b}$ is consistent.

$\Leftrightarrow$ $[A \mid \vec{e_1} \ \vec{e_2} \ \cdots \ \vec{e_n}] \xrightarrow[\text{row operation}]{} [\overset{= I_n}{\text{RREF}} \mid \overset{\text{Inverse}}{\vec{x_1} \cdots \vec{x_n}}]$

$\uparrow$ Every $\binom{\text{row}}{\text{column}}$ pivot

$\Leftrightarrow$ $T: \mathbb{R}^n \to \mathbb{R}^n$ is onto $\Leftrightarrow$ T is 1-1

$\Leftrightarrow$ Columns in A spans $\mathbb{R}^n$

$\Leftrightarrow$ Every $\binom{\text{row}}{\text{column}}$ is pivotal

$\Leftrightarrow$ Columns in A lin. indep.

$\Leftrightarrow$ $T(\vec{x}) = 0$ implies $\vec{x} = \vec{0}$

An Algorithm for Computing A^{-1}

If $A \in \mathbb{R}^{n\times n}$, and $n > 2$, how do we calculate A^{-1}? Here's an algorithm we can use:

1. Row reduce the augmented matrix $(A \,|\, I_n)$
2. If reduction has form $(I_n \,|\, B)$ then A is invertible and $B = A^{-1}$. Otherwise, A is not invertible.

Example

Compute the inverse of $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 0 & 0 & 1 \end{bmatrix}$.

$$[\,A \mid I\,] = \left[\begin{array}{ccc|ccc} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[R_1 \leftrightarrow R_2]{} \left[\begin{array}{ccc|ccc} 1 & 0 & 3 \to 0 & 0 & 1 & 0 \\ 0 & 1 & 2 \to 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[\substack{R_1 \to R_1 - 3R_3 \\ R_2 \to R_2 - 2R_3}]{} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & -3 \\ 0 & 1 & 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$= A^{-1}$

Why Does This Work?

We can think of our algorithm as simultaneously solving n linear systems:

$$\begin{aligned} A\vec{x}_1 &= \vec{e}_1 \\ A\vec{x}_2 &= \vec{e}_2 \\ &\vdots \\ A\vec{x}_n &= \vec{e}_n \end{aligned}$$

Each column of A^{-1} is $A^{-1}\vec{e}_i = \vec{x}_i$.

Over the next few slides we explore another explanation for how our algorithm works. This other explanation uses elementary matrices.

$$I_n \xrightarrow[\text{1 row op.}]{} E$$

Elementary matrix

Ex

$$I_n \xrightarrow[R_1 \to R_1+R_2]{} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Elementary Matrices

An elementary matrix, E, is one that differs by I_n by one row operation. Recall our elementary row operations:

1. swap rows
2. multiply a row by a non-zero scalar
3. add a multiple of one row to another

We can represent each operation by a matrix multiplication with an **elementary matrix**.

Ex

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \xrightarrow[R_1 \to R_1+R_2]{} \underbrace{\begin{bmatrix} 5 & 7 & 9 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}}_{=\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}}$$

A has inverse

$$\iff A \underset{\substack{(1)\\ \uparrow \\ E_1}}{\overset{E_1^{-1}}{\rightleftarrows}} \boxed{E_1 \cdot A} \underset{\substack{(2)\\ \uparrow \\ E_2}}{\overset{E_2^{-1}}{\rightleftarrows}} \boxed{E_2 \cdot E_1 \cdot A} \overset{E_3^{-1}}{\rightleftarrows} \cdots \underset{\substack{(k)\\ \uparrow \\ E_k}}{\overset{E_k^{-1}}{\rightleftarrows}} I$$

$$\iff \underbrace{E_k \cdots E_2 E_1}\, A = I \quad \leftarrow \quad A^{-1} = E_k E_{k-1} \cdots E_2 \cdot E_1$$

$\Leftrightarrow \quad A = E_1^{-1} E_2^{-1} \cdots E_k^{-1}$

A is a product of elementary matrices

T/F : If A, B invertible, A · B invertible?

Example

Suppose

$$E \begin{bmatrix} 1 & 1 & 1 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

By inspection, what is E? How does it compare to I_3?

?

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 \to R_2 + 2 \cdot R_1} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = E.$$

Theorem

Returning to understanding why our algorithm works, we apply a sequence of row operations to A to obtain I_n:

$$(E_k \cdots E_3 E_2 E_1)A = I_n$$

Thus, $E_k \cdots E_3 E_2 E_1$ is the inverse matrix we seek.

Our algorithm for calculating the inverse of a matrix is the result of the following theorem.

Theorem

Matrix A is invertible if and only if it is row equivalent to the identity. In this case, the any sequence of elementary row operations that transforms A into I, applied to I, generates A^{-1}.

Using The Inverse to Solve a Linear System

- We could use A^{-1} to solve a linear system,

$$A\vec{x} = \vec{b}$$

We would calculate A^{-1} and then:

$$\vec{x} = A^{-1} \cdot \vec{b}$$

- As our textbook points out, A^{-1} is seldom used: computing it can take a very long time, and is prone to numerical error.
- So why did we learn how to compute A^{-1}? Later on in this course, we use elementary matrices and properties of A^{-1} to derive results.
- A recurring theme of this course: just because we **can** do something a certain way, doesn't that we **should**.

Section 2.3 : Invertible Matrices

Chapter 2 : Matrix Algebra

Math 1554 Linear Algebra

"A synonym is a word you use when you can't spell the other one."
- Baltasar Gracián

The theorem we introduce in this section of the course gives us many ways of saying the same thing. Depending on the context, some will be more convenient than others.

Topics and Objectives

Topics

We will cover these topics in this section.

1. The invertible matrix theorem, which is a review/synthesis of many of the concepts we have introduced.

Objectives

For the topics covered in this section, students are expected to be able to do the following.

1. Characterize the invertibility of a matrix using the Invertible Matrix Theorem.
2. Construct and give examples of matrices that are/are not invertible.

