Sample sapces having equally likely (Sec 2.5)

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Recall

Axiom of Probability

- $0 \leq P(E) \leq 1$
- $P(S) = 1$ $\qquad$ $(E_i \cap E_j = \phi$ for $i \neq j)$
- If $E_1, E_2, \cdots$ are mutually exclusive events

$$P\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P(E_i)$$

$\Rightarrow$ (i) $P(\phi) = 0$

(ii) $P(E) \leq P(F)$ if $E \subseteq F$.

(iii) $P(E^c) = 1 - P(E)$

(iv) $P(E \cup F) = P(E) + P(F) - P(E \cap F)$

Let S be a sample space with finitely many outcomes. For convenience, let $S = \{1, 2, 3, \cdots, N\}$.

In many cases, it is natural to assume that all outcomes in S are equally likey to occur. In other words, we assume

$$\mathbb{P}(\underbrace{\{1\}}_{E_1}) = \mathbb{P}(\underbrace{\{2\}}_{E_2}) = \cdots = \mathbb{P}(\underbrace{\{N\}}_{E_N}).$$

$\mathbb{P}(S)=1$ $\quad$ $\mathbb{P}(\cup E_i) = \Sigma\, \mathbb{P}(E_i)$

By the axioms (ii) and (iii), we have

$$1 = \mathbb{P}(\underbrace{S}_{=\cup E_i}) = \underbrace{\mathbb{P}(\{1\})}_{\frac{1}{N}} + \underbrace{\mathbb{P}(\{2\})}_{\frac{1}{N}} + \cdots + \underbrace{\mathbb{P}(\{N\})}_{\frac{1}{N}}.$$

Therefore, $\mathbb{P}(\{i\}) = \frac{1}{N}$ for each $i = 1, 2, \cdots, N$. Define the probability of an event E by $\quad \mathbb{P}(E) = \mathbb{P}(\{2, 3, 8, 10 \cdots\}) = \{a\} \cup \{b\} \cdots$

$$\mathbb{P}(E) = \sum_{i \in E} \mathbb{P}(\{i\}) = \frac{\text{Number of Outcomes in } E}{\text{Number of Outcomes in } S}. \quad = \Sigma\, \mathbb{P}(\{a\}) \cdots = \frac{1}{N} \cdot \# E$$

Then one can see that $(S, \mathbb{P})$ is a probability space.

$P(E) = \left[\binom{5}{3} + \binom{6}{3} + \binom{8}{3}\right] \Big/ \binom{19}{3}$

Example 1-A

all balls are distinct.

An urn contains 5 red, 6 blue, and 8 green balls. If a set of 3 balls is randomly selected, what is the probability that each of the balls will be

1. of the same color? E
2. of different colors?

distinct 19

R R R R R
B B B B B B
G ⋯ G

3

all Outcomes $:=$ choose 3 out of 19

$= \binom{19}{3}$

$S = \{$ RRR, GBR, RBG, ⋯ $\}$

don't care about ordering.

Outcomes in E = 3 red of 5 or 3 blue of 6 or 3 green of 8

$= \binom{5}{3} + \binom{6}{3} + \binom{8}{3}$

Example 1-A

$S = \binom{19}{3} \times 3!$

An urn contains 5 red, 6 blue, and 8 green balls. If a set of 3 balls is randomly selected, what is the probability that each of the balls will be

1. of the same color?
2. of different colors? F

no ordering

$$P(F) = \frac{\binom{5}{1} \overset{\text{and}}{\times} \binom{6}{1} \overset{\text{and}}{\times} \binom{8}{1}}{\binom{19}{3}}$$

Example 1-B

$$\mathbb{P}(E) = \frac{5^3 + 6^3 + 8^3}{19^3}.$$

An urn contains 5 red, 6 blue, and 8 green balls. Suppose we draw a ball, note its color, and replace it into the urn. If we draw 3 balls in this way, what is the probability that each of the balls are of the same color? or of different colors?

E F

$S = 19 \times 19 \times 19$ outcomes $\leftarrow$ Consider ordering.

$E = 5 \times 5 \times 5 \leftarrow 3R$

or +

$6 \times 6 \times 6 \leftarrow 3B$ outcoms

or +

$8 \times 8 \times 8 \leftarrow 3G$

Example 1-B

Ordering

An urn contains 5 red, 6 blue, and 8 green balls. Suppose we draw a ball, note its color, and replace it into the urn. If we draw 3 balls in this way, what is the probability that each of the balls are of the same color? or of different colors?

$$F$$

$$\begin{array}{ccc} R & B & G \end{array}$$

$$P(F) = \frac{1}{19^3} \cdot \Big(\underbrace{5 \times 6 \times 8}_{\text{perm.}} \times 3! \Big)$$

$$B \quad R \quad G = 6 \times 5 \times 8$$

$$B \quad G \quad R = 6 \times 8 \times 5$$

$$\vdots$$

Example 1-B

An urn contains 5 red, 6 blue, and 8 green balls. Suppose we draw a ball, note its color, and replace it into the urn. If we draw 3 balls in this way, what is the probability that each of the balls are of the same color? or of different colors?

Example 2-A

A football team consists of 20 offensive and 20 defensive players. The players are to be paired in groups of 2 for the purpose of determining roommates. If the pairing is done at random, what is the probability that there are no offensive–defensive roommate pairs?

no group ordering

S: 40 people $\longrightarrow$ 20 groups of size 2.

$$\frac{1}{20!}\underbrace{\binom{40}{2}\cdot\binom{38}{2}\cdot\binom{36}{2}\cdots\binom{4}{2}\binom{2}{2}}$$

$$= \frac{1}{20!}\binom{40}{\underbrace{2,2,2,\cdots,2}_{20}}$$

$$= \frac{1}{20!}\,\frac{40!}{2!\,2!\cdots 2!} = \frac{40!}{20!\cdot 2^{20}}$$

Example 2-A

$P(E) = \dfrac{\left(\dfrac{20!}{2^{10}\cdot 10!}\right)^2}{\left(\dfrac{40!}{2^{20}\cdot 20!}\right)}$

A football team consists of 20 offensive and 20 defensive players. The players are to be paired in groups of 2 for the purpose of determining roommates. If the pairing is done at random, what is the probability that there are no offensive–defensive roommate pairs?

20 D

10 group of 2

20 O 10 gps of 2

$\frac{1}{10!}\binom{20}{2,2,\cdots,2}$

$\frac{1}{10!}\binom{20}{2,2,\cdots,2}$

$\left(\dfrac{20!}{2^{10}\cdot 10!}\right)^2$

Example 2-B

$$\left(\frac{16!}{2^8 \cdot 8!}\right)^2 \cdot \binom{20}{4}^2 \cdot 4! \Big/ \left(\frac{40!}{2^{20} \cdot 20!}\right)$$

A football team consists of 20 offensive and 20 defensive players. The players are to be paired in groups of 2 for the purpose of determining roommates. What is the probability that there are 4 offensive–defensive roommate pairs?

20 D $\binom{20}{4}$ 4

$\binom{20}{4}$ 20 O 4

16 → 8 gp of D $\frac{16!}{2^8 \cdot 8!}$

4 groups: D_1 D_2 D_3 D_4 4!

16 → 8 gp of O $\frac{16!}{2^8 \cdot 8!}$

Example 3

If 4 married couples are arranged in a row, find the probability that no husband sits next to his wife.

Example 4-A

Suppose there are n distinct balls and r distinct urns. How many ways are there to distribute balls into urns?

Example 4-B

What if the balls are indistinguishable?