# Chapter 3. Continuous Distribution

Math 3215 Spring 2024

Georgia Institute of Technology

## Section 1.

## Random Variables of the Continuous Type

For Discrete RV

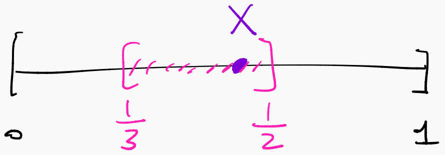
$$\mathbb{P}(X = x) = \frac{\overset{\text{"Count"}}{\#} \text{ of outcomes}}{\# \text{ of total outcomes}}$$

## Continuous Random Variables

Let the random variable $X$ denote the outcome when a point is selected at random from an interval $[0, 1]$.

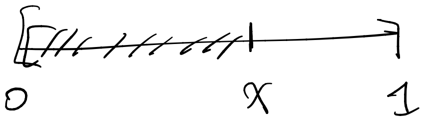
If the experiment is performed in a fair manner, it is reasonable to assume that the probability that the point is selected from an interval $[\frac{1}{3}, \frac{1}{2}]$ is

The CDF of $X$ is



$$\mathbb{P}(X \in [\tfrac{1}{3}, \tfrac{1}{2}]) = \frac{\text{length of } [\frac{1}{3}, \frac{1}{2}]}{\text{length of } [0, 1]} = \frac{1}{6}.$$

$$\text{CDF} = F(x) = \mathbb{P}(X \leq x) = \frac{\text{length of } [0, x]}{\text{length of } [0, 1]}$$



$$= x.$$

$$\mathbb{P}(X = 0.55) = 0.$$

## Continuous Random Variables

**Definition**

We say a random variable $X$ on a sample space $S$ is a continuous random variable if there exists a function $f(x)$ such that

- $f(x) \geq 0$ for all $x$,
- $\int_{S(X)} f(x)\,dx = 1$, and
- For any interval $(a, b) \subset \mathbb{R}$,

$$\mathbb{P}(a < X < b) = \int_a^b f(x)\,dx.$$

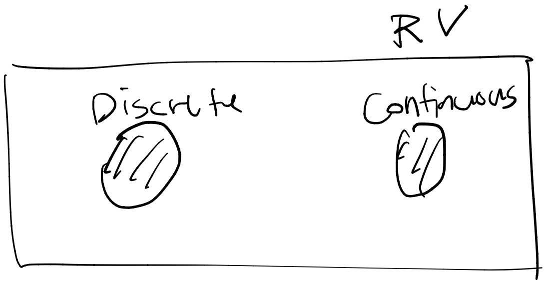
The function $f(x)$ is called **the probability density function (PDF)** of $X$.

(density)

Note • $\underbrace{\mathbb{P}(X=a)}_{\|} = 0$.

$$\left( \because \lim_{\varepsilon \to 0} \mathbb{P}(a-\varepsilon < X < a+\varepsilon) = \lim_{\varepsilon \to 0} \int_{a-\varepsilon}^{a+\varepsilon} f(x)\,dx = 0 \right)$$

•



X : conti. RV. $\Rightarrow$ we have a PDF $\underline{\underline{f(x)}}$

① $f(x) \geq 0$

② $\int_{-\infty}^{\infty} f(x)\,dx = 1$

③ $\int_{a}^{b} f(x)\,dx = P(X \in (a,b)) = P(a \leq X \leq b) = P(a < X < b)$

$= P(a < X \leq b) = P(a \leq X < b)$

## Continuous Random Variables

The CDF of $X$ is $F(x) = P(X \leq x) = P(X \in (-\infty, x]) = \int_{-\infty}^{x} f(t)\,dt$

The expectation (mean) of $X$ is $E[X] = \int_{-\infty}^{\infty} x f(x)\,dx = \mu$

The variance of $X$ is $Var(X) = E[(X-\mu)^2] = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx$

The standard deviation of $X$ is $Std(X) = \sqrt{Var(X)}$

The moment generating function of $X$ is

$$M(t) = E[e^{tX}] = \int_{-\infty}^{\infty} e^{tx} f(x)\,dx$$

Discrete Case : $E[u(X)] = \underline{\sum} u(x) \cdot f(x)$

Conti. Case : $E[u(X)] = \int_{-\infty}^{\infty} u(x) \cdot f(x)\,dx$.

$$\left\{ \begin{array}{l} e^{x},\ \sin x,\ \cos x,\ \ln x,\ \frac{1}{x},\ x^{n},\ ---- \\ \text{Change of variables, Integration by Parts.} \end{array} \right.$$

$$\underset{\text{PDF}}{\underline{f(x)}} = P(X=x)$$

$$P(a<X<b) = \int_a^b f(x)\,dx$$

## Continuous Random Variables

$$\underset{\text{PMF}}{} \text{PMF} = \overset{\swarrow \text{ a prob.}}{P(X=x)} \leq 1 \quad // \quad \overset{\swarrow \text{ Not a prob.}}{\text{PDF}}$$

**Properties**

① The PMF of a discrete random variable is bounded by 1. But for PDF, $f(x)$ can be greater than 1.

② For CDF $F$, we have $\boxed{F'(x) = f(x)}$ where $F$ is differentiable at $x$.

- What is PDF.

$$\lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} P(x-\varepsilon < X < x+\varepsilon) = \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_{x-\varepsilon}^{x+\varepsilon} f(t)\,dt$$

$$= f(x)$$

- $$\frac{d}{dx} F(x) = \frac{d}{dx} \int_{-\infty}^{x} f(t)\,dt \underset{\uparrow}{=} f(x)$$

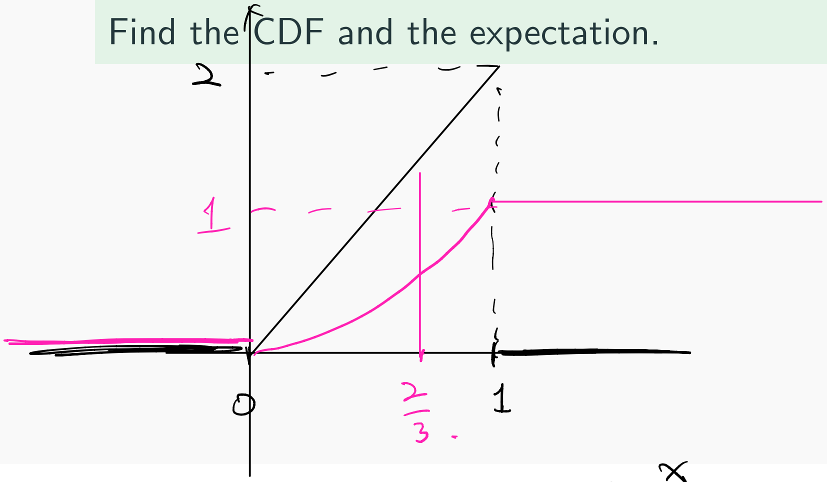
( Fundamental Thm of Calculus)

## Continuous Random Variables

**Example**

Let $X$ be a continuous random variable with a PDF $\overset{f}{g}(x) = 2x$ for $0 < x < 1$.

Find the CDF and the expectation.



$$f(x) = \begin{cases} 2x & , \ 0 < x < 1 \\ 0 & , \ o.w. \end{cases}$$

CDF : $F(x) = \int_{-\infty}^{x} f(t)\,dt = \begin{cases} 0 & , \ x \leq 0 \\ 1 & , \ x \geq 1 \\ x^2 & , \ 0 < x < 1 \end{cases}$

$$\int_0^x 2t\,dt = \left[t^2\right]_0^x$$
$$= x^2 - 0^2$$
$$= x^2$$

Exp : $E[X] = \int_{-\infty}^{\infty} x\,f(x)\,dx = \int_0^1 x \cdot (2x)\,dx = \left[\frac{2}{3}x^3\right]_0^1$

$$= \frac{2}{3}.$$

$$(u\cdot v)' = u'\cdot v + u\cdot v'$$

$$u'\cdot v = (u\cdot v)' - u\cdot v'$$

$$\int u'\cdot v = u\cdot v - \int u\cdot v' \quad \leftarrow \text{IBP (Integration by Parts)}$$

$$\begin{cases} v = x \\ u' = e^{-x} \end{cases} \qquad v' = 1 \qquad u = -e^{-x}$$

$$\int_0^\infty \boxed{x}\cdot e^{-x}\,dx = \lim_{N\to\infty}\Big[\underbrace{x(-e^{-x})}\Big]_0^N - \int_0^\infty 1\cdot(-e^{-x})\,dx$$

$$= \lim_{N\to\infty}\left(-\underbrace{N\cdot e^{-N}} - \underbrace{0\cdot(-e^{-0})}\right) + \int_0^\infty e^{-x}\,dx$$

$$= 0 + \left[-e^{-x}\right]_0^{\infty 0} = 1.$$

# Continuous Random Variables

$$f(x) = \begin{cases} x e^{-x}, & x > 0 \\ 0, & x \le 0. \end{cases}$$

for $x > 0$

**Example**

Let X have the PDF $f(x) = xe^{-x}$. Find the MGF.