Motivating Question

When is a square matrix invertible? Let me count the ways!

The Invertible Matrix Theorem (IMT)

$AC = CA = I$

Invertible matrices enjoy a rich set of equivalent descriptions.

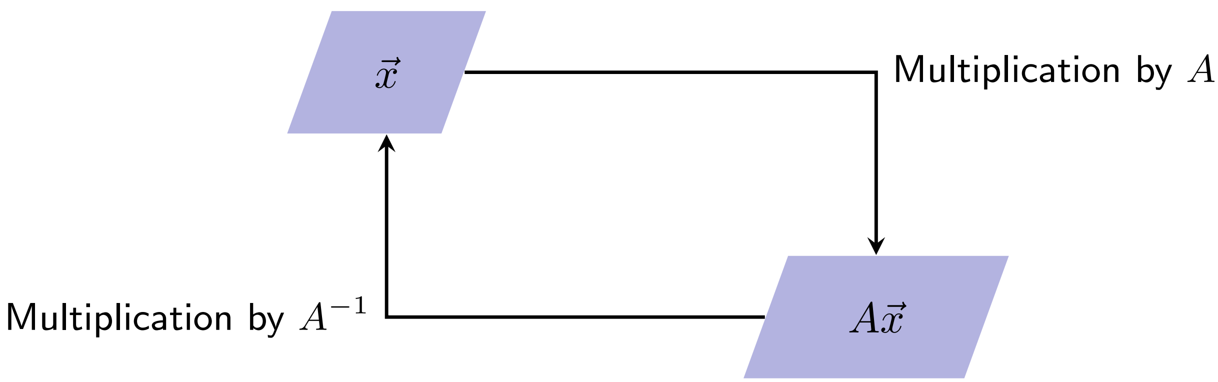
Theorem

Let A be an $n \times n$ matrix. These statements are all equivalent.

a) A *is invertible.*
b) A *is row equivalent to* I_n.
c) A *has* n *pivotal columns. (All columns are pivotal.)*
d) $A\vec{x} = \vec{0}$ *has only the trivial solution.*
e) *The columns of* A *are linearly independent.*
f) *The linear transformation* $\vec{x} \mapsto A\vec{x}$ *is one-to-one.*
g) *The equation* $A\vec{x} = \vec{b}$ *has a solution for all* $\vec{b} \in \mathbb{R}^n$.
h) *The columns of* A *span* $\mathbb{R}^n$.
i) *The linear transformation* $\vec{x} \mapsto A\vec{x}$ *is onto.*
j) *There is a* $n \times n$ *matrix* C *so that* $CA = I_n$. *(*A *has a left inverse.)*
k) *There is a* $n \times n$ *matrix* D *so that* $AD = I_n$. *(*A *has a right inverse.)*
l) A^T *is invertible.*

Invertibility and Composition

The diagram below gives us another perspective on the role of A^{-1}.



The matrix inverse A^{-1} transforms Ax back to $\vec{x}$. This is because:

$$A^{-1}(A\vec{x}) = (A^{-1}A)\vec{x} =$$

The Invertible Matrix Theorem: Final Notes

- Items j and k of the invertible matrix theorem (IMT) lead us directly to the following theorem.

> **Theorem**
>
> If A and B are $n \times n$ matrices and $AB = I$, then A and B are invertible, and $B = A^{-1}$ and $A = B^{-1}$.

- The IMT is a set of equivalent statements. They divide the set of all square matrices into two separate classes: invertible, and non-invertible.
- As we progress through this course, we will be able to add additional equivalent statements to the IMT (that deal with determinants, eigenvalues, etc).

Example 1

Is this matrix invertible?

$$\begin{bmatrix} 1 & 0 & -2 \\ 3 & 1 & -2 \\ -5 & -1 & 9 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 3 & 1 & -2 & 0 & 1 & 0 \\ -5 & -1 & 9 & 0 & 0 & 1 \end{array}\right] \longrightarrow \left[\, I \;\middle|\; \cancel{A^{-1}} \,\right]$$

$$A \longrightarrow \cancel{I}$$

$A \downarrow$ EF with 3 pivots

$$\begin{bmatrix} 1 & 0 & -2 \\ 3 & 1 & -2 \\ -5 & -1 & 9 \end{bmatrix} \xrightarrow[R_3 \to R_3 + 5 \cdot R_1]{R_2 \to R_2 - 3R_1} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 4 \\ 0 & -1 & -1 \end{bmatrix}$$

$$\downarrow R_3 \to R_2 + R_2$$

$$\underline{\text{Invertible}} \Longleftarrow \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 4 \\ 0 & 0 & 3 \end{bmatrix}$$

Example 2

If possible, fill in the missing elements of the matrices below with numbers so that each of the matrices are singular. If it is not possible to do so, state why.

Not Invertible

Lin. dep.

$\Downarrow$

Not invertible

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & * & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

Lin. Dep.

$\Downarrow$

Not Invertible

Always Invertible

Section 2.4 : Partitioned Matrices

Chapter 2 : Matrix Algebra

Math 1554 Linear Algebra

"Mathematics is not about numbers, equations, computations, or algorithms. Mathematics is about understanding."
- William Paul Thurston

Multiple perspectives of the same concept is a theme of this course; each perspective deepens our understanding. In this section we explore another way of representing matrices and their algebra that gives us another way of thinking about them.

Topics and Objectives

Topics

We will cover these topics in this section.

1. Partitioned matrices (or block matrices)

Objectives

For the topics covered in this section, students are expected to be able to do the following.

1. Apply partitioned matrices to solve problems regarding matrix invertibility and matrix multiplication.

What is a Partitioned Matrix?

Example

This matrix:

$= I$

$$\left[\begin{array}{ccc|cc} 3 & 1 & 4 & 1 & 0 \\ 1 & 6 & 1 & 0 & 1 \\ \hline 0 & 0 & 0 & 4 & 2 \end{array}\right]$$

$= 0$

can also be written as:

$$\begin{bmatrix} \begin{bmatrix} 3 & 1 & 4 \\ 1 & 6 & 1 \end{bmatrix} & \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ \\ \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 4 & 2 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{bmatrix}$$

We partitioned our matrix into four **blocks**, each of which has different dimensions.

