

Supplementary Problems - Ch. 7 and Google PageRank

The following problems are sample problems meant to show the type of problems that might appear on the final exam for the sections after Exam 3, namely Google PageRank and the content of Chapter 7 (symmetric matrices, spectral theorem, quadratic forms, constrained optimization, and Singular Value Decomposition).

WARNING: The problems are **not** meant to be an exhaustive list. Please be aware that these are not necessarily the same problems that will appear on the final exam, and these problems may not cover every aspect of what may be asked. These are meant only to be a sample of problems that could appear.


1. (a) Suppose A is an $m \times n$ matrix and $\vec{b} \in \mathbb{R}^m$ unless otherwise stated. Select **true** if the statement is true for all choices of A and \vec{b} . Otherwise, select **false**.

true false

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|-----------------------|----------------------------------|--|
| <input type="radio"/> | <input checked="" type="radio"/> | If A is a singular symmetric matrix then the minimum value of the quadratic form $Q(\vec{x}) = \vec{x}^T A \vec{x}$ subject to the constraint $\ \vec{x}\ = 1$ is zero. |
| <input type="radio"/> | <input checked="" type="radio"/> | Suppose the least of the singular values of A is 2. Then, the quadratic form $Q(\vec{x}) = \ A\vec{x}\ ^2 - \ \vec{x}\ ^2$ is indefinite. |
| <input type="radio"/> | <input checked="" type="radio"/> | If $Q : \mathbb{R}^3 \rightarrow \mathbb{R}$ is a quadratic form having <i>only</i> cross terms, then it is indefinite. |
| <input type="radio"/> | <input checked="" type="radio"/> | If the singular values of a square $n \times n$ matrix A are $\sigma_1, \sigma_2, \dots, \sigma_n$, then $\det(A) = \sigma_1 \sigma_2 \cdots \sigma_n$. |
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- (b) Indicate whether the following situations are possible or impossible.

possible impossible

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|-----------------------|----------------------------------|---|
| <input type="radio"/> | <input checked="" type="radio"/> | A symmetric matrix A which is not diagonalizable. |
| <input type="radio"/> | <input checked="" type="radio"/> | An $m \times n$ matrix A that does not have a singular value decomposition. |
| <input type="radio"/> | <input checked="" type="radio"/> | An indefinite ^{Positive semi-definite} quadratic form $Q(\vec{x}) = \vec{x}^T A \vec{x}$ such that $Q(\vec{x}) \geq 0$ for all inputs \vec{x} . |
| <input type="radio"/> | <input checked="" type="radio"/> | An invertible $n \times n$ matrix A such that the <u>minimum value of $Q(\vec{x}) = \vec{x}^T A^T A \vec{x} = \ A\vec{x}\ ^2$</u> subject to the constraint $\ \vec{x}\ = 1$ is <u>less than 0</u> . |
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 You do not need to justify your reasoning for questions on this page.

2. Give an example of a symmetric 3×3 matrix A such that:

- the corresponding quadratic form $Q(\vec{x}) = \vec{x}^T A \vec{x}$ is negative semi-definite and has no cross product terms;
- the range of the corresponding linear transformation $T(\vec{x}) = A\vec{x}$ is 2-dimensional; and
- one of the eigenvalues of A has algebraic multiplicity equal to 2.

$\lambda_i \leq 0$

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\phi(\lambda) = \det(A - \lambda I) = \det \begin{bmatrix} -\lambda & 0 & 0 \\ 0 & -\lambda & 2 \\ 0 & 2 & -\lambda \end{bmatrix} = -\lambda(\lambda^2 - 4) = -\lambda(\lambda - 2)(\lambda + 2) = 0$$

= smallest eigenvalue.

3. Find the minimum value of $Q(\vec{x}) = 4x_2x_3$ on the unit sphere in \mathbb{R}^3 .

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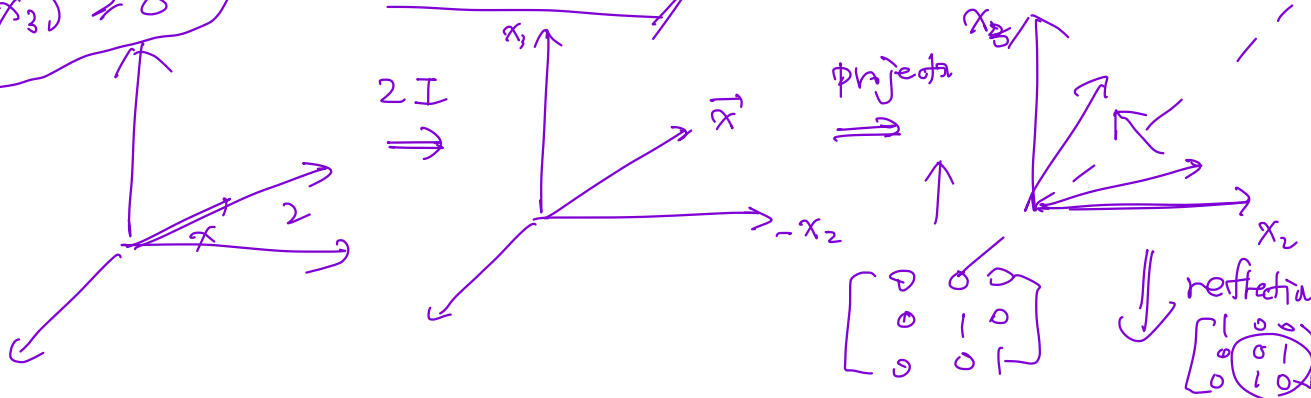
$$4x_2x_3 \geq -2x_2^2 - 2x_3^2$$

