

Sample spaces having equally likely (Sec 2.5)

University of Illinois at Urbana–Champaign

Math 461 Spring 2022

Instructor: Daesung Kim

Recall

Axiom of Probability

- $0 \leq P(E) \leq 1$
- $P(\mathcal{S}) = 1$ ($E_i \cap E_j = \emptyset$ for $i \neq j$)
- If E_1, E_2, \dots are mutually exclusive events

$$P\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P(E_i)$$

- \Rightarrow
- (i) $P(\emptyset) = 0$
 - (ii) $P(E) \leq P(F)$ if $E \subseteq F$.
 - (iii) $P(E^c) = 1 - P(E)$
 - (iv) $P(E \cup F) = P(E) + P(F) - P(E \cap F)$

Let S be a sample space with finitely many outcomes. For convenience, let $S = \{1, 2, 3, \dots, N\}$.

In many cases, it is natural to assume that all outcomes in S are equally likely to occur. In other words, we assume

$$\mathbb{P}(\{1\}) = \mathbb{P}(\{2\}) = \dots = \mathbb{P}(\{N\}).$$

$$\mathbb{P}(S) = 1 \quad \mathbb{P}(\cup E_i) = \sum \mathbb{P}(E_i)$$

By the axioms (ii) and (iii), we have

$$1 = \mathbb{P}(S) = \mathbb{P}(\{1\}) + \mathbb{P}(\{2\}) + \dots + \mathbb{P}(\{N\}).$$

Therefore, $\mathbb{P}(\{i\}) = \frac{1}{N}$ for each $i = 1, 2, \dots, N$. Define the probability of an event E by

$$\mathbb{P}(E) = \sum_{i \in E} \mathbb{P}(\{i\}) = \frac{\text{Number of Outcomes in } E}{\text{Number of Outcomes in } S} = \frac{1}{N} \cdot \# E$$

Then one can see that (S, \mathbb{P}) is a probability space.

$$P(E) = \binom{5}{3} + \binom{6}{3} + \binom{8}{3} / \binom{19}{3}$$

Example 1-A

all balls are distinct.

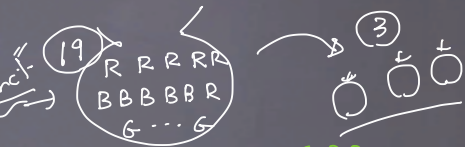
An urn contains 5 red, 6 blue, and 8 green balls. If a set of 3 balls is randomly selected, what is the probability that each of the balls will be

1. of the same color? E

2. of different colors?

all
Outcomes := Choose 3 out of 19

$$= \binom{19}{3}$$



$$\Omega = \{ RRR, RBG, \dots \}$$

don't care about ordering.

Outcomes in $E =$ 3 red of 5 or 3 blue of 6 or 3 green of 8

$$= \binom{5}{3} + \binom{6}{3} + \binom{8}{3}$$

Example 1-A

$$\zeta = \binom{19}{3} \times 3!$$

An urn contains 5 red, 6 blue, and 8 green balls. If a set of 3 balls is randomly selected, what is the probability that each of the balls will be

1. of the same color?
2. of different colors? F

no ordering
↙

$$\binom{5}{1} \text{ and } \binom{6}{1} \text{ and } \binom{8}{1}$$

$$P(F) = \frac{\binom{5}{1} \times \binom{6}{1} \times \binom{8}{1}}{\binom{19}{3}}$$

Example 1-B

$$P(E) = \frac{5^3 + 6^3 + 8^3}{19^3}$$

An urn contains 5 red, 6 blue, and 8 green balls. Suppose we draw a ball, note its color, and replace it into the urn. If we draw 3 balls in this way, what is the probability that each of the balls are of the same color? or of different colors?

