Math 3215: Intro to Probability and Statistics

Final Exam Solution, Summer 2023

1. Suppose A, B, and C are events with

$$\begin{aligned} \mathbb{P}(A) = 0.58, \quad & \mathbb{P}(B) = 0.55, \quad \mathbb{P}(C) = 0.4, \\ \mathbb{P}(A \cap B) = 0.28, \quad & \mathbb{P}(A \cap C) = 0.23, \quad \mathbb{P}(B \cap C) = 0.23, \\ & \mathbb{P}(A \cap B \cap C) = 0.13. \end{aligned}$$

   (a) (4 points) Find $\mathbb{P}(C|A \cap B)$.

   (b) (4 points) Find $\mathbb{P}(A^c \cap B^c \cap C^c)$.

Solution:

(a) $\mathbb{P}(C|A \cap B) = \mathbb{P}(A \cap B \cap C)/\mathbb{P}(A \cap B) = 0.13/0.28 = 13/28$.

(b) Since

$$\begin{aligned} \mathbb{P}(A \cup B \cup C) &= \mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(C) - \mathbb{P}(A \cap B) - \mathbb{P}(B \cap C) - \mathbb{P}(C \cap A) + \mathbb{P}(A \cap B \cap C) \\ &= 0.58 + 0.55 + 0.4 - 0.28 - 0.23 - 0.23 + 0.13 \\ &= 0.92, \end{aligned}$$

Thus, $\mathbb{P}(A^c \cap B^c \cap C^c) = 1 - \mathbb{P}(A \cup B \cup C) = 0.08$.

2. Suppose you roll two 4 faced dice, with faces labeled 1,2,3,4, and each equally likely to appear on top. Let X_1 be the number from the first die and X_2 the number from the second die. Let $Y_1 = \min\{X_1, X_2\}$ and $Y_2 = \max\{X_1, X_2\}$.

   (a) (4 points) Find the PMF of Y_2.

   (b) (4 points) Find $\mathbb{E}[Y_1 + Y_2]$.

Solution:

(a) $f_{Y_2}(1) = 1/16$, $f_{Y_2}(2) = 3/16$, $f_{Y_2}(3) = 5/16$, $f_{Y_2}(4) = 7/16$, otherwise 0.

(b) $\mathbb{E}[Y_1 + Y_2] = \mathbb{E}[X_1 + X_2] = \mathbb{E}[X_1] + \mathbb{E}[X_2] = 5$.

3. Suppose X is a random variable taking values in $S = \{0, 1, 2, 3, \dots\}$ with PMF $f_X(k) = \dfrac{C \cdot 2^k}{k!}$.

   (a) (4 points) Find the constant C.

   (b) (4 points) Find $\mathbb{E}[e^{3X}]$.

Solution:

(a) Since $\sum_{k=0}^{\infty} f_X(k) = C \sum_{k=0}^{\infty} 2^k/k! = Ce^2 = 1$, $C = e^{-2}$. That is, X is a Poisson random variable with $\lambda = 2$.

(b) $\mathbb{E}[e^{3X}] = \sum_{k=0}^{\infty} e^{3k} e^{-2} 2^k/k! = e^{-2} e^{2e^3} = \exp(2(e^3 - 1))$.

4. Consider an urn containing 10 balls, of which 5 are black and 5 are white. Suppose two balls are drawn at random without replacement. Let A be the event that the first ball is black, and B the event that the second ball is white.
   (a) (4 points) Find $\mathbb{P}(B)$.
   (b) (4 points) Find $\mathbb{P}(A|B)$.

Solution:
(a) $\mathbb{P}(B) = \mathbb{P}(A \cap B) + \mathbb{P}(A^c \cap B) = \mathbb{P}(B|A)\mathbb{P}(A) + \mathbb{P}(B|A^c)\mathbb{P}(A^c) = 5/9 \cdot 1/2 + 4/9 \cdot 1/2 = 1/2$.
(b) $\mathbb{P}(A|B) = \mathbb{P}(A \cap B)/\mathbb{P}(B) = 5/9$.