Another Example of a Partitioned Matrix

Example: The reduced echelon form of a matrix. We can use a partitioned matrix to

$$\begin{bmatrix} 1 & 0 & 0 & 0 & * & \cdots & * \\ 0 & 1 & 0 & 0 & * & \cdots & * \\ 0 & 0 & 1 & 0 & * & \cdots & * \\ 0 & 0 & 0 & 1 & * & \cdots & * \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix} = \begin{bmatrix} I_4 & F \\ 0 & 0 \end{bmatrix}$$

This is useful when studying the **null space** of A, as we will see later in this course.

Row Column Method

Recall that a row vector times a column vector (of the right dimensions) is a scalar. For example,

$$\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = 1 \cdot 1 + 1 \cdot 0 + 1 \cdot 2 .$$

This is the **row column** matrix multiplication method from Section 2.1.

Theorem

Let A be $m \times n$ and B be $n \times p$ matrix. Then, the (i, j) entry of AB is

$$\mathrm{row}_i\, A \cdot \mathrm{col}_j\, B.$$

This is the **Row Column Method** for matrix multiplication.

Partitioned matrices can be multiplied using this method, as if each block were a scalar (provided each block has appropriate dimensions).

$$4\left\{\begin{array}{l} 2\{ \\ 2\{ \end{array}\right. \overbrace{\begin{bmatrix} \underbrace{A}_{2} & \underbrace{B}_{2} \\ C & D \end{bmatrix}}^{4} \cdot \overbrace{\begin{bmatrix} \underbrace{P}_{2} & \underbrace{Q}_{2} \\ R & S \end{bmatrix}}^{4} \begin{array}{l} \}2 \\ \}2 \end{array} 4$$

$$= \left[\begin{array}{c|c} A\cdot P + B\cdot R & AQ + BS \\ \hline CP + DR & CQ + DS \end{array}\right]$$

$$-\frac{1}{a}\cdot b\cdot\frac{1}{c} = -\frac{b}{ac}$$

$$\begin{bmatrix} \frac{1}{a} & -\frac{b}{ac} \\ 0 & \frac{1}{c} \end{bmatrix}$$

Example of Row Column Method

Recall, using our formula for a 2×2 matrix, $\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}^{-1} = \frac{1}{ac}\begin{bmatrix} c & -b \\ 0 & a \end{bmatrix}$.

Example: Suppose $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times n}$, and $C \in \mathbb{R}^{n\times n}$ are invertible matrices. Construct the inverse of $\begin{bmatrix} A & B \\ 0 & C \end{bmatrix} \in \mathbb{R}^{2n\times 2n}$.

$A \in \mathbb{R}^{n\times n}$

$$\left[\begin{array}{c|c} A & B \\ \hline 0 & C \end{array}\right]\left[\begin{array}{c|c} P & Q \\ \hline R & S \end{array}\right] = I_{2n} = \left[\begin{array}{c|c} I_n & 0 \\ \hline 0 & I_n \end{array}\right]$$

$$\left[\begin{array}{c|c} A\cdot P + BR & AQ + BS \\ \hline C\cdot R & C\cdot S \end{array}\right] = \left[\begin{array}{c|c} I & 0 \\ \hline 0 & I \end{array}\right]$$

- $C\cdot S = I$, $S = C^{-1}$
  (C invertible)
- $C\cdot R = 0$ Choose $R = 0$
- $AP + BR = AP = I$, $P = A^{-1}$
- $AQ + BS = 0$, $A^{-1}AQ = -A^{-1}B\cdot C^{-1}$, $Q = -A^{-1}BC^{-1}$

$$\Rightarrow \left[\begin{array}{c|c} A^{-1} & -A^{-1}BC^{-1} \\ \hline 0 & C^{-1} \end{array}\right]$$

Section 2.5 : Matrix Factorizations

Chapter 2 : Matrix Algebra

Math 1554 Linear Algebra

"Mathematical reasoning may be regarded rather schematically as the exercise of a combination of two facilities, which we may call intuition and ingenuity." - Alan Turing

The use of the LU Decomposition to solve linear systems was one of the areas of mathematics that Turing helped develop.

Topics and Objectives

Topics
We will cover these topics in this section.

1. The LU factorization of a matrix
2. Using the LU factorization to solve a system
3. Why the LU factorization works

Objectives
For the topics covered in this section, students are expected to be able to do the following.

1. Compute an LU factorization of a matrix.
2. Apply the LU factorization to solve systems of equations.
3. Determine whether a matrix has an LU factorization.

Motivation

- Recall that we **could** solve $A\vec{x} = \vec{b}$ by using

$$\vec{x} = A^{-1}\vec{b}$$

- This requires computation of the inverse of an $n \times n$ matrix, which is especially difficult for large n.
- Instead we could solve $A\vec{x} = \vec{b}$ with Gaussian Elimination, but this is not efficient for large n
- There are more efficient and accurate methods for solving linear systems that rely on matrix factorizations.

Matrix Factorizations

- A **matrix factorization**, or **matrix decomposition** is a factorization of a matrix into a product of matrices.
- Factorizations can be useful for solving $A\vec{x} = \vec{b}$, or understanding the properties of a matrix.
- We explore a few matrix factorizations throughout this course.
- In this section, we factor a matrix into **lower** and into **upper** triangular matrices.

Triangular Matrices

- A rectangular matrix A is **upper triangular** if $a_{i,j} = 0$ for $i > j$. Examples:

$$\begin{pmatrix} 1 & 5 & 0 \\ 0 & 2 & 4 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

- A rectangular matrix A is **lower triangular** if $a_{i,j} = 0$ for $i < j$. Examples:

$$\begin{pmatrix} 1 & 0 & 0 \\ 3 & 2 & 0 \end{pmatrix}, \quad \begin{pmatrix} 3 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 2 \\ 1 \\ 2 \end{pmatrix}$$

Ask: Can you name a matrix that is both upper and lower triangular?