Check if $f$ is a PDF $\left\langle \begin{array}{l} \cdot\; f(x) \ge 0 \\ \cdot\; \int_{-\infty}^{\infty} f(x)\,dx = 1 \end{array}\right.$

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_0^\infty x e^{-x}\,dx =$$

$$\text{MGF}: \quad \mathbb{E}[e^{tX}] = \int_{-\infty}^{\infty} e^{tx} f(x)\,dx$$

$$= \int_0^\infty e^{tx}\cdot x e^{-x}\,dx = \int_0^\infty x\cdot e^{-(1-t)\cdot x}\,dx$$

$$= \int_0^\infty \frac{u}{(1-t)} e^{-u} \frac{du}{(1-t)}$$

$$= \frac{1}{(1-t)^2}\int_0^\infty u e^{-u}\,du$$

$$= \begin{cases} \frac{1}{(1-t)^2}. & t < 1 \\ \text{DNF}. & t \ge 1 \end{cases}$$

①

$$\begin{cases} v = x \\ u' = \cancel{e^{-x}}\; e^{-(1-t)x} \end{cases} \qquad v' = 1 \qquad u = \underbrace{-\cancel{e^{-x}}}\; -\frac{1}{(1-t)} e^{-(1-t)x}$$

$\leftarrow$ IBP (Integration by Parts)

②

$$u = (1-t)x$$

$$du = (1-t)\cdot dx$$

# Uniform Random Variables

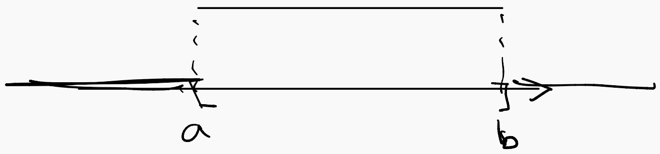
**Definition**

$X$ is a uniform random variable if its PDF is constant on its support.

If its support is $[a, b]$, then the PDF is

$$f(x) = \begin{cases} \frac{1}{b-a} & a \leq x \leq b \\ 0 \; , & x > b \text{ or } x < a \end{cases}$$

We denote by $X \sim U(a, b)$. $=$ Unif $(a, b)$



# Uniform Random Variables

$$f(x) = \begin{cases} \frac{1}{b-a} , & x \in [a,b] \\ 0 , & o.w \end{cases}$$

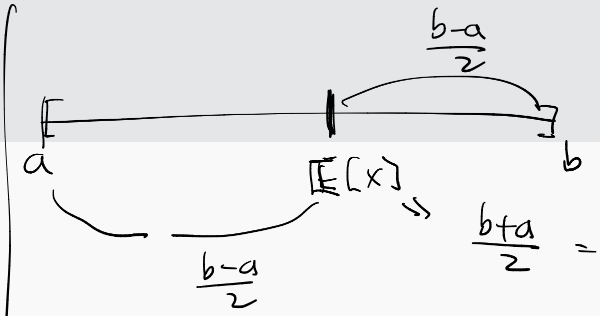
**Theorem**

If $X \sim U(a, b)$, then

$\mathbb{E}[X] = \frac{a+b}{2}$

$\text{Var}[X] = \frac{(b-a)^2}{12}$

$$M(t) = \begin{cases} \frac{e^{tb} - e^{ta}}{t(b-a)} , & t \neq 0 \\ 1 , & t = 0 \end{cases}$$



$\Rightarrow \frac{b+a}{2} = a + \frac{b-a}{2}$

$$\mathbb{E}[x] = \int_a^b x \cdot \underset{''f(x)}{\frac{1}{b-a}} \, dx = \frac{1}{(b-a)} \cdot \left[\frac{1}{2} x^2\right]_a^b$$

$$= \frac{1}{\cancel{b-a}} \frac{1}{2} \underline{(b^2 - a^2)} \rightarrow = (\cancel{b-a})(b+a) \qquad = \frac{1}{2}(a+b) = \mu$$

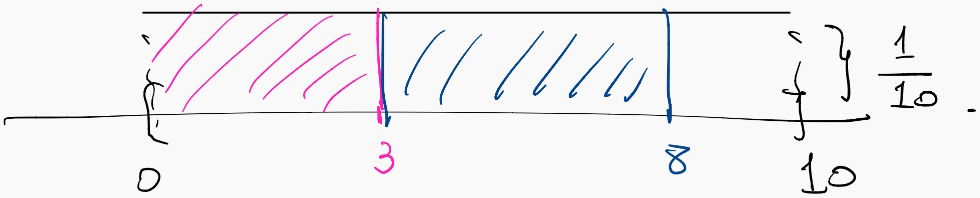
$$\text{Var}(x) = \int_a^b (x-\mu)^2 \cdot \frac{1}{b-a} \, dx = \frac{(b-a)^2}{12} .$$

## Uniform Random Variables

**Example**

If $X$ is uniformly distributed over $(0, 10)$, calculate $\mathbb{P}(X < 3)$, $\mathbb{P}(X > 6)$, and $\mathbb{P}(3 < X < 8)$.

$\mathbb{P}(3 < X < 8) = \frac{5}{10}$.



$\mathbb{P}(X < 3) = \frac{3}{10}$.

Recall

- X is conti. RV if it has a PDF.
- $f(x)$ is a PDF if $\begin{cases} f(x) \geq 0 \text{ for all } x, \\ \int_{\mathbb{R}} f(x)\,dx = 1 \\ P(a < X < b) = \int_a^b f(x)\,dx. \end{cases}$
- $X \sim U(a,b)$ $\quad f(x) = \begin{cases} \frac{1}{b-a} & a < x < b \\ 0 & \text{otherwise.} \end{cases}$

## Uniform Random Variables

**Example**

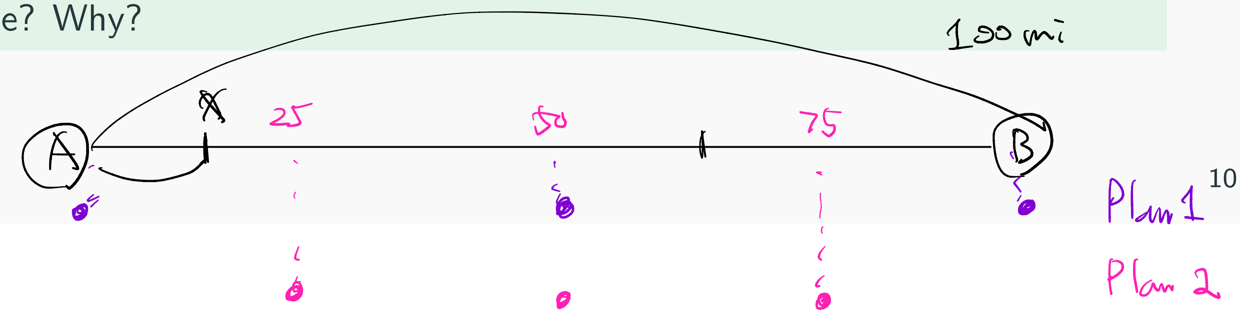
A bus travels between the two cities A and B, which are 100 miles apart.

If the bus has a breakdown, the distance from the breakdown to city A has a $U(0, 100)$ distribution.

There are bus service stations in city A, in B, and in the center of the route between A and B.

It is suggested that it would be more efficient to have the three stations located 25, 50, and 75 miles, respectively, from A.

Do you agree? Why?



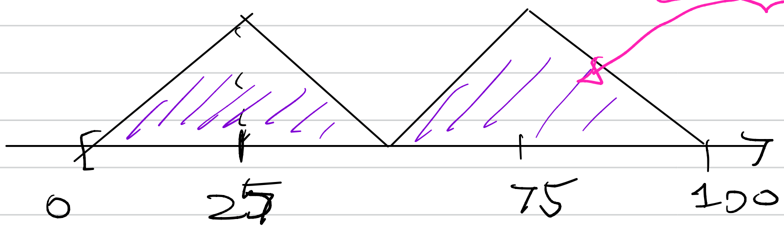
$X$ : breakdown point $\sim U(0,100)$

$u(X)$ : distance from $X$ to nearest Station.