5. Let X and Y be two random variables with joint pdf $f_{X,Y}(x,y) = 2e^{-x-y}$ for $0 \le x \le y < \infty$ and otherwise 0.
   (a) (4 points) Find the marginal PDF of X.
   (b) (4 points) Find the conditional expectation $\mathbb{E}[Y|X=x]$ for $x > 0$.

Solution:
(a) $f_X(x) = \int_x^\infty 2e^{-x-y}\,dy = 2e^{-2x}$ for $x \ge 0$.
(b) Since $f_{Y|X}(y|x) = e^{-(y-x)}$ for $y \ge x$, $\mathbb{E}[Y|X=x] = \int_x^\infty ye^{-(y-x)}\,dy = x+1$.

6. Let X and Y be two random variables with joint pdf $f_{X,Y}(x,y) = 2$ for $0 < x+y < 1$, $x > 0$, $y > 0$, and otherwise 0.
   (a) (5 points) Find the covariance $\mathrm{Cov}(X,Y)$.
   (b) (5 points) Compute $\mathbb{P}(X \le \frac{1}{2})$, $\mathbb{P}(Y \le \frac{1}{2})$, and $\mathbb{P}(X \le \frac{1}{2}, Y \le \frac{1}{2})$.

Solution:
(a) $\mathbb{E}[XY] = \int_0^1 \int_0^{1-y} 2xy\,dxdy = \int_0^1 y(1-y)^2\,dy = 1/3 - 1/4 = 1/12$ and $\mathbb{E}[X] = \int_0^1 \int_0^{1-y} 2x\,dxdy = \int_0^1 (1-y)^2\,dy = 1/3 = \mathbb{E}[Y]$. Thus $\mathrm{Cov}(X,Y) = 1/12 - 1/3^2 = -1/36$.
(b) $\mathbb{P}(X \le \frac{1}{2}) = \mathbb{P}(Y \le \frac{1}{2}) = 3/4$ and $\mathbb{P}(X \le \frac{1}{2}, Y \le \frac{1}{2}) = 1/2$.

7. Let (X,Y) be a bivariate normal random vector.
   (a) (5 points) Suppose that both X and Y have mean 2 and variance 3, while the correlation coefficient of X and Y is $\rho = -\frac{1}{6}$. Find $\mathrm{Var}(X-Y)$.
   (b) (5 points) Assume now that X and Y have mean 7 and variance 4, but that the correlation coefficient has changed and is now given by $\rho = 0$. Write $\mathbb{P}(4 \le X \le 9, Y \le 8)$ in terms of $\Phi(z) = \mathbb{P}(Z \le z)$ for $z \ge 0$ where Z is the standard normal random variable.

Solution:
(a) $\mathrm{Var}(X-Y) = \mathrm{Var}(X) + \mathrm{Var}(Y) - 2\,\mathrm{Cov}(X,Y) = \mathrm{Var}(X) + \mathrm{Var}(Y) - 2\rho\sqrt{\mathrm{Var}(X)\,\mathrm{Var}(Y)} = 3 + 3 - 2 \cdot (-1/6) \cdot 3 = 7$.

(b) Since $\rho = 0$, X and Y are independent. Thus

$$\begin{aligned}\mathbb{P}(4 \le X \le 9, Y \le 8) &= \mathbb{P}(4 \le X \le 9)\mathbb{P}(Y \le 8) \\ &= \mathbb{P}((4-7)/2 \le (X-7)/2 \le (9-7)/2)\mathbb{P}((Y-7)/2 \le (8-7)/2) \\ &= \mathbb{P}(-1.5 \le Z \le 1)\mathbb{P}(Z \le 0.5) \\ &= (\Phi(1) - \Phi(-1.5))\Phi(0.5) = (\Phi(1) + \Phi(1.5) - 1)\Phi(0.5).\end{aligned}$$

8. Let X be a uniform random variable on $(-1,1)$ and $Y = X^3$.
   (a) (4 points) Find the pdf of Y.
   (b) (4 points) Compute $\mathrm{Cov}(X,Y)$.