The LU Factorization

Theorem

If A is an $m \times n$ matrix that can be row reduced to echelon form without row exchanges, then $A = LU$. L is a lower triangular $m \times m$ matrix with 1's on the diagonal, U is an **echelon** form of A.

upper

Example: If $A \in \mathbb{R}^{3\times 2}$, the LU factorization has the form:

$$A = LU = \begin{pmatrix} 1 & 0 & 0 \\ * & 1 & 0 \\ * & * & 1 \end{pmatrix} \begin{pmatrix} * & * \\ 0 & * \\ 0 & 0 \end{pmatrix}$$

Idea

E_1 ① $E_1 \cdot A$ E_2 ② ... ⓚ

$$A \xrightarrow{\text{row operations}} \text{Echelon form} = U$$

replacement only

$$E_k^{-1} E_k \cdots E_2 E_1 A = E_k^{-1} U$$

$$E_{k-1} \cdots E_1 A = E_k^{-1} \cdot U$$

$$\vdots$$

$$A = \underbrace{(E_1^{-1} \cdot E_2^{-1} \cdots E_k^{-1})}_{= L} U$$

lower triangular

Why We Can Compute the LU Factorization

Suppose A can be row reduced to echelon form U without interchanging rows. Then,

$$E_p \cdots E_1 A = U$$

where the E_j are matrices that perform elementary row operations. They happen to be lower triangular and invertible, e.g.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

Therefore,

$$A = \underbrace{E_1^{-1} \cdots E_p^{-1}}_{=L} U = LU.$$

Using the LU Decomposition

Goal: given A and $\vec{b}$, solve $A\vec{x} = \vec{b}$ for $\vec{x}$.

Algorithm: construct $A = LU$, solve $A\vec{x} = LU\vec{x} = \vec{b}$ by:

1. Forward solve for $\vec{y}$ in $L\vec{y} = \vec{b}$.
2. Backwards solve for x in $U\vec{x} = \vec{y}$.

Example: Solve the linear system whose LU decomposition is given.

$$A = LU = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} 2 \\ 3 \\ 2 \\ 0 \end{pmatrix}$$

① $A\vec{x} = \vec{b}$ $\quad L \cdot \underbrace{\boxed{U\vec{x}}}_{=\vec{y}} = \vec{b}$ Solve $L\boxed{\vec{y}} = \vec{b}$ first

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 2 \\ 0 \end{pmatrix}$$

① $y_1 = 2$

② $y_1 + y_2 = 2 + y_2 = 3$, $y_2 = 1$

③ $2 \cdot y_2 + y_3 = 2 + y_3 = 2$, $y_3 = 0$

④ $y_4 = 0$

(2) $U\vec{x} = \vec{y}$ Solve for $\vec{x}$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$2 \cdot x_3 = 0$

$2x_2 + \underbrace{x_3}_{=0} = 1$

$x_3 = 0$

$x_2 = \frac{1}{2}$

$x_1 = ?$

Ex

$A \in \mathbb{R}^{7\times 9}$

$A = LU$

Size of $U = ?$ 7×9

" $L = 7\times 7$

An Algorithm for Computing LU

To compute the LU decomposition:

1. Reduce A to an echelon form U by a sequence of row replacement operations, if possible.
2. Place entries in L such that the same sequence of row operations reduces L to I.

Note that

- In MATH 1554, the only row replacement operation we can use is to *replace a row with a multiple of a row above it*.
- More advanced linear algebra courses address this limitation.

Example: Compute the LU factorization of A.

$$A = \overset{3\times 4}{\begin{pmatrix} 4 & -3 & -1 & 5 \\ -16 & 12 & 2 & -17 \\ 8 & -6 & -12 & 22 \end{pmatrix}} = \underset{L}{\underline{3\times 3}} \cdot \underset{U}{\underline{3\times 4}}$$

Echelon form

Replacement

① $$\begin{pmatrix} 4 & -3 & -1 & 5 \\ \underbrace{-16}_{=0} & 12 & 2 & -17 \\ \underbrace{8}_{=0} & -6 & -12 & 22 \end{pmatrix} \xrightarrow[\substack{1: R_2 \to R_2 + 4R_1 \\ 2: R_3 \to R_3 - 2R_1}]{} \begin{pmatrix} \boxed{4} & -3 & -1 & 5 \\ 0 & 0 & \boxed{-2} & 3 \\ 0 & 0 & \underbrace{-10}_{\to 0} & 12 \end{pmatrix}$$

$$\xrightarrow[3: R_3 \to R_3 - 5R_2]{} \begin{pmatrix} 4 & -3 & -1 & 5 \\ 0 & 0 & -2 & 3 \\ 0 & 0 & 0 & -3 \end{pmatrix} = U$$

② $$E_1^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$E_2^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -(-2) & 0 & 1 \end{pmatrix} \qquad E_3^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -(-5) & 1 \end{pmatrix}$$

$$\underbrace{E_3} \cdot E_2 \cdot E_1 A = U$$

$$E_2 \cdot E_1 A = E_3^{-1} \cdot U$$

$$\vdots$$

$$A = \underbrace{(E_1^{-1} \cdot E_2^{-1} \cdot E_3^{-1})}_{= L} \cdot U$$

$$\begin{pmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 5 & 1 \end{pmatrix} \overset{\text{check}}{=} \begin{pmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 2 & 5 & 1 \end{pmatrix} = L$$

Summary

- To solve $A\vec{x} = LU\vec{x} = \vec{b}$,

  $\vec{y}$ $\parallel$

  1. Forward solve for $\vec{y}$ in $L\vec{y} = \vec{b}$. top $\rightarrow$ bottom y_1 y_2 $\vdots$
  2. Backwards solve for $\vec{x}$ in $U\vec{x} = \vec{y}$. bottom $\rightarrow$ to $x_n, x_{n-1}, \dots, x_1$
- To compute the LU decomposition:
  1. Reduce A to an echelon form U by a sequence of row replacement operations, if possible.
  2. Place entries in L such that the same sequence of row operations reduces L to I. Elementary matrices
- The textbook offers a different explanation of how to construct the LU decomposition that students may find helpful.
- Another explanation on how to calculate the LU decomposition that students may find helpful is available from MIT OpenCourseWare: www.youtube.com/watch?v=rhNKncraJMk

Section 2.8 : Subspaces of $\mathbb{R}^n$

Chapter 2 : Matrix Algebra

Math 1554 Linear Algebra

Topics and Objectives

Topics

We will cover these topics in this section.