$E[u(X)] \leftarrow$ minimize this

Plan 1 : $\quad u(X) = \begin{cases} x & 0 < X < 25 \\ |x-50| & 25 < x < 75 \\ 100 - x & 75 < x < 100 \end{cases}$

$$E[\underbrace{u(X)}_{\uparrow \text{ RV.}}] = \int u(x)\cdot f(x)\,dx = \frac{1}{100}\underbrace{\int_0^{100} u(x)\,dx}.$$

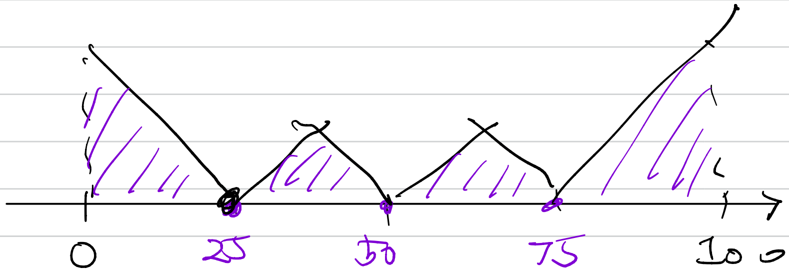


$$= \frac{1}{100}\cdot 50\cdot 25 = \frac{25}{2}.$$

Plan 2 :

↑

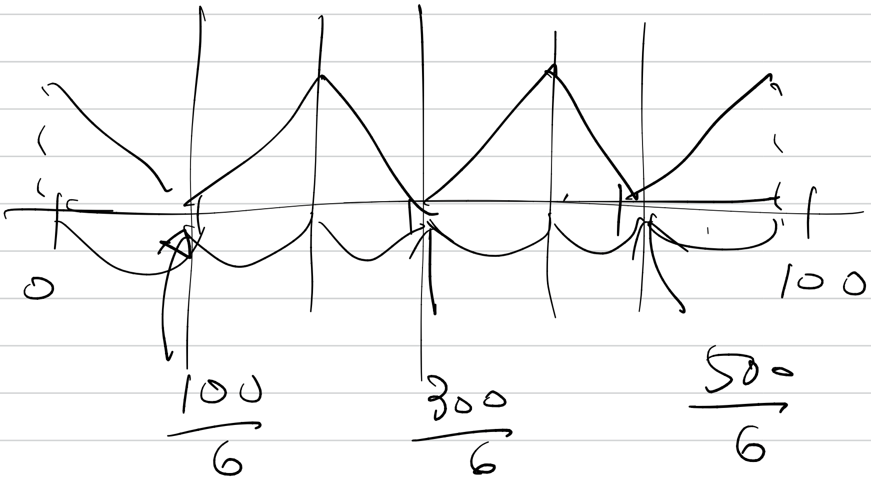
Better!



$$E[u(X)] = \left(25\cdot 25 + 25\cdot\frac{25}{2}\right)\frac{1}{100}$$

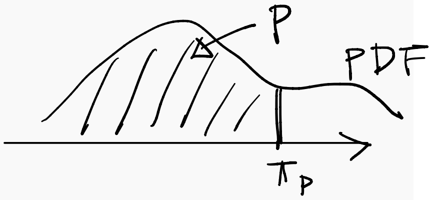
$$= 25\cdot \cancel{25}\cdot\left(1+\frac{1}{2}\right)\cdot\frac{1}{\cancel{100}\,4}$$

$$= 25\cdot\frac{3}{8}$$



# Percentile

$0 \leq p \leq 1 \quad 0 \leq 100p \leq 100.$



CDF ↓

**The (100$p$)-th percentile** is a number $\pi_p$ such that $F(\pi_p) = p$.

For example, the 50th percentile is the number $\pi_{\frac{1}{2}} = q_2$ such that $F(\pi_{\frac{1}{2}}) = \frac{1}{2}$ and this is called the median.

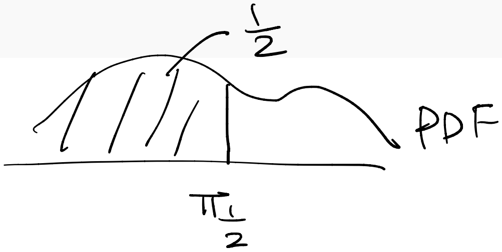
The 25th and 75th percentiles are called the first and third quartiles, respectively, and are denoted by $q_1 = \pi_{0.25}$ and $q_3 = \pi_{0.75}$.

Ex.

$= q_2$

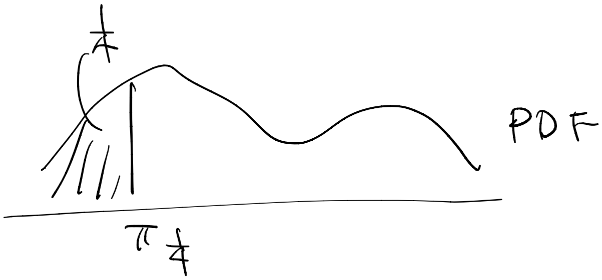
$= 2^{nd}$ quartile

$50^{th}$ percentile $= (100 \cdot \frac{1}{2})^{th}$ percentile $=$ median



$25^{th}$ percentile $= (100 \cdot \frac{1}{4})^{th}$ percentile $= 1^{st}$ quartile

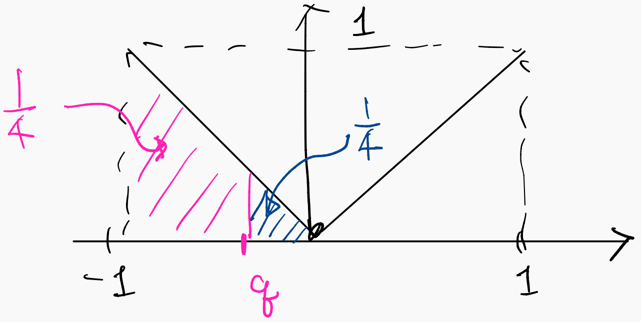
$= q_1$

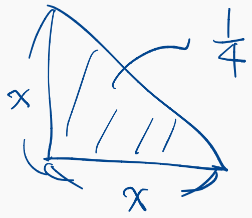


# Percentile

**Example**

Let $X$ be a continuous random variable with PDF $f(x) = |x|$ for $-1 < x < 1$. Find $q_1, q_2, q_3$.





$\frac{1}{2} \cdot x^2 = \frac{1}{4}$

$x = \frac{1}{\sqrt{2}}$

$q_1 = -\frac{1}{\sqrt{2}}$.

$q_2 = 0$

$q_3 = \frac{1}{\sqrt{2}}$ } by symmetry.

## Exercise

Let $f(x) = c\sqrt{x}$ for $0 \leq x \leq 4$ be the PDF of a random variable $X$.

Find $c$, the CDF of $X$, and $\mathbb{E}[X]$.

# Section 2.

# The Exponential, Gamma, and Chi-Square Distributions

$X$ = # of customers in $[0, 1]$

Exp. # $= \lambda$ in 1hr

$W$ = waiting time for 1st customer

Find CDF of $W$: $F(t) = \mathbb{P}(W \leq t) = 1 - \mathbb{P}(\underbrace{W > t}_{\text{no customer in } [0,t]})$

## Exponential random variables

Consider a Poisson random variable $X$ with parameter $\lambda$.

This represents the number of occurrances in a given interval, say $[0, 1]$.

If $\lambda = 5$, that means the expected number of occurrances in $[0, 1]$ is 5.

Let $W$ be the waiting time for the first occurrence. Then,

$$\mathbb{P}(W > t) = \mathbb{P}(\underline{\text{no occurrences in } [0, t]}) = \mathbb{P}(X_t = 0)$$

for $t > 0$.