$$2x_2^2 + 4x_2x_3 + 2x_3^2 \geq 0$$

$$2(x_2 + x_3)^2 \geq 0$$

$= \vec{x}^T \cdot A \cdot \vec{x}$

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix} \in \mathbb{R}^{3 \times 3}$$



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4. Let $A = U\Sigma V^T$, where

$$U = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix},$$

$$\Sigma = \begin{bmatrix} \sqrt{6} & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad V = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ -1/\sqrt{3} & 0 & 2/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \end{bmatrix}.$$

(a) What is the rank of A ? 2

(b) $\max_{\|\vec{x}\|=1} \|A\vec{x}\| = \sqrt{6}$

(c) $\det(A^T A) = 0$

(d) What are the eigenvalues of $A^T A$? 6, 4, 0

(e) Find an orthonormal basis for $\text{Row}(A)$.

$$\left[\begin{array}{c} 1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{array} \right], \left[\begin{array}{c} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{array} \right]$$

(f) Find an orthonormal basis for $\text{Nul}(A^T)$.

$$\frac{1}{\sqrt{2}} \left[\begin{array}{c} -1 \\ 0 \\ 1 \\ 0 \end{array} \right], \frac{1}{\sqrt{2}} \left[\begin{array}{c} 0 \\ 1 \\ 0 \\ -1 \end{array} \right]$$

$$\textcircled{1} \quad \|Ax\| = \left\| \underbrace{U}_{\text{orthogonal}} \Sigma V^T x \right\| = \left\| \Sigma \underbrace{(V^T x)}_y \right\| \leftarrow \text{max.}$$

$$\|x\| = 1 = \left\| \underbrace{V^T x}_y \right\|$$

$$\max \{ \|Ax\| : \|x\| = 1 \} = \max \{ \|\Sigma y\| : \|y\| = 1 \}$$

$$= \text{largest } \sigma_i = \sigma_1$$

$$\textcircled{2} \quad \max \|Ax\|^2 = \max \{ x^T \cdot (A^T A) x \} = \text{largest eigenvalue of } A^T A$$

$$= \lambda_1$$

$$\max \|Ax\| = \sqrt{\lambda_1} = \sigma_1$$

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5. **Show work** on this page with work under the problem, and **your answer in the box**.

Consider the quadratic form $Q(x_1, x_2, x_3) = 3x_1^2 + x_2^2 + x_3^2 - 6x_2x_3$.

- (a) There is a change of variables $\vec{x} = P\vec{y}$, where P is an orthogonal matrix, such that the resulting quadratic form $Q(y_1, y_2, y_3)$ has no cross product terms. What is $Q(y_1, y_2, y_3)$ where $\vec{x} = P\vec{y}$?

$$Q(y_1, y_2, y_3) = 3y_1^2 + 4y_2^2 - 2y_3^2$$

(b) Is Q positive-definite, negative-definite, or indefinite? *Explain in a few words.*

positive-definite negative-definite indefinite

Extra problem: in case you want to practice doing least-squares with non-linear models.

6. *Show work on this page with work under the problem, and your answer in the box.*

In this problem, you will use the least-squares method to find the values α and β which best fit the curve

$$y = \alpha x^3 + \beta x^2$$

to the data points $(-1, 2)$, $(0, 1)$, $(1, 4)$ using the parameters α and β .

(i) What is the augmented matrix for the linear system of equations associated to this least squares problem?

$$\left[\begin{array}{cc|c} -1 & 1 & 2 \\ 0 & 0 & 1 \\ 1 & 1 & 4 \end{array} \right]$$

(ii) What is the augmented matrix for the normal equations for this system.

$$\left[\begin{array}{cc|c} 2 & 0 & 2 \\ 0 & 2 & 6 \end{array} \right]$$

(iii) Find a least-squares solution to the linear system from (i) to determine the parameters α and β of the best fitting curve.

$$\alpha = \boxed{1} \quad \beta = \boxed{3}$$