1. Subspaces, Column space, and Null spaces
2. A basis for a subspace.

Objectives

For the topics covered in this section, students are expected to be able to do the following.

1. Determine whether a set is a subspace.
2. Determine whether a vector is in a particular subspace, or find a vector in that subspace.
3. Construct a basis for a subspace (for example, a basis for Col(A))

Motivating Question

Given a matrix A, what is the set of vectors $\vec{b}$ for which we can solve $A\vec{x} = \vec{b}$?

Subsets of $\mathbb{R}^n$

+ Structure

Definition

A **subset of** $\mathbb{R}^n$ is any collection of vectors that are in $\mathbb{R}^n$.

$$\text{Span}\{\vec{v}_1, \vec{v}_2\} = \{ c_1\vec{v}_1 + c_2\vec{v}_2 : c_1, c_2 \in \mathbb{R} \}$$

$$(c_1\vec{v}_1 + c_2\vec{v}_2) + (c_1'\vec{v}_1 + c_2'\vec{v}_2)$$

$$= (c_1 + c_1')\vec{v}_1 + (c_2 + c_2')\vec{v}_2 \in \text{Span}\{\vec{v}_1, \vec{v}_2\}$$

subset. & Subspace

Subspaces in $\mathbb{R}^n$

("vector space")

subset + structure.

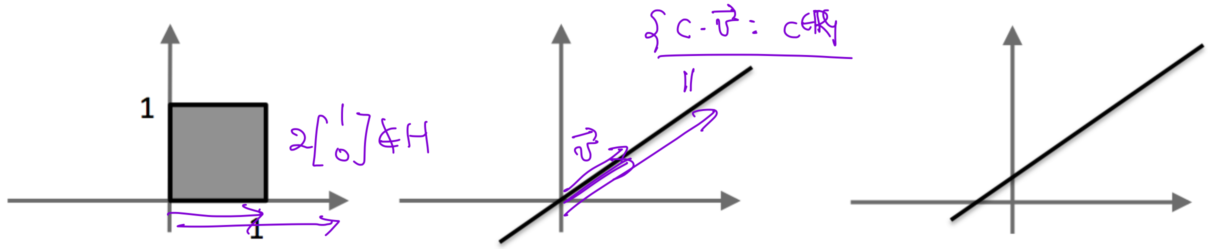
Definition

A subset H of $\mathbb{R}^n$ is a **subspace** if it is closed under scalar multiplies and vector addition. That is: for any $c \in \mathbb{R}$ and for $\vec{u}, \vec{v} \in H$,

1. $c\,\vec{u} \in H$ ✓
2. $\vec{u} + \vec{v} \in H$ ✓

Note that condition 1 implies that the zero vector must be in H. ⇒ smallest subspace $\{\vec{0}\}$

Example 1: Which of the following subsets could be a subspace of $\mathbb{R}^2$?



a) the unit square

Not a subspace.

b) a line passing through the origin

Subspace

c) a line that doesn't pass through the origin

does not contain $\vec{0}$

Not a subsp.

The Column Space and the Null Space of a Matrix

Recall: for $\vec{v}_1, \ldots, \vec{v}_p \in \mathbb{R}^n$, that $\mathrm{Span}\{\vec{v}_1, \ldots, \vec{v}_p\}$ is: a Subspace of $\mathbb{R}^n$.

This is a **subspace**, spanned by $\vec{v}_1, \ldots, \vec{v}_p$.

Definition

Given an $m \times n$ matrix $A = [\vec{a}_1 \quad \cdots \quad \vec{a}_n]$

1. The **column space of** A, $\mathrm{Col}\, A$, is the subspace of $\mathbb{R}^m$ spanned by $\vec{a}_1, \ldots, \vec{a}_n$.
2. The **null space of** A, $\mathrm{Null}\, A$, is the subspace of $\mathbb{R}^n$ spanned by the set of all vectors $\vec{x}$ that solve $A\vec{x} = \vec{0}$.

set of solutions

$$\mathrm{Col}(A) = \mathrm{Span}\{\vec{a}_1, \ldots, \vec{a}_n\}$$
$$= \{c_1\vec{a}_1 + \cdots + c_n\vec{a}_n : c_1, \ldots, c_n \in \mathbb{R}\}$$
$$= \{\underbrace{A \cdot \vec{x}}_{=T(\vec{x})} : \vec{x} \in \mathbb{R}^n\}$$
$$= \mathrm{Range}(T)$$

$$\mathrm{Null}(A) = \{\vec{x} \in \mathbb{R}^n : \underbrace{A\vec{x} = \vec{0}}_{\text{Eqn}}\} = \text{Solution set.}$$
$$= \{\vec{x} \in \mathbb{R}^n : T(\vec{x}) = \vec{0}\}$$

Why subspace?