$X_t$ : # of customers in $[0,t]$

$\sim \text{Pois}(\lambda t)$

$$= e^{-\lambda t} \cdot \frac{(\lambda t)^0}{0!} = e^{-\lambda t}$$

$F(t) = 1 - \mathbb{P}(W > t) = 1 - e^{-\lambda t}$

$F'(t) = \lambda e^{-\lambda t} = f(t) \qquad t \geq 0$

# Exponential random variables

Exp # of customer

Exp. waiting Tim

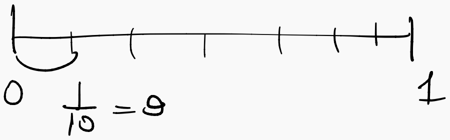
**Definition**

We say $X$ is **an exponential random variable** with parameter $\lambda$ (or mean $\theta$ where $\lambda = \frac{1}{\theta}$) if its pdf is

$$f(x) = \lambda e^{-\lambda x} = \frac{1}{\theta} e^{-\frac{x}{\theta}},$$

for $\boxed{x \geq 0}$ and otherwise 0. Here, $\lambda$ is the parameter and $\theta$ is the mean.

$\lambda = 10$



# Exponential random variables

**Theorem**

Suppose that $X$ is an exponential random variable with parameter $\lambda = \frac{1}{\theta}$.

$\mathbb{E}[X] = \frac{1}{\lambda} = \theta$

$\mathrm{Var}[X] = \frac{1}{\lambda^2} = \theta^2$ Exercise

$M(t) = \frac{\lambda}{\lambda - t} = \frac{1}{1-\theta t}$ $\quad t < \lambda \quad$ or $\quad t < \frac{1}{\theta}$

$$\mathbb{E}[X] = \int_{\mathbb{R}} x f(x)\,dx = \int_0^\infty x \cdot \lambda e^{-\lambda x}\,dx = \int_0^\infty (\lambda x) \cdot e^{-\lambda x}\,dx$$

$u = \lambda x$

$du = \lambda dx$

$$= \frac{1}{\lambda}\int_0^\infty \underset{1}{\underset{\downarrow}{u}} \cdot \overset{-e^{-u}}{\overset{\uparrow}{e^{-u}}}\,du$$

$$= \frac{1}{\lambda}\left[ \underbrace{\left[u \cdot (-e^{-u})\right]_0^\infty - \int_0^\infty 1 \cdot (-e^{-u})\,du}_{=1} \right]$$

$$= \frac{1}{\lambda}.$$

# Exponential random variables

$$f(x) = \frac{1}{\theta} e^{-\frac{x}{\theta}} = \lambda e^{-\lambda x}, \quad x \geq 0.$$

**Example**

Let $X$ have an exponential distribution with a mean $\theta = 20$.

Find $\mathbb{P}(X < 18)$.

$$F(18) = P(X < 18) = \int_0^{18} \frac{1}{20} \cdot e^{-\frac{x}{20}} dx = \left[ - e^{-\frac{x}{20}} \right]_0^{18}$$

$$= 1 - e^{-\lambda \cdot 18} \qquad = 1 - e^{-\frac{18}{20}}.$$

$$F(t) = 1 - e^{-\lambda t}$$

$$P(X > t) = e^{-\lambda t}.$$

# Exponential random variables

**Example**

Customers arrive in a certain shop according to an approximate Poison process at a mean rate of 20 per hour.

What is the probability that the shopkeeper will have to wait more than five minutes for the arrival of the first customer?

$W$ = waiting time (hr) $\sim$ Exp $\lambda = 20$.

( " (min) $\sim$ Exp $\lambda = \frac{1}{3}$ )

$$\mathbb{P}(\text{more than } 5 \text{ min}) = \mathbb{P}\left(W > \frac{5}{60}\right) = e^{-20 \cdot \frac{5}{60}} = e^{-\frac{5}{3}}$$

$$= \int_{\frac{5}{60}}^{\infty} 20 \cdot e^{-20x}\, dx$$

Bin : # of success in $n$ trials

Geom : # of trials until $1^{st}$ success.

Neg Bin : # of trials until $r^{th}$ success

---

Poisson : # of customers in $[0, 1]$

Exp : Waiting time for $1^{st}$ customer

??

## Gamma random variables

Consider a Poisson random variable $X$ with $\lambda$.

$W \sim \text{Gamma}(\lambda, \alpha)$

Let $W$ be the waiting time until $\alpha$-th occurrences, then its CDF is

$$F(t) = \mathbb{P}(W \le t) = 1 - \mathbb{P}(W > t) = 1 - \sum_{k=0}^{\alpha-1} \frac{(\lambda t)^k e^{-\lambda t}}{k!}.$$

↑ PMF of Poisson.

Thus, the PDF is

$$F'(x) = f(x) = \frac{\lambda(\lambda x)^{\alpha-1}}{(\alpha-1)!} e^{-\lambda x}. \quad , \text{ for } x \geq 0.$$

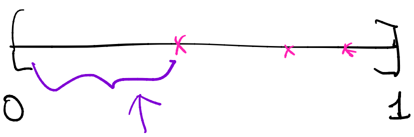
This random variable is called **a gamma random variable** with $\lambda$ and $\alpha$ where $\lambda = \frac{1}{\theta} > 0$.

This can be extended to non-integer $\alpha > 0$.

Ex. $\alpha = \frac{1}{2}$

Q: What is $(\frac{1}{2})!$, $(\frac{7}{3})!$, $\pi!$, ....

Exponential RV



\# of customers $= X \sim \text{Pois}(\lambda)$

$W$ = Waiting time for 1st customer $\sim \text{Exp}(\lambda)$

$$f_W(t) = \lambda e^{-\lambda t} \quad , \quad t \geq 0$$

$$\begin{cases} E[X] = \frac{1}{\lambda} = \theta \\ \text{Var}(X) = \frac{1}{\lambda^2} = \theta^2 \\ P(W > t) = e^{-\lambda t} \end{cases}$$

$W$ = Waiting time for 3rd ($\alpha$th) customer $\sim \text{Gamma}(\lambda, \alpha)$

rate parameter

shape parameter

## Gamma functions

The gamma function is defined by

$$\Gamma(t) = \int_0^\infty y^{t-1} e^{-y} \, dy$$

for $t > 0$.

By integration by parts, we have

$$\Gamma(t) = \int_0^\infty \underset{(t-1)\,y^{t-2}}{y^{t-1}} \; \overset{(-e^{-y})}{e^{-y}} \, dy = \left[ y^{t-1} \cdot (-e^{-y}) \right]_0^\infty - \int_0^\infty (t-1) y^{t-2} (-e^{-y}) \, dy$$

$$= \lim_{N \to \infty} \left[ y^{t-1} (-e^{-y}) \right]_0^N + (t-1) \boxed{\int_0^\infty y^{(t-1)-1} e^{-y} \, dy}$$

$= \Gamma(t-1)$

$$= 0 + (t-1) \cdot \Gamma(t-1)$$

$$\boxed{\Gamma(t) = (t-1)\,\Gamma(t-1)}$$

## Gamma functions

In particular, $\Gamma(1) = \int_0^\infty y^{1-1} e^{-y} dy = \int_0^\infty e^{-y} dy = [-e^{-y}]_0^\infty = 1$.

$\Gamma(2) = 1 \cdot \Gamma(1) = 1$

$\Gamma(3) = 2 \cdot \Gamma(2) = 2 \cdot 1 = 2$ , $\Gamma(4) = 3 \cdot \Gamma(3) = 3 \cdot 2 \cdot 1 = 6$

$\Gamma(n) = (n-1)\Gamma(n-1) = (n-1)(n-2)\Gamma(n-2) = \cdots = (n-1)(n-2)\cdots 1$

$= (n-1)!$

for integers $n$.

$$\Gamma(\underset{\substack{\uparrow \\ \text{could be non-integer number}}}{t}) = \int_0^\infty y^{t-1} e^{-y} dy$$

= Generalized Factorial.

Ex

$$\Gamma\left(\tfrac{1}{2}\right) = \text{“}\left(-\tfrac{1}{2}\right)!\text{”} = \int_0^\infty y^{\frac{1}{2}-1} e^{-y} dy = \int_0^\infty \frac{e^{-y}}{\sqrt{y}} dy$$

$$\left(\begin{array}{l} y = z^2 \\ z = \sqrt{y} \\ dz = \frac{1}{2} \cdot \frac{1}{\sqrt{y}} dy \end{array}\right) \qquad = 2\boxed{\int_0^\infty e^{-z^2} dz} = A$$

polar coordinate ↙

$$A^2 = \int_0^\infty e^{-z^2} dz \int_0^\infty e^{-w^2} dw = \int_0^\infty\int_0^\infty e^{-(z^2+w^2)} dz\, dw$$

## Gamma random variables

Expectation of Exp. RV

**Theorem**

$\mathbb{E}[X] = \frac{\alpha}{\lambda}$ $= \alpha \cdot \boxed{\frac{1}{\lambda}}$

$\lambda = \frac{1}{\theta}$.