Solution:

(a) $f_Y(y) = \frac{1}{6}y^{-\frac{2}{3}}$ for $y \in (-1,1)$ and otherwise 0.

(b) $\mathbb{E}[XY] = \mathbb{E}[X^4] = \int_{-1}^{1} \frac{1}{2}x^4\, dx = 1/5$ and $\mathbb{E}[X] = \mathbb{E}[X^3] = 0$ by symmetry. Thus, $\mathrm{Cov}(X,Y) = 1/5$.

9. A fair die will be rolled 720 times independently.
   (a) (5 points) What is the probability that among the 180 rolls the number 6 will appear between 27 and 32 times inclusively? That is, what is $\mathbb{P}(27 \le X \le 32)$? Write down the probability without using the tables and approximations.
   (b) (5 points) Using a normal approximation, **with half-unit correction**, write down an expression for the probability that among the 180 rolls the number 6 will appear between 27 and 32 times inclusively. Use the corresponding tables to find an approximate value for this probability.

Solution:

(a) $\mathbb{P}(27 \le X \le 32) = \sum_{k=27}^{32} \binom{180}{k} \left(\frac{1}{6}\right)^k \left(\frac{5}{6}\right)^{180-k}$.

(b)

$$\begin{aligned}\mathbb{P}(27 \le X \le 32) &= \mathbb{P}(26.5 < X < 32.5) \\ &= \mathbb{P}\left(\frac{26.5 - np}{\sqrt{np(1-p)}} < \frac{X - np}{\sqrt{np(1-p)}} < \frac{32.5 - np}{\sqrt{np(1-p)}}\right) \\ &= \mathbb{P}\left(\frac{26.5 - 30}{5} < \frac{X - 30}{5} < \frac{32.5 - 30}{5}\right) \\ &\approx \mathbb{P}(-0.7 < Z < 0.5) \\ &\approx 0.6915 + 0.7580 - 1 = 0.4495.\end{aligned}$$

10. (6 points) Let $\overline{X}$ be the mean of a random sample of size $n = 25$ from a distribution with mean $\mu = 16$ and variance $\sigma^2 = 63$. Use Chebvshev's inequality to find a lower bound for $\mathbb{P}(14 < \overline{X} < 18)$.

Solution: $\mathbb{P}(14 < \overline{X} < 18) = 1 - \mathbb{P}(|\overline{X} - 16| \geq 2) \geq 1 - \frac{\sigma^2/n}{2^2} = 1 - 0.63 = 0.37.$

11. Let $W_1 < W_2 < \cdots < W_6$ be the order statistics of n independent observations from a $U(0,1)$ distribution.
    (a) (5 points) Find the pdfs of W_1 and W_6.
    (b) (5 points) Find $\mathbb{E}[W_1]$ and $\mathbb{E}[W_6]$.

Solution:

(a) Let $F(t)$ be the cdf of the uniform random varialbe on $(0,1)$. Then, $F(t) = t$ for $t \in (0,1)$, $F(t) = 0$ for $t \leq 0$ and $F(t) = 1$ for $t \geq 1$. Let $f(t)$ be the pdf of the unifrom RV.

Since $f_{W_1}(t) = 6(1-F(t))^5 f(t)$, $f_{W_1}(t) = 6(1-t)^5$ for $t \in (0,1)$ and otherwise 0.

Since $f_{W_6}(t) = 6F(t)^5 f(t)$, $f_{W_1}(t) = 6t^5$ for $t \in (0,1)$ and otherwise 0.

(b) $\mathbb{E}[W_1] = 1 - 6/7 = 1/7$ and $\mathbb{E}[W_6] = 6/7$.

12. (6 points) A random sample of size 100 from the normal distribution $N(\mu, 25)$ yielded $\overline{X} = 35.6$. Find a two-sided 95% confidence interval for μ.

Solution: Since $z_{\alpha/2} = 1.96$, the confidence interval is $[35.6 - z_{\alpha/2} \cdot 5/\sqrt{100}, 35.6 + z_{\alpha/2} \cdot 5/\sqrt{100}] = [35.6 - 0.98, 35.6 + 0.98] = [34.62, 36.58]$.