$\vec{u}, \vec{v} \in \mathrm{Null}(A)$ $\qquad A\vec{u} = \vec{0}$, $A\vec{v} = \vec{0}$

$\vec{u} + \vec{v} \in \mathrm{Null}(A) \Leftarrow A(\vec{u} + \vec{v}) = \vec{0}$

Example

Is $\vec{b}$ in the column space of A?

$$A = \begin{bmatrix} 1 & -3 & -4 \\ -4 & 6 & -2 \\ -3 & 7 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & -4 \\ 0 & -6 & -18 \\ 0 & 0 & 0 \end{bmatrix}, \quad \vec{b} = \begin{pmatrix} 3 \\ 3 \\ -4 \end{pmatrix}$$

① $\vec{b} \in Col(A)$

$\Leftrightarrow$ $\vec{b}$ is a lin. combi. of columns

$\Leftrightarrow$ $A\vec{x} = \vec{b}$ is consistent

$\Leftrightarrow$ $[A \mid b]$ last columns is not pivot

---

② $\vec{v} = \lambda \begin{pmatrix} -5 \\ -3 \\ 1 \end{pmatrix} \in Null(A)$?

$\lambda \in \mathbb{R}$

$\Leftrightarrow$ $\vec{v}$ is a solution for $A\vec{x} = \vec{0}$

$\Leftrightarrow$ $A\vec{v} = \vec{0}$

check

$\Leftrightarrow$ $\begin{bmatrix} 1 & -3 & -4 \\ 0 & -6 & -18 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -5 \\ -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Example 2 (continued)

Using the matrix on the previous slide: is $\vec{v}$ in the null space of A?

$$\vec{v} = \begin{pmatrix} -5\lambda \\ -3\lambda \\ \lambda \end{pmatrix}, \quad \lambda \in \mathbb{R}$$

Basis

Definition

A **basis** for a subspace H is a set of linearly independent vectors in H that span H.

Example

$A = [1 \ 2 \ 1 \ 5]$

The set $H = \{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \in \mathbb{R}^4 \mid x_1 + 2x_2 + x_3 + 5x_4 = 0 \}$ is a subspace.

$$[1 \ 2 \ 1 \ 5] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

a) H is a null space for what matrix A?
b) Construct a basis for H.

$H = \text{Null}(A)$

9/25/23

H is a <u>subspace</u> of $\mathbb{R}^n$ if
$$\begin{cases} H \subseteq \mathbb{R}^n & \\ \vec{u}+\vec{v} \in H & \text{for } \vec{u},\vec{v} \in H \\ c\cdot\vec{u} \in H & \text{for } \vec{u} \in H,\ c \in \mathbb{R} \end{cases}$$

Examples: $\{\vec{0}\}$, $\mathbb{R}^n$, $\operatorname{Span}\{\vec{v}_1, \dots, \vec{v}_p\}$ are subspaces

$A \in \mathbb{R}^{m\times n}$, $A = [\vec{a}_1, \dots, \vec{a}_n]$ $\vec{a}_1, \dots, \vec{a}_n \in \mathbb{R}^m$

$$\operatorname{Col}(A) = \operatorname{Span}\{\vec{a}_1, \dots, \vec{a}_n\} \subseteq \mathbb{R}^m$$

$$\operatorname{Null}(A) = \{\vec{x} \in \mathbb{R}^n : A\vec{x} = \vec{0}\} \subseteq \mathbb{R}^n$$

$\mathcal{B} \subset H$ is a <u>basis</u> if $\begin{cases} H = \operatorname{Span} \mathcal{B} = \operatorname{Span}\{\vec{v}_1, \dots, \vec{v}_n\} \\ \mathcal{B} \text{ is linearly indep.} \end{cases}$

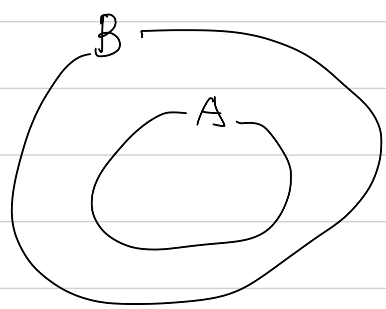
$\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_n\}$

Ex: $H = \mathbb{R}^2$, $\mathcal{B}_1 = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$

$\mathcal{B}_2 = \left\{ \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right\}$

Notation: $A \subset B$ means

if $x \in A$ then $x \in B$



Example

Construct a basis for Null$A$ and a basis for Col$A$.

free

$$A = \begin{bmatrix} -3 & 6 & -1 & 0 \\ 1 & -2 & 2 & 0 \\ 2 & -4 & 5 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{l} \to x_1 - 2x_2 = 0 \\ \to x_3 = 0 \\ x_2, x_4 : \text{free} \end{array}$$

$$\text{Null}(A) = \{ \vec{x} \in \mathbb{R}^4 : A\vec{x} = \vec{0} \}$$

$$= \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \underbrace{\begin{bmatrix} 2x_2 \\ x_2 \\ 0 \\ x_4 \end{bmatrix}}_{} \right\} = \left\{ x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} : x_2, x_4 \in \mathbb{R} \right\}$$

$$= \text{Span} \left\{ \underbrace{\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}}_{} \right\} \qquad \begin{bmatrix} 2x_2 \\ x_2 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ x_4 \end{bmatrix}$$

lin. indep.

$$\Rightarrow \left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\} \text{ is a basis for Null}(A).$$

Example

Construct a basis for Null$A$ and a basis for Col$A$.

$$A = \begin{array}{c} \begin{bmatrix} -3 & 6 & -1 & 0 \\ 1 & -2 & 2 & 0 \\ 2 & -4 & 5 & 0 \end{bmatrix} \\ \begin{array}{cccc} \shortparallel & \shortparallel & \shortparallel & \shortparallel \\ \vec{a}_1 & \vec{a}_2 & \vec{a}_3 & \vec{a}_4 \end{array} \end{array} \sim \begin{bmatrix} 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathrm{Col}(A) = \mathrm{Span}\{\vec{a}_1, \cdots, \vec{a}_4\} = \mathrm{Span}\{\underbrace{\vec{a}_1, \vec{a}_3}_{\text{lin. indep.}}\}$$

$\left\{ \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix} \right\}$ is a basis for Col(A).