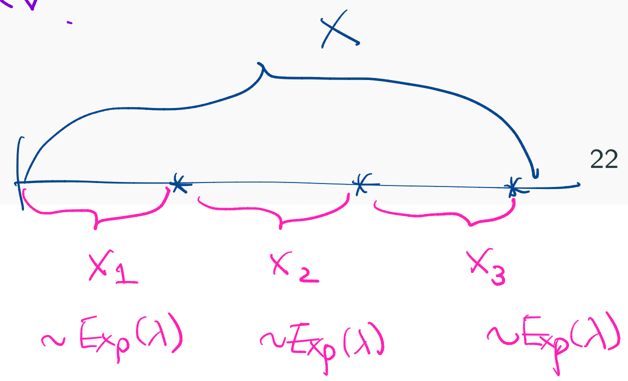
Variance of Exp. RV

$\text{Var}[X] = \frac{\alpha}{\lambda^2}$ $= \alpha \cdot \boxed{\frac{1}{\lambda^2}}$

$M(t) = \frac{1}{(1-\theta t)^\alpha}$ for $t \leq \frac{1}{\theta}$.

$= \left(\frac{1}{1-\theta t}\right)^\alpha$ MGF of Exp. RV.

$X \sim \text{Gamma}(\lambda, 3)$



$X = X_1 + X_2 + X_3$

$=$ Indep. Sum of $\text{Exp}(\lambda)$

# Gamma random variables

**Example**

Suppose the number of customers per hour arriving at a shop follows a Poisson random variable with mean 20.

That is, if a minute is our unit, then $\lambda = \frac{1}{3}$.

What is the probability that the second customer arrives more than five minutes after the shop opens for the day?

$$W = \text{waiting time for } 2^{nd} \text{ customer} \sim \text{Gamma}\left(\frac{1}{3}, 2\right)$$

$$f(t) = \frac{\lambda^{\alpha} t^{\alpha-1}}{(\alpha-1)!} e^{-\lambda t} = \left(\frac{1}{3}\right)^2 t e^{-\frac{t}{3}}$$

$$P(W > 5) = \int_5^{\infty} \frac{1}{9} \underset{\downarrow 1}{t} \, \underset{\searrow -3e^{-\frac{t}{3}}}{e^{-\frac{t}{3}}} dt$$

$$= \frac{1}{9} \cdot \left( \left[ t \cdot (-3e^{-\frac{t}{3}}) \right]_5^{\infty} - \int_5^{\infty} (-3e^{-\frac{t}{3}}) \, dt \right)$$

$$= \frac{1}{9} \left( 5 \cdot 3 \cdot e^{-\frac{5}{3}} + 3 \left[ -3 e^{-\frac{t}{3}} \right]_5^{\infty} \right)$$

$$= \frac{1}{9} \left( 15 e^{-\frac{5}{3}} + 9 e^{-\frac{5}{3}} \right) = \frac{24}{9} e^{-\frac{5}{3}}.$$

$X = \#$ of customer $\sim \text{Pois}(\frac{5}{3})$

$$\mathbb{P}(W > 5) = \mathbb{P}(\# \text{ of customer in } [0,5] = 0, 1)$$
$$= \mathbb{P}(X=0) + \mathbb{P}(X=1)$$
$$= e^{-\frac{5}{3}} \cdot \frac{(\frac{5}{3})^0}{0!} + e^{-\frac{5}{3}} \frac{(\frac{5}{3})^1}{1!}$$
$$= e^{-\frac{5}{3}} \cdot \left(1 + \frac{5}{3}\right)$$
$$= \frac{8}{3} \cdot e^{-\frac{5}{3}}.$$

$$\mathbb{P}(X = k) = e^{-\lambda} \frac{\lambda^k}{k!}$$

---

In general, $W \sim \text{Gamma}(\lambda, n)$

$X_t \sim \text{Poisson}(\lambda \cdot t)$

$$\mathbb{P}(W > t) = \mathbb{P}(X_t < n)$$

$$\boxed{\text{"Similar" property for Bin, NegBin}}$$ //

# Chi-square distribution

$\left(\lambda = \frac{1}{2}\right)$

Let $X$ have a gamma distribution with $\theta = 2$ and $\alpha = r/2$, where $r$ is a positive integer.

The pdf of $X$ is

$$f(x) = \frac{1}{\Gamma(\frac{r}{2})2^{\frac{r}{2}}} x^{\frac{r}{2}-1} e^{-\frac{x}{2}}$$

for $x > 0$.

We say that $X$ has **a chi-square distribution** with $r$ degrees of freedom and we use the notation $X \sim \chi^2(r)$.

$$\mathbb{P}(X > t) = e^{-\frac{t}{\theta}}.$$

## Exercise

$$f_X(t) = \frac{1}{\theta} e^{-\frac{t}{\theta}}, \quad t \geqslant 0.$$

Let $X$ have an exponential distribution with mean $\theta$.

Compute $\underbrace{\mathbb{P}(X > 15|X > 10)}$ and $\mathbb{P}(X > 5)$. $= e^{-\frac{5}{\theta}}$

$$= \frac{\mathbb{P}(X > 15)}{\mathbb{P}(X > 10)} = \frac{e^{-\frac{15}{\theta}}}{e^{-\frac{10}{\theta}}} = e^{-\frac{15}{\theta} + \frac{10}{\theta}} = e^{-\frac{5}{\theta}}$$

⟨Memorylessness Property⟩

$$\mathbb{P}(X > t+s \mid X > s) = \mathbb{P}(X > t)$$

## Section 3.
## The Normal Distribution

# Gaussian random variables

**Definition**

We say $X$ is **a Gaussian random variable** or has **a normal distribution** if its PDF is given by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right).$$

Here $\mu$ is the mean and $\sigma$ is the standard deviation. We use the notation $X \sim N(\underset{\substack{\uparrow\\ \text{mean}}}{\mu}, \underset{\substack{\nwarrow\\ \text{variance.}}}{\sigma^2})$.

# Gaussian random variables

$$f(x) = \frac{1}{\sqrt{2\pi}\cdot\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad -\infty < x < \infty$$

**Theorem**

$\int_{\mathbb{R}} f(x)\,dx = 1 \longrightarrow (✱)$

$\mathbb{E}[X] = \mu$

$\text{Var}[X] = \sigma^2$

$M(t) = \exp\left(\mu t + \frac{\sigma^2 t^2}{2}\right)$

To show (✱),

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\cdot\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx \quad \left(\overset{\text{"Goal"}}{=} 1\right)$$

$z = \frac{x-\mu}{\sigma}$

$dz = \frac{1}{\sigma}\cdot dx \longrightarrow$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz.$$

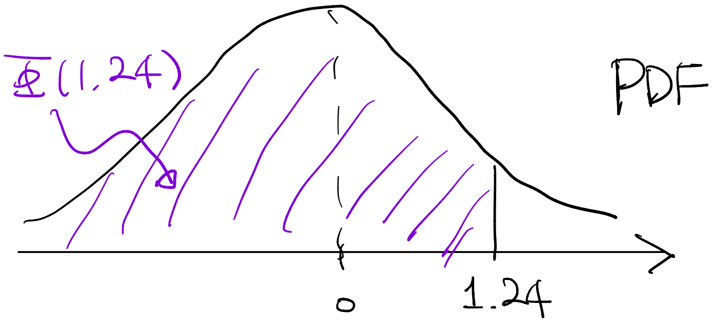
$$= \frac{2}{\sqrt{2\pi}} \underbrace{\int_0^{\infty} e^{-\frac{z^2}{2}} dz}_{\text{related to } \Gamma(\frac{1}{2})}$$

$X \sim N(\mu, \sigma^2)$ $\qquad f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-\frac{|x-\mu|^2}{2\sigma^2}}, \quad -\infty < x < \infty$

$\mathbb{E}[X] = \mu \qquad Var(X) = \sigma^2$

$Z \sim N(0, 1)$ $\quad (\mu = 0, \ \sigma^2 = 1)$ Standard Normal.

$\Phi(x) = \underbrace{\mathbb{P}(Z \leqslant x)}$ CDF of $Z$



## Standard normal distribution

In particular, if $\mu = 0$ and $\sigma = 1$, then $Z \sim N(0, 1)$ is called **the standard normal random variable**.

**Example**

Let $Z$ ~~is~~ be $N(0, 1)$.

