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Conditional Expectation

X, Y discrete RVs with joint pmf $p(x,y)$. If $p_Y(y) > 0$,

$$\mathbb{E}[X \mid Y=y] = \sum_x x \cdot P(X=x \mid Y=y) = \sum x \, p_{X|Y}(x|y)$$

$$= \sum_x x \cdot \frac{p(x,y)}{p_Y(y)} .$$

<u>Example</u> $X, Y \sim \mathrm{Bin}(n,p)$ indep.

$\mathbb{E}[X \mid X+Y=m] = ?$

Let $Z = X+Y \sim \mathrm{Bin}(2n, p)$.

$$P(X=j, Z=k) = P(X=j, Y=k-j) = P(X=j)\,P(Y=k-j)$$

$$= \binom{n}{j} p^j (1-p)^{n-j} \binom{n}{k-j} p^{k-j} (1-p)^{n-k+j}$$

$$= \binom{n}{j}\binom{n}{k-j} p^k (1-p)^{n-k} \quad \text{if } j \le k$$

$$P(Z=k) = \sum_{j=0}^{k} P(X=j, Z=k) = \binom{2n}{k} p^k (1-p)^{n-k}$$

$$p_{X|Z}(j|k) = \frac{\binom{n}{j}\binom{n}{k-j}}{\binom{2n}{k}} \quad \text{for } j = 0, \cdots, k.$$

$$: \ \text{HyperGeom}(2n, n, k)$$

$$\Rightarrow \mathbb{E}[X \mid Z=m] = \frac{n \cdot m}{2n} = \frac{m}{2}.$$

Suppose X, Y are jointly continuous with $f(x,y)$.

If $f_Y(y) > 0$ then

$$\mathbb{E}[X \mid Y=y] = \int x \cdot f_{X|Y}(x|y)\, dx.$$

<u>Example</u>

$$f(x,y) = \begin{cases} \frac{1}{y} \cdot e^{-\frac{x}{y}} \cdot e^{-y}, & 0 < x, y < \infty \\ 0, & \text{o.w.} \end{cases}$$

$$f_Y(y) = e^{-y} \cdot \mathbb{1}_{(0,\infty)}(y) \quad \Rightarrow \quad f_{X|Y}(x|y) = \frac{1}{y} e^{-x/y} \, \mathbb{1}_{(0,\infty)}(x).$$

i.e $X \mid Y=y \sim \text{Exp}\left(\frac{1}{y}\right)$. Thus $\mathbb{E}[X \mid Y=y] = y$.

Conditional Expectation as a RV

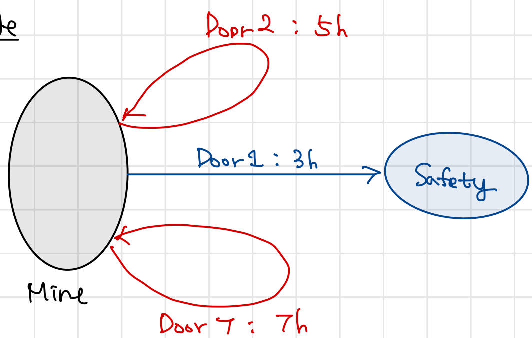
Previous examples show that $\mathbb{E}[X \mid Y=y]$ is a function of y. That is, $\mathbb{E}[X \mid Y=y] =: g(y)$

If $Y=y$ w/ some prob. then $\mathbb{E}[X \mid Y=y]$ returns a number in terms of y w/ the same prob.

<u>Def</u> $\mathbb{E}[X \mid Y] := g(Y)$.

<u>Conditioning</u> $\mathbb{E}[X] = \mathbb{E}[\,\mathbb{E}[X \mid Y]\,]$

<u>Example</u>



$X =$ exit time

$Y =$ 1st choice of Door

$$= \begin{cases} 1 & \text{with } 1/3 \\ 2 & \text{"} \\ 3 & \text{"} \end{cases}$$

$$\mathbb{E}[\text{exit time}] = \mathbb{E}[\,\mathbb{E}[X \mid Y]\,]$$

$\mathbb{E}[X \mid Y=1] = 3$, $\mathbb{E}[X \mid Y=2] = \mathbb{E}[X] + 5$

$\mathbb{E}[X \mid Y=3] = \mathbb{E}[X] + 7$

$$\mathbb{E}[X] = \frac{1}{3}(3 + \mathbb{E}X + 5 + \mathbb{E}X + 7) = \frac{2}{3}\mathbb{E}X + 5$$

$\therefore \quad \mathbb{E}X = 15$.

Note h: a function

$$\mathbb{E}[(X-h(Y))^2] \geq \mathbb{E}[(X-\mathbb{E}[X|Y])^2]$$

$\because$ $\mathbb{E}[X \cdot h(Y) | Y] = h(Y) \cdot \mathbb{E}[X|Y]$

$\mathbb{E}[h(Y)^2 | Y] = h(Y)^2$

$\mathbb{E}[(X-h(Y))^2 | Y] = \mathbb{E}[X^2|Y] - 2h(Y)\,\mathbb{E}[X|Y] + h(Y)^2$

$$= \mathbb{E}[(X-\mathbb{E}[X|Y])^2 | Y] + \mathbb{E}[X|Y]^2 + h(Y)^2 - 2h(Y)\cdot\mathbb{E}[X|Y]$$

$\Rightarrow$ $\mathbb{E}[(X-h(Y))^2] = \mathbb{E}[(X-\mathbb{E}[X|Y])^2] + \mathbb{E}[(\mathbb{E}[X|Y]-h(Y))^2]$. $\lrcorner$

How to interprete this ?

Y : given information $\Rightarrow$ Knowledge from Y $\ni \mathbb{E}[X|Y]$

$= \{ h(Y) : h \text{ is a function} \}$

Best possible estimate for X given Y ?

Find h s.t. $X - h(Y)$ is as small as possible.

($\mathbb{E}((X-h(Y))^2)$)

$\Rightarrow$ the optimal h is $h(Y) = \mathbb{E}[X|Y]$.