E

F

$$S = 19 \times 19 \times 19 \quad \text{outcomes} \quad \leftarrow$$

Consider ordering.

$$F = 5 \times 5 \times 5 \quad \leftarrow \quad 3R$$

$$6 \times 6 \times 6 \quad \leftarrow \quad 3B$$

$$8 \times 8 \times 8 \quad \leftarrow \quad 3G$$

outcomes

Example 1-B

Ordering



An urn contains 5 red, 6 blue, and 8 green balls. Suppose we draw a ball, note its color, and replace it into the urn. If we draw 3 balls in this way, what is the probability that each of the balls are of the same color? or of different colors?

$$P(F) = \frac{1}{19^3} \cdot \left(\begin{array}{ccc} & F & \\ R & B & G \\ \sim & \sim & \\ \underbrace{5 \times 6 \times 8}_{\text{perm.}} & \times & 3! \end{array} \right)$$

$$B \quad R \quad G = 6 \times 5 \times 8$$

$$B \quad G \quad R = 6 \times 8 \times 5$$

⋮
}

Example 1-B

An urn contains 5 red, 6 blue, and 8 green balls. Suppose we draw a ball, note its color, and replace it into the urn. If we draw 3 balls in this way, what is the probability that each of the balls are of the same color? or of different colors?

Example 2-A

A football team consists of 20 offensive and 20 defensive players. The players are to be paired in groups of 2 for the purpose of determining roommates. If the pairing is done at random, what is the probability that there are no offensive–defensive roommate pairs?

no group ordering.

§: 40 people \rightarrow 20 groups of size 2.

$$\frac{1}{20!} \binom{40}{2} \cdot \binom{38}{2} \cdot \binom{36}{2} \cdots \binom{4}{2} \binom{2}{2}$$

$$= \frac{1}{20!} \binom{40}{2, 2, 2, \dots, 2}$$

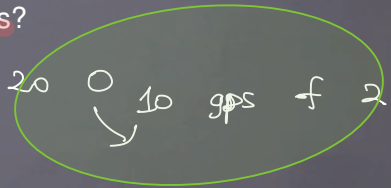
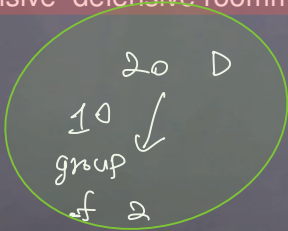
$$= \frac{1}{20!} \frac{40!}{2! \cdot 2! \cdots 2!} = \frac{40!}{20! \cdot 2^{20}}$$

Example 2-A

$$P(E) =$$

$$\frac{\left(\frac{20!}{2^{10} \cdot 10!}\right)^2}{\left(\frac{40!}{2^{20} \cdot 20!}\right)}$$

A football team consists of 20 offensive and 20 defensive players. The players are to be paired in groups of 2 for the purpose of determining roommates. If the pairing is done at random, what is the probability that **there are no offensive-defensive roommate pairs?**



$$\frac{1}{10!} \binom{20}{2, 2, \dots, 2}$$

$$\frac{1}{10!} \binom{20}{2, 2, \dots, 2} \cdot \frac{1}{10!} \binom{20}{2, 2, \dots, 2}$$

$$\frac{20!}{2^{10} \cdot 10!}$$

Example 2-B

$$\left(\frac{16!}{2^8 \cdot 8!} \right)^2 \cdot \binom{20}{4}^2 \cdot 4! \Big/ \left(\frac{40!}{2^{20} \cdot 20!} \right)$$

A football team consists of 20 offensive and 20 defensive players. The players are to be paired in groups of 2 for the purpose of determining roommates. What is the probability that there are 4 offensive-defensive roommate pairs?

20 D $\binom{20}{4}$ $\binom{20}{4}$ 20 O

16 \swarrow

4 \swarrow 4 \swarrow

8 grp. of D 4 groups 8 grp. of O

$\frac{16!}{2^8 \cdot 8!}$ $\begin{matrix} D_1 & D_2 & D_3 & D_4 \\ - & - & - & - \end{matrix}$ $\frac{16!}{2^8 \cdot 8!}$

4!

Example 3

If 4 married couples are arranged in a row, find the probability that no husband sits next to his wife.

Example 4-A

Suppose there are n distinct balls and r distinct urns. How many ways are there to distribute balls into urns?

Example 4-B

What if the balls are indistinguishable?