Find $\mathbb{P}(Z \leq 1.24)$, $\mathbb{P}(1.24 \leq Z \leq 2.37)$, and $\mathbb{P}(-2.37 \leq Z \leq -1.24)$.

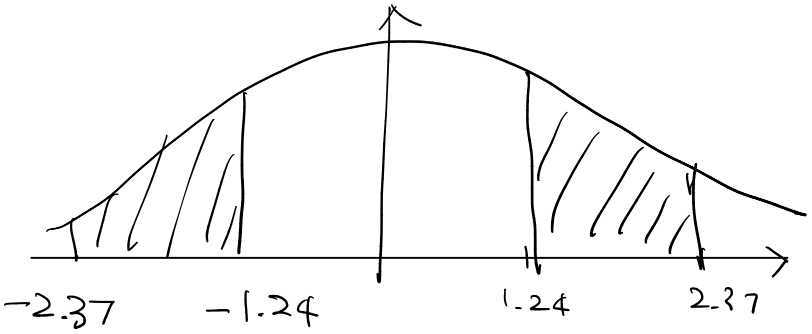
$$\Phi(1.24) = \mathbb{P}(Z \leqslant 1.24) = \int_{-\infty}^{1.24} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz$$

$$\mathbb{P}(1.24 \leqslant Z \leqslant 2.37) = \mathbb{P}(Z \leqslant 2.37) - \mathbb{P}(Z \leqslant 1.24)$$

$$= \Phi(2.37) - \Phi(1.24)$$

(by symm.) $\parallel$

$$\mathbb{P}(-2.37 \leqslant Z \leqslant -1.24)$$



## Standard normal distribution

**Theorem**

If $X \sim N(\mu, \sigma^2)$, then $Z = \frac{X-\mu}{\sigma}$ is the standard normal.

① $X$ : Normal $\Rightarrow$ $aX + b$ : Normal

② $Z$ : normal $Z = \left(\frac{1}{\sigma}\right) \cdot X + \left(-\frac{\mu}{\sigma}\right)$

$$\mathbb{E}[Z] = \mathbb{E}\left[\frac{X-\mu}{\sigma}\right]$$

$$= \frac{1}{\sigma}\mathbb{E}[X-\mu] = \frac{1}{\sigma}\left(\mathbb{E}[X] - \mu\right) = 0$$

$$\mathrm{Var}(Z) = \mathrm{Var}\left(\frac{X-\mu}{\sigma}\right) = \frac{1}{\sigma^2}\mathrm{Var}(X-\mu) = \frac{1}{\sigma^2}\mathrm{Var}(X)$$

$$= 1.$$

## Standard normal distribution

**Example**

Let $X \sim N(3, 16)$.

Find $\mathbb{P}(4 \leq X \leq 8)$, $\mathbb{P}(0 \leq X \leq 5)$, and $\mathbb{P}(-2 \leq X \leq 1)$.

$\mu = 3$ , $\sigma^2 = 16$ , $\sigma = 4$ $\qquad Z = \dfrac{X-3}{4} \sim N(0,1)$

$$\mathbb{P}(4 \leqslant X \leqslant 8) = \mathbb{P}\left(\frac{4-3}{4} \leqslant Z \leqslant \frac{8-3}{4}\right)$$

$$= \mathbb{P}(0.25 \leqslant Z \leqslant 1.25)$$

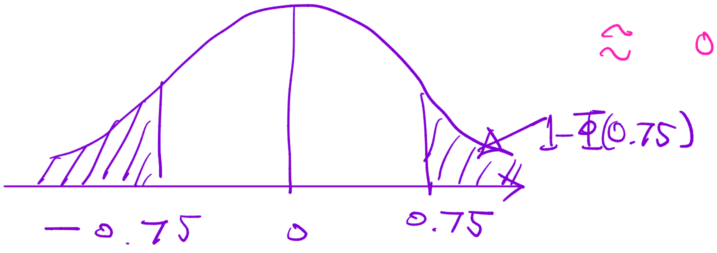
$$= \Phi(1.25) - \Phi(0.25)$$

$$\approx 0.8944 - 0.5987 .$$

$$\mathbb{P}(0 \leqslant x \leqslant 5) = \mathbb{P}\left(\frac{0-3}{4} \leqslant Z \leqslant \frac{5-3}{4}\right)$$

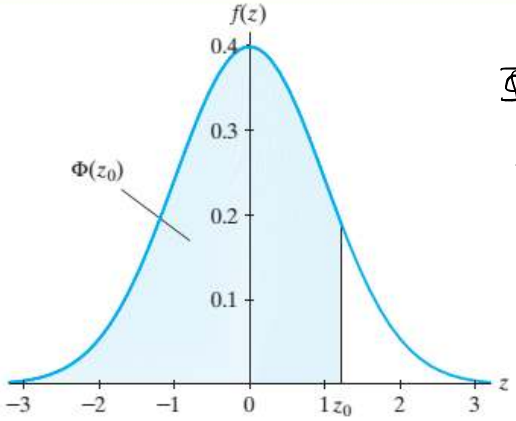
$$= \Phi(0.5) - \underline{\Phi(-0.75)} = 1 - \Phi(0.75)$$

$$\approx 0.6915 - (1 - 7734)$$



In general, $\Phi(-z) = 1 - \Phi(z)$.

**Table Va** The Standard Normal Distribution Function



$\Phi(z) = \mathbb{P}(Z \leq z)$

$\mathbb{P}(Z \leq 1.24) = \Phi(1.24)$

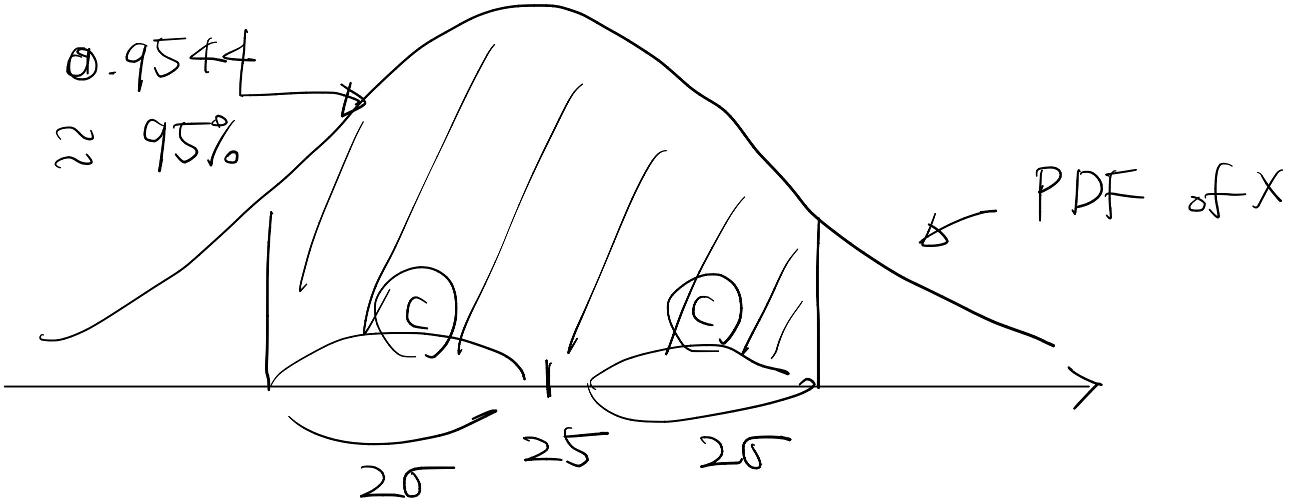
$\approx 0.8925$.

