Chapter 6. Point Estimation

Math 3670 Summer 2024

Georgia Institute of Technology

Section 1.
Some General Concepts of Point Estimation

Statistics and Point Estimators

Definition

A statistic is any quantity whose value can be calculated from sample data.

Prior to obtaining data, there is uncertainty as to what value of any particular statistic will result.

Therefore, a statistic is a random variable and will be denoted by an uppercase letter; a lowercase letter is used to represent the calculated or observed value of the statistic.

A point estimate of a parameter θ is a single number that can be regarded as a sensible value for θ.

A point estimate is obtained by selecting a suitable statistic and computing its value from the given sample data.

The selected statistic is called the point estimator of θ

Example

Let μ (a parameter) denote the true average breaking strength of wire connections used in bonding semiconductor wafers.

A random sample of $n = 10$ connections might be made, and the breaking strength of each one determined, resulting in observed strengths $x_1, x_2, \cdots, x_{10}$.

The sample mean breaking strength $\bar{x}$ could then be used to draw a conclusion about the value of μ.

Example

An automobile manufacturer has developed a new type of bumper, which is supposed to absorb impacts with less damage than previous bumpers. The manufacturer has used this bumper in a sequence of 25 controlled crashes against a wall, each at 10 mph, using one of its compact car models.

Let X be the number of crashes that result in no visible damage to the automobile. The parameter to be estimated is

$$p = \mathbb{P}(\text{no damage in a single crash}).$$

If X is observed to be $x = 15$, then

$$\hat{p} = \qquad\qquad\qquad\qquad\qquad .$$

Example

The article "Is a Normal Distribution the Most Appropriate Statistical Distribution for Volumetric Properties in Asphalt Mixtures?" reported the following observations on $X =$ voids filled with asphalt (%) for 52 specimens of a certain type of hot-mix asphalt:

74.33	71.07	73.82	77.42	79.35	82.27	77.75	78.65	77.19
74.69	77.25	74.84	60.90	60.75	74.09	65.36	67.84	69.97
68.83	75.09	62.54	67.47	72.00	66.51	68.21	64.46	64.34
64.93	67.33	66.08	67.31	74.87	69.40	70.83	81.73	82.50
79.87	81.96	79.51	84.12	80.61	79.89	79.70	78.74	77.28
79.97	75.09	74.38	77.67	83.73	80.39	76.90		

Possible estimators for σ^2 are

For a parameter θ, there are many different possible estimators.

Among them, which one would be best? How can we choose the best possible estimator for θ?

Unbiased Estimators

Definition

A point estimator $\widehat{\theta}$ is said to be an unbiased estimator of θ if

$$\mathbb{E}[\widehat{\theta}] = \theta$$

for every possible value of θ.

If $\widehat{\theta}$ is not unbiased, the difference

$$\mathbb{E}[\widehat{\theta}] - \theta$$

is called the bias of $\widehat{\theta}$.

Principle of Unbiased Estimation: When choosing among several different estimators of u, select one that is unbiased.

Unbiased Estimators

Example

Let $X \sim \text{Bin}(n, p)$.

Find an unbiased estimator of p.

Unbiased Estimators

Example

Let $X \sim \text{Unif}(0, \theta)$.

Find an unbiased estimator of θ.

Unbiased Estimators

Proposition

If $X_1, X_2, \cdots, X_n$ is a random sample from a distribution with mean μ and variance σ^2, then

$$\bar{X} =$$
$$S^2 =$$

are unbiased estimators of μ and σ^2.

Estimators with Minimum Variance

Principle of Minimum Variance Unbiased Estimation

Among all estimators of θ that are unbiased, choose the one that has minimum variance.

The resulting $\widehat{\theta}$ is called the **minimum variance unbiased estimator (MVUE)** of θ.

Example

Let $X_1, \cdots, X_n$ be a random sample from a uniform distribution on $[0, \theta]$. Consider

$$\widehat{\theta}_1 = \frac{n+1}{n} \max X_i$$
$$\widehat{\theta}_2 = 2\overline{X}.$$

Are they unbiased?

Find the variances of $\widehat{\theta}_1$ and $\widehat{\theta}_2$

Estimators with Minimum Variance

Theorem

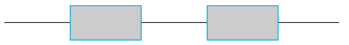
Let $X_1, \cdots, X_n$ be a random sample from a normal distribution with parameters μ and σ^2.

Then the estimator $\overline{X}$ is the MVUE for μ.

Exercise

(6.1-8) In a random sample of 80 components of a certain type, 12 are found to be defective.

1. Give a point estimate of the proportion of all such components that are not defective.
2. A system is to be constructed by randomly selecting two of these components and connecting them in series, as shown here.

   The series connection implies that the system will function if and only if neither component is defective (i.e., both components work properly).
   Estimate the proportion of all such systems that work properly.

Section 2.
Methods of Point Estimation

The Method of Moments

Definition

Let $X_1, \cdots, X_n$ be a random sample from a PMF or PDF $f(x)$.

For $k = 1, 2, \cdots$, the k-th population moment, or k-th moment of the distribution $f(x)$, is $\mathbb{E}[X^k]$.

The k-th sample moment is

The Method of Moments

Definition

Let $X_1, \cdots, X_n$ be a random sample from a distribution with PMF or PDF $f(x; \theta_1, \cdots, \theta_m)$, where $\theta_1, \cdots, \theta_m$ are parameters whose values are unknown.

Then the moment estimators

$$\widehat{\theta}_1, \cdots, \widehat{\theta}_m$$

are obtained by equating the first m sample moments to the corresponding first m population moments and solving for $\theta_1, \cdots, \theta_m$.

Example

Let $X_1, \cdots, X_n$ represent a random sample of service times of n customers at a certain facility, where the underlying distribution is assumed exponential with parameter λ.

Find the moment estimator for λ.

Example

Let $X_1, \cdots, X_n$ be a random sample of size n from a Gamma distribution.

Find the moment estimators for α, β.

Maximum Likelihood Estimation

Definition

Let $X_1, \cdots, X_n$ have joint PMF or PDF

$$f(x_1, x_2, \cdots, x_n; \theta_1, \cdots, \theta_m)$$

where the parameters $\theta_1, \cdots, \theta_m$ have unknown values.

When $x_1, \cdots, x_n$ are the observed sample values and f is regarded as a function of $\theta_1, \cdots, \theta_m$, it is called the likelihood function.

The maximum likelihood estimates (MLE) $\widehat{\theta}_1, \cdots, \widehat{\theta}_m$ are those values of the θ_i's that maximize the likelihood function

When the X_i's are substituted in place of the x_i's, the maximum likelihood estimators result.

Maximum Likelihood Estimation

Example

Let $X_1, \cdots, X_n$ be a random sample from Bernoulli distribution.

Find the Likelihood function and the MLE for p.

Example

Let $X_1, \cdots, X_n$ be a random sample from exponential distribution.

Find the Likelihood function and the MLE for λ.

Maximum Likelihood Estimation

Example

Let $X_1, \cdots, X_n$ be a random sample from normal distribution.

Find the Likelihood function and the MLEs for μ, σ^2.

Exercise

(6.2-23) Two different computer systems are monitored for a total of n weeks.

Let X_i denote the number of breakdowns of the first system during the i-th week, and suppose the X_i's are independent and drawn from a Poisson distribution with parameter μ_1.

Similarly, let Y_i denote the number of breakdowns of the second system during the i-th week, and assume independence with each Y_i Poisson with parameter μ_2.

Derive the MLE's of μ_1, μ_2, and $\mu_1 - \mu_2$