<u>Note</u>

"dimension"

\# of elements in a "basis for Col (A)

= # of pivots

\# of elements in a basis for Null (A)

= # of free variables

non - pivots

Additional Example

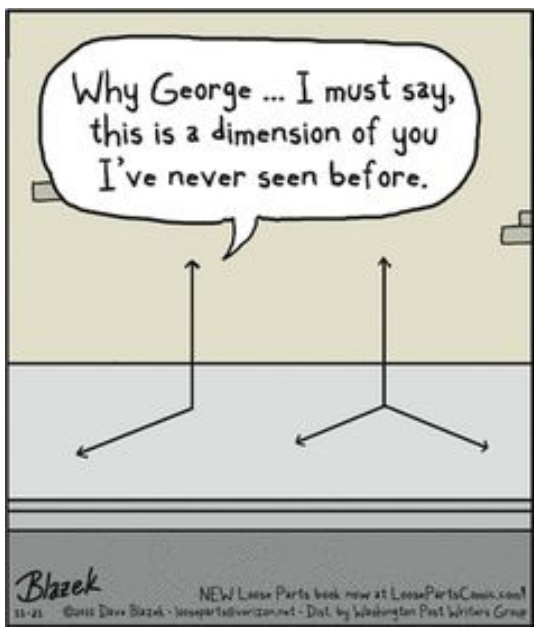
Let $V = \left\{ \begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{R}^2 \mid ab = 0 \right\}$.

1. Give an example of a vector that is in V.
2. Give an example of a vector that is not in V.
3. Is the zero vector in V?
4. Is V a subspace?

Section 2.9 : Dimension and Rank

Chapter 2 : Matrix Algebra

Math 1554 Linear Algebra



Topics and Objectives

Topics

We will cover these topics in this section.

1. Coordinates, relative to a basis.
2. Dimension of a subspace.
3. The Rank of a matrix

Objectives

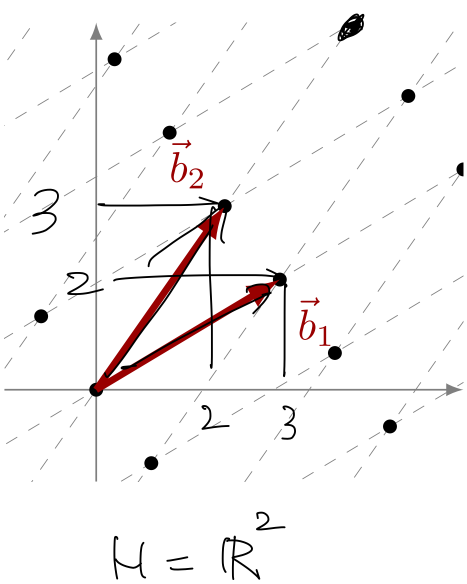
For the topics covered in this section, students are expected to be able to do the following.

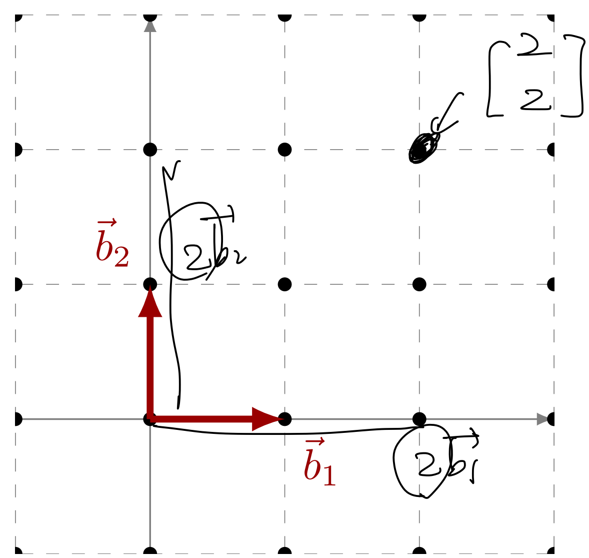
1. Calculate the coordinates of a vector in a given basis.
2. Characterize a subspace using the concept of dimension (or cardinality).
3. Characterize a matrix using the concepts of rank, column space, null space.
4. Apply the Rank, Basis, and Matrix Invertibility theorems to describe matrices and subspaces.

Choice of Basis

Key idea: There are many possible choices of basis for a subspace. Our choice can give us dramatically different properties.

Example: sketch $\vec{b}_1 + \vec{b}_2$ for the two different coordinate systems below.





Span $\mathcal{B} = H$

$\mathcal{B}$ is lin. indep.

Coordinates

$\Rightarrow$ For any $\vec{x} \in H$,

$\vec{x} = c_1 \vec{b}_1 + \cdots + c_p \vec{b}_p$ unique

Definition

Let $\mathcal{B} = \{\vec{b}_1, \ldots, \vec{b}_p\}$ be a basis for a subspace H. If $\vec{x}$ is in H, then **coordinates of $\vec{x}$ relative $\mathcal{B}$** are the weights (scalars) $c_1, \ldots, c_p$ so that

$$\vec{x} = \underbrace{c_1\vec{b}_1 + \cdots + c_p\vec{b}_p} = B \cdot [x]_{\mathcal{B}}$$