$$P(Z \le z) = \Phi(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-w^2/2}\, dw$$

$$\Phi(-z) = 1 - \Phi(z)$$

| $z$ | 0.00 | 0.01 | 0.02 | 0.03 | 0.04 | 0.05 | 0.06 | 0.07 | 0.08 | 0.09 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0.0 | 0.5000 | 0.5040 | 0.5080 | 0.5120 | 0.5160 | 0.5199 | 0.5239 | 0.5279 | 0.5319 | 0.5359 |
| 0.1 | 0.5398 | 0.5438 | 0.5478 | 0.5517 | 0.5557 | 0.5596 | 0.5636 | 0.5675 | 0.5714 | 0.5753 |
| 0.2 | 0.5793 | 0.5832 | 0.5871 | 0.5910 | 0.5948 | 0.5987 | 0.6026 | 0.6064 | 0.6103 | 0.6141 |
| 0.3 | 0.6179 | 0.6217 | 0.6255 | 0.6293 | 0.6331 | 0.6368 | 0.6406 | 0.6443 | 0.6480 | 0.6517 |
| 0.4 | 0.6554 | 0.6591 | 0.6628 | 0.6664 | 0.6700 | 0.6736 | 0.6772 | 0.6808 | 0.6844 | 0.6879 |
| 0.5 | 0.6915 | 0.6950 | 0.6985 | 0.7019 | 0.7054 | 0.7088 | 0.7123 | 0.7157 | 0.7190 | 0.7224 |
| 0.6 | 0.7257 | 0.7291 | 0.7324 | 0.7357 | 0.7389 | 0.7422 | 0.7454 | 0.7486 | 0.7517 | 0.7549 |
| 0.7 | 0.7580 | 0.7611 | 0.7642 | 0.7673 | 0.7703 | 0.7734 | 0.7764 | 0.7794 | 0.7823 | 0.7852 |
| 0.8 | 0.7881 | 0.7910 | 0.7939 | 0.7967 | 0.7995 | 0.8023 | 0.8051 | 0.8078 | 0.8106 | 0.8133 |
| 0.9 | 0.8159 | 0.8186 | 0.8212 | 0.8238 | 0.8264 | 0.8289 | 0.8315 | 0.8340 | 0.8365 | 0.8389 |
| 1.0 | 0.8413 | 0.8438 | 0.8461 | 0.8485 | 0.8508 | 0.8531 | 0.8554 | 0.8577 | 0.8599 | 0.8621 |
| 1.1 | 0.8643 | 0.8665 | 0.8686 | 0.8708 | 0.8729 | 0.8749 | 0.8770 | 0.8790 | 0.8810 | 0.8830 |
| 1.2 | 0.8849 | 0.8869 | 0.8888 | 0.8907 | 0.8925 | 0.8944 | 0.8962 | 0.8980 | 0.8997 | 0.9015 |
| 1.3 | 0.9032 | 0.9049 | 0.9066 | 0.9082 | 0.9099 | 0.9115 | 0.9131 | 0.9147 | 0.9162 | 0.9177 |
| 1.4 | 0.9192 | 0.9207 | 0.9222 | 0.9236 | 0.9251 | 0.9265 | 0.9279 | 0.9292 | 0.9306 | 0.9319 |
| 1.5 | 0.9332 | 0.9345 | 0.9357 | 0.9370 | 0.9382 | 0.9394 | 0.9406 | 0.9418 | 0.9429 | 0.9441 |
| 1.6 | 0.9452 | 0.9463 | 0.9474 | 0.9484 | 0.9495 | 0.9505 | 0.9515 | 0.9525 | 0.9535 | 0.9545 |
| 1.7 | 0.9554 | 0.9564 | 0.9573 | 0.9582 | 0.9591 | 0.9599 | 0.9608 | 0.9616 | 0.9625 | 0.9633 |
| 1.8 | 0.9641 | 0.9649 | 0.9656 | 0.9664 | 0.9671 | 0.9678 | 0.9686 | 0.9693 | 0.9699 | 0.9706 |
| 1.9 | 0.9713 | 0.9719 | 0.9726 | 0.9732 | 0.9738 | 0.9744 | 0.9750 | 0.9756 | 0.9761 | 0.9767 |
| 2.0 | 0.9772 | 0.9778 | 0.9783 | 0.9788 | 0.9793 | 0.9798 | 0.9803 | 0.9808 | 0.9812 | 0.9817 |
| 2.1 | 0.9821 | 0.9826 | 0.9830 | 0.9834 | 0.9838 | 0.9842 | 0.9846 | 0.9850 | 0.9854 | 0.9857 |
| 2.2 | 0.9861 | 0.9864 | 0.9868 | 0.9871 | 0.9875 | 0.9878 | 0.9881 | 0.9884 | 0.9887 | 0.9890 |
| 2.3 | 0.9893 | 0.9896 | 0.9898 | 0.9901 | 0.9904 | 0.9906 | 0.9909 | 0.9911 | 0.9913 | 0.9916 |
| 2.4 | 0.9918 | 0.9920 | 0.9922 | 0.9925 | 0.9927 | 0.9929 | 0.9931 | 0.9932 | 0.9934 | 0.9936 |
| 2.5 | 0.9938 | 0.9940 | 0.9941 | 0.9943 | 0.9945 | 0.9946 | 0.9948 | 0.9949 | 0.9951 | 0.9952 |
| 2.6 | 0.9953 | 0.9955 | 0.9956 | 0.9957 | 0.9959 | 0.9960 | 0.9961 | 0.9962 | 0.9963 | 0.9964 |
| 2.7 | 0.9965 | 0.9966 | 0.9967 | 0.9968 | 0.9969 | 0.9970 | 0.9971 | 0.9972 | 0.9973 | 0.9974 |
| 2.8 | 0.9974 | 0.9975 | 0.9976 | 0.9977 | 0.9977 | 0.9978 | 0.9979 | 0.9979 | 0.9980 | 0.9981 |
| 2.9 | 0.9981 | 0.9982 | 0.9982 | 0.9983 | 0.9984 | 0.9984 | 0.9985 | 0.9985 | 0.9986 | 0.9986 |
| 3.0 | 0.9987 | 0.9987 | 0.9987 | 0.9988 | 0.9988 | 0.9989 | 0.9989 | 0.9989 | 0.9990 | 0.9990 |

| $\alpha$ | 0.400 | 0.300 | 0.200 | 0.100 | 0.050 | 0.025 | 0.020 | 0.010 | 0.005 | 0.001 |
|---|---|---|---|---|---|---|---|---|---|---|
| $z_\alpha$ | 0.253 | 0.524 | 0.842 | 1.282 | 1.645 | 1.960 | 2.054 | 2.326 | 2.576 | 3.090 |
| $z_{\alpha/2}$ | 0.842 | 1.036 | 1.282 | 1.645 | 1.960 | 2.240 | 2.326 | 2.576 | 2.807 | 3.291 |



## Standard normal distribution

**Example**

Let $X \sim N(25, 36)$.

Find a constant $c$ such that $\mathbb{P}(|X - 25| \leq c) = 0.9544$.

$\mu = 25$, $\sigma^2 = 36$, $\sigma = 6$, $Z = \frac{X-25}{6} \sim N(0,1)$

$$\mathbb{P}\left( \frac{|X-25|}{6} \leqslant \frac{c}{6} \right) = \mathbb{P}\left( |Z| \leqslant \frac{c}{6} \right)$$

$$= \mathbb{P}\left( -\frac{c}{6} \leqslant Z \leqslant \frac{c}{6} \right)$$

$$= \Phi\left(\frac{c}{6}\right) - \Phi\left(-\frac{c}{6}\right)$$

$$= \Phi\left(\frac{c}{6}\right) - \left(1 - \Phi\left(\frac{c}{6}\right)\right)$$

$$= 2 \cdot \Phi\left(\frac{c}{6}\right) - 1 = 0.9544$$

$$2 \cdot \Phi\left(\frac{c}{6}\right) = 1.9544$$

$$\Phi\left(\frac{c}{6}\right) = 0.9772 = \Phi(2) \quad \therefore \frac{c}{6} = 2$$

$$c = 12.$$

## Standard normal distribution

$e^Z$

**Theorem**

If $Z$ is the standard normal, then $Z^2$ is $\chi^2(1)$. $= \text{Gamma}\left(\frac{1}{2}, \frac{1}{2}\right)$

$X = Z^2 \leftarrow$ A New RV.

Q: How to find the distribution of $X$?