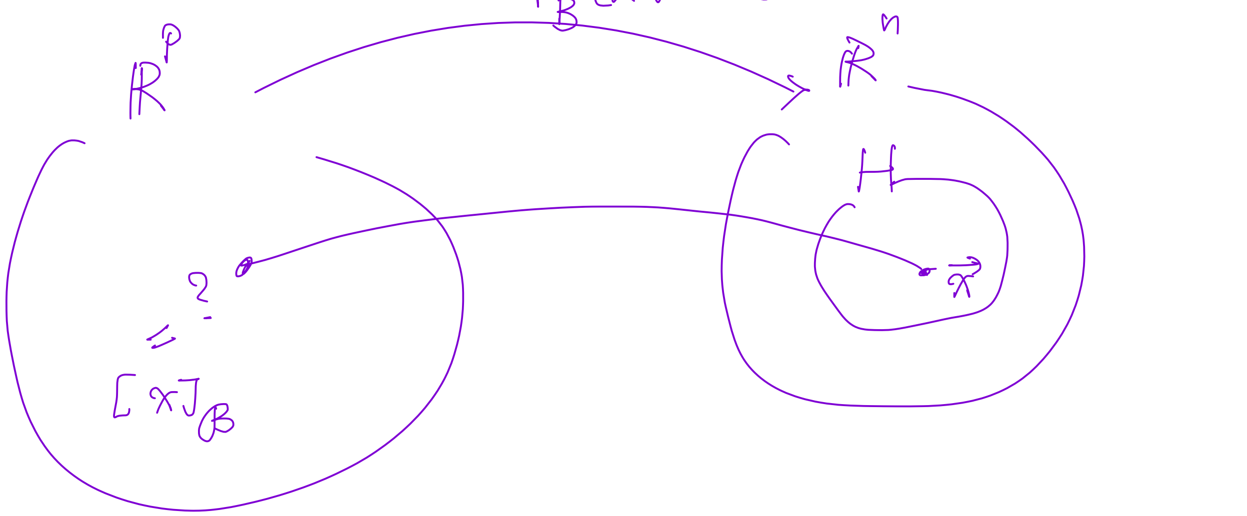
And

$$[x]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_p \end{bmatrix}$$

is the **coordinate vector of $\vec{x}$ relative to $\mathcal{B}$**, or the $\mathcal{B}$-**coordinate vector of $\vec{x}$**

$B = [\vec{b}_1, \ldots \vec{b}_p]$

$T_B(\vec{x}) = B\vec{x}$



Example 1

Let $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, and $\vec{x} = \begin{bmatrix} 5 \\ 3 \\ 5 \end{bmatrix}$. Verify that $\vec{x}$ is in the span of $\mathcal{B} = \{\vec{v}_1, \vec{v}_2\}$, and calculate $[\vec{x}]_{\mathcal{B}}$.

Dimension

Definition

The **dimension** (or cardinality) of a non-zero subspace H, $\dim H$, is the number of vectors in a basis of H. We define $\dim\{0\} = 0$.

Basis Theorem

Any two choices of bases $\mathcal{B}_1$ and $\mathcal{B}_2$ of a non-zero subspace H have the same dimension.

Examples:

$\mathcal{B} = \{ \vec{e_1}, \vec{e_2}, \cdots, \vec{e_n} \}$

Span $\mathcal{B} = \mathbb{R}^n$

lin. indep

$\rightarrow \mathcal{B}$ is a basis for $\mathbb{R}^n$

1. $\dim \mathbb{R}^n =$ n

   $=$ Null (A)

2. $H = \{(x_1, \ldots, x_n) \ : \ x_1 + \cdots + x_n = 0\}$ has dimension $n-1$

   $A = [1 \ 1 \ \cdots \ 1]$

3. $\dim(\mathrm{Null}\, A)$ is the number of free var.

4. $\dim(\mathrm{Col}\, A)$ is the number of pivot columns

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ \vdots \end{bmatrix}, \cdots \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \right\}$$

$A \in \mathbb{R}^{m \times n}$

full rank.

Rank

$0 \leq \text{Rank}(A) = \dim(\text{Col}(A)) \leq n$

> **Definition**
>
> The **rank** of a matrix A is the dimension of its column space.

Example 2: Compute rank(A) and dim(Nul(A)).

$$A = \begin{bmatrix} 2 & 5 & -3 & -4 & 8 \\ 4 & 7 & -4 & -3 & 9 \\ 6 & 9 & -5 & 2 & 4 \\ 0 & -9 & 6 & 5 & -6 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} 2 & 5 & -3 & -4 & 8 \\ 0 & -3 & 2 & 5 & -7 \\ 0 & 0 & 0 & 4 & -6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\text{Rank}(A) = 2$ or 3

$\dim(\text{Null}(A)) = 2$ or 3

$A \in \mathbb{R}^{n \times n}$ $\quad T(\vec{x}) = A\vec{x}$

T is $1-1$ $\Leftrightarrow$ $A\vec{x} = T(\vec{x}) = 0$ implies $\vec{x} = 0$

$\Leftrightarrow$ $\text{Null}(A) = \{ \vec{x} : A\vec{x} = \vec{0} \} = \{\vec{0}\}$

$\Leftrightarrow$ $\dim(\text{Null}(A)) = 0$.

$\Leftrightarrow$ $\text{Rank}(A) = n$

Rank, Basis, and Invertibility Theorems

Theorem (Rank Theorem)

If a matrix A has n columns, then $\mathrm{Rank}\, A + \dim(\mathrm{Nul}\, A) = n$.

Theorem (Basis Theorem)

Any two bases for a subspace have the same dimension.

Theorem (Invertibility Theorem)

Let A be a $n \times n$ matrix. These conditions are equivalent.

1. *A is invertible.*
2. *The columns of A are a basis for $\mathbb{R}^n$.*
3. $\mathrm{Col}\, A = \mathbb{R}^n$.
4. $\mathrm{rank}\, A = \dim(\mathrm{Col}\, A) = n$.
5. $\mathrm{Null}\, A = \{0\}$.

Examples

$A = \begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix}$

If possible give an example of a 2×3 matrix A, that is in RREF and has the given properties.

a) $\underset{\#\text{ of pivots } \leq 2}{\underset{\parallel}{\text{rank}(A)}} = 3$

Not possible.

b) rank$(A) = 2$

$$\begin{bmatrix} 1 & 0 & * \\ 0 & 1 & * \end{bmatrix}$$

c) dim(Null(A)) $= 2$

$$\begin{bmatrix} 1 & * & * \\ 0 & 0 & 0 \end{bmatrix}$$

d) Null$A = \{0\}$

Not possible.

dim (Null (A))
$\parallel$
# of free variables

$\begin{bmatrix} 1 & 0 & \end{bmatrix}$ ← free.