CDF $\Rightarrow$

$$F_X(x) = P(Z^2 \le x) = P(-\sqrt{x} \le Z \le \sqrt{x})$$

$$= \Phi(\sqrt{x}) - \Phi(-\sqrt{x})$$

$$= 2 \cdot \Phi(\sqrt{x}) - 1$$

$$f_X(x) = \frac{d}{dx} F_X(x) = 2 \cdot \Phi'(\sqrt{x}) \cdot \frac{1}{2\sqrt{x}}$$

$$= \frac{1}{\sqrt{x}} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{(\sqrt{x})^2}{2}} = \frac{1}{\sqrt{x}} \cdot \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{x}{2}}$$

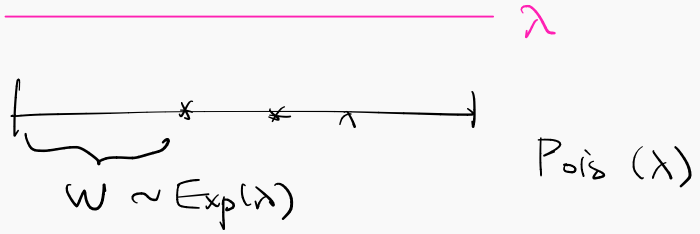
# Section 4.
# Additional Models

## Weibull distribution

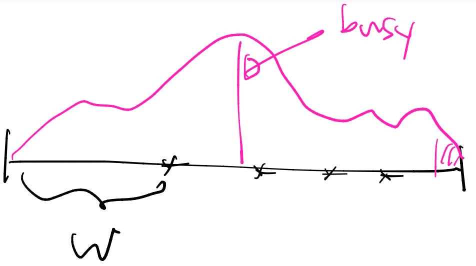
Recall the postulates of an approximate Poisson:

- The numbers of occurrences in nonoverlapping subintervals are independent.
- The probability of two or more occurrences in a sufficiently short subinterval is essentially zero.
- The probability of exactly one occurrence in a sufficiently short subinterval of length $h$ is approximately $\lambda h$.

$$\mathbb{P}(W > t) = e^{-\lambda t} \quad \Longleftarrow$$



$$\mathbb{P}(W > t) = e^{-\int_0^t \lambda(w)\, dw}$$



## Weibull distribution

One can think the event occurrence as a failure and so $\lambda$ can be understood as the failure rate.

Poisson distribution and its waiting time (exponential distribution) has a constant failure rate.

Sometimes, it is more natural to choose $\lambda$ as a function of $t$ in the last assumption.

Then the waiting time $W$ for the first occurrence satisfies

$$\mathbb{P}(W > t) = \exp\left(-\int_0^t \lambda(w)\,dw\right).$$

# Weibull distribution

$$\lambda(t) = (\text{Const.})\ t^{\text{power}}$$

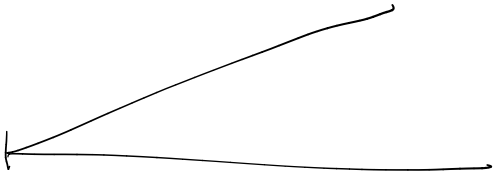
**Definition**

If $\lambda(t) = \alpha\frac{t^{\alpha-1}}{\beta^{\alpha}}$, then the waiting time $W$ for the first occurrence has the density

$$g(t) = \lambda(t)\exp\left(-\int_0^t \lambda(w)\,dw\right) = \alpha\frac{t^{\alpha-1}}{\beta^{\alpha}}\exp\left(-(\frac{t}{\beta})^{\alpha}\right).$$

$W$ is called **the Weibull random variable**.

$$\mathbb{P}(W > t) = e^{-\int_0^t \lambda(w)dw} = e^{-(\frac{t}{\beta})^{\alpha}}$$



## Weibull distribution

$$\lambda(t) = \frac{\alpha \, t^{\alpha-1}}{\beta^{\alpha}} \qquad P(W>t) = e^{-t^2}$$

**Example**

If $\lambda(t) = 2t$, then the waiting time $W$ has the density

and it is a Weibull random variable with $\alpha = 2$ and $\beta = 1$.

If $W_1, W_2$ are independent (Exp) Weibull with $\alpha$ and $\beta$ above, is the (max) minimum of $W_1, W_2$ Weibull?

$$X = \min\{W_1, W_2\}$$

$$P(X \le t) = 1 - P(X > t)$$

$$P(X > t) = P(\{W_1 > t\} \text{ and } \{W_2 > t\})$$

$$= P(W_1 > t)\, P(W_2 > t)$$

$$= e^{-t^2} \cdot e^{-t^2} = e^{-2t^2}$$

$$= e^{-(\sqrt{2}\, t)^2} \sim \text{Weibull.}$$

$$\left(\alpha = 2, \quad \beta = \frac{1}{\sqrt{2}}\right)$$

## Order Statistics

# Weibull distribution

**Theorem**

The mean of $W$ is $\mu = \beta\Gamma(1 + \frac{1}{\alpha})$.

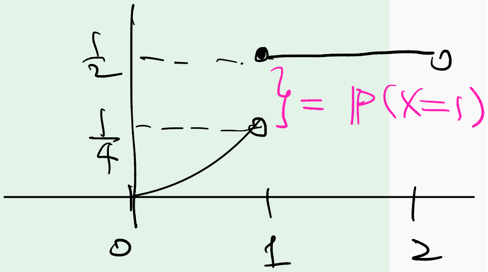
The variance is $\sigma^2 = \beta^2 \left(\Gamma(1 + \frac{2}{\alpha}) - \Gamma(1 + \frac{1}{\alpha})^2\right)$.

# Mixed type random variables

**Example**

Suppose $X$ has a CDF

$$F(x) = \begin{cases} 0, & x < 0 \\ \boxed{\frac{x^2}{4}}, & \boxed{0 \le x < 1} \\ \frac{1}{2}, & 1 \le x < 2 \\ \frac{x}{3}, & 2 \le x < 3 \\ 1, & x \ge 3. \end{cases}$$



Find $\mathbb{P}(0 < X < 1)$, $\mathbb{P}(0 < X \le 1)$, and $\mathbb{P}(X = 1)$.

$$\mathbb{P}(0 < X < 1)$$

$$= \mathbb{P}(X \le 1) - \mathbb{P}(X \le 0) - \underbrace{\mathbb{P}(X = 1)}$$

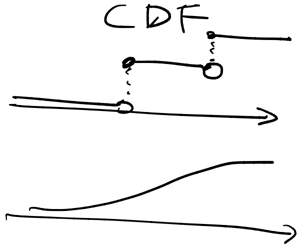
$$= F(1) - F(0) - \underbrace{\mathbb{P}(X = 1)}$$

$$= \frac{1}{2} - 0 - \left(\frac{1}{2} - \frac{1}{4}\right) = \frac{1}{4}.$$

Discrete RVs : Countable $S$

Conti. RVs : PDF

More RVs ''



↑ All have $\boxed{\text{CDF}}$ ''

## Mixed type random variables

**Example**

Consider the following game: A fair coin is tossed.

If the outcome is heads, the player receives $2.

If the outcome is tails, the player spins a balanced spinner that has a scale from 0 to 1.

The player then receives that fraction of a dollar associated with the point selected by the spinner.

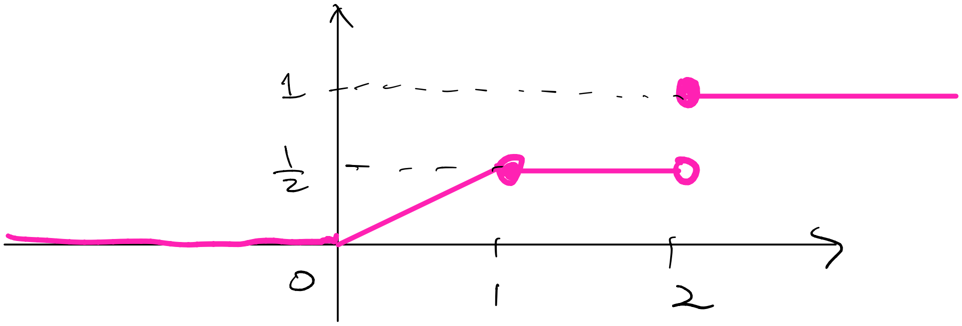
Let $X$ be the amount received. Draw the graph of the cdf $F(x)$.

$$X = \begin{cases} 2 & \text{if Heads} \\ U & \text{if Tails} \end{cases}, \quad U \sim \text{Unif}(0,1)$$

$$F(x) = P(X \leq x) = \begin{cases} 0 & , x < 0 \\ \frac{1}{2} \cdot x & 0 \leq x < 1 \\ \frac{1}{2} & 1 \leq x < 2 \\ 1 & x \geq 2 \end{cases}$$

$x = -1$

$x = \frac{1}{2}$ $\quad P(X \leq \frac{1}{2}) = ?$



## Exercise

The cdf of $X$ is given by

$$F(x) = \begin{cases} 0, & x < -1 \\ \frac{x}{4} + \frac{1}{2}, & -1 \leq x < 1 \\ 1, & x \geq 1. \end{cases}$$

Find $\mathbb{P}(X < 0)$, $\mathbb{P}(X < -1)$, and $\mathbb{P}(-1 \leq X < \frac{1}{2})$.