Chapter 4. Bivariate Distributions

Math 3215 Spring 2024

Georgia Institute of Technology

Section 1.
Bivariate Distributions of the Discrete Type

Motivation

Suppose that we observe the maximum daily temperature, X, and maximum relative humidity, Y, on summer days at a particular weather station.

Random.

We want to determine a relationship between these two variables.

For instance, there may be some pattern between temperature and humidity that can be described by an appropriate curve $Y = u(X)$.

PMF (Prob. Mass Function) : $f(x) = \mathbb{P}(X = x)$

Joint distribution

discrete

Let X and Y be two random variables defined on a discrete sample space.

Let S denote the corresponding two-dimensional space of X and Y, the two random variables of the discrete type.

Definition

" and "

The function $f(x, y) = \mathbb{P}(X = x, Y = y)$ is called the joint probability mass function (joint PMF) of X and Y.

$$f(x, y) = \mathbb{P}(\{X = x\} \cap \{Y = y\})$$

Joint distribution

Joint PMF = " prob. "

= $\mathbb{P}$ (——→)

Note that

- $0 \leq f(x, y) \leq 1$ $\mathbb{P}(\ S\ )$
- $\sum_{(x,y) \in S} f(x, y) = 1$
- $\mathbb{P}((X, Y) \in A) = \sum_{(x,y) \in A} f(x, y)$

Joint distribution

Example (4 faced)

Roll a pair of fair dice.

Let X denote the smaller and Y the larger outcome on the dice.

Find the joint PMF of (X, Y).

$$f(x,y) = \begin{cases} \frac{1}{16}, & (x,y) = (1,1) \\ \frac{1}{8}, & (1,2) \\ \frac{1}{8}, & (1,3) \\ \frac{1}{8}, & (1,4) \\ \frac{1}{16}, & \underline{(2,2)} \\ \vdots & (2,3) \\ & (2,4) \\ & (3,3) \\ & (3,4) \\ & (4,4) \end{cases}$$

Marginal distribution

Definition

Let X and Y have the joint probability mass function $f(x,y)$.

The probability mass function of X, which is called the marginal probability mass function of X, is defined by

$$f_X(x) = \sum_y f(x,y) = \mathbb{P}(X = x).$$

$$f_X(x) = \mathbb{P}(X = x)$$

$$= \sum_{\text{all possible } y} \mathbb{P}(\underbrace{\{X = x\} \cap \{Y = y\}})$$

$$= \sum_y f(x,y)$$

$$f_Y(y) = \sum_x f(x,y)$$

Def X, Y RVs

General

Indep. if $\mathbb{P}(X \in A \cap Y \in B)$ and, $\cap$

$= \mathbb{P}(X \in A)\,\mathbb{P}(Y \in B)$

for all possible A, B.

Marginal distribution

Definition (X, Y : discrete)

We say X and Y are independent if

Joint PMF $= \mathbb{P}(X = x, Y = y) = \mathbb{P}(X = x)\mathbb{P}(Y = y)$ = Product of Marginal PMFs

for all $(x, y) \in S$.

Equivalently, $f(x, y) = f_X(x) f_Y(y)$ for all x, y.

Otherwise, we say X and Y are dependent.

Marginal distribution

Example

Let the joint PMF of X and Y be defined by

$$f(x, y) = \frac{x+y}{21}$$

for $x = 1, 2, 3$ and $y = 1, 2.$

Find the marginal PMFs of X and Y.

Determine whether they are independent.

$$f_X(x) = \mathbb{P}(X = x) = \sum_y f(x,y) = f(x,1) + f(x,2)$$

$$= \frac{1}{21}(x+1) + \frac{1}{21}(x+2) = \frac{1}{21}(2x+3)$$

$$f_Y(y) = \mathbb{P}(Y = y) = \sum_x f(x,y) = f(1,y) + f(2,y) + f(3,y)$$

$$= \frac{1}{21} \cdot \left((y+1) + (y+2) + (y+3) \right) = \frac{1}{21}(3y+6) = \frac{1}{7}(y+2)$$

$$f(x,y) = f_X(x) \cdot f_Y(y)$$

$$\underline{\frac{1}{21}(x+y)} \neq \underline{\frac{1}{21}(2x+3) \cdot \frac{1}{7}(y+2)}$$

why?

dependent.

Marginal distribution

Example

Let the joint PMF of X and Y be defined by

$$f(x, y) = \frac{xy^2}{30} = (\text{a function of } x) \cdot (\text{function of } y)$$

for $x = 1, 2, 3$ and $y = 1, 2$.

Find the marginal PMFs of X and Y.

Determine whether they are independent.

$$f_X(x) = f(x,1) + f(x,2) = \frac{x \cdot 1^2}{30} + \frac{x \cdot 2^2}{30}$$

$$= \frac{x}{30} \cdot (5) = \frac{x}{6}$$

$$f_Y(y) = f(1,y) + f(2,y) + f(3,y)$$

$$= \frac{1 \cdot y^2}{30} + \frac{2 \cdot y^2}{30} + \frac{3 \cdot y^2}{30} = \frac{6}{30} \cdot y^2 = \frac{y^2}{5}$$

$$f(x,y) \overset{?}{=} f_X(x) \cdot f_Y(y)$$

$$\frac{xy^2}{30} = \frac{x}{6} \cdot \frac{y^2}{5}$$

Expectations

Definition

Let X_1 and X_2 be random variables of the discrete type with the joint PMF $f(x_1, x_2)$ on the space S. If $u(X_1, X_2)$ is a function of these two random variables, then

$$\mathbb{E}[u(X_1, X_2)] = \sum_{(x_1,x_2)\in S} u(x_1, x_2) f(x_1, x_2).$$

In particular, if $u(x_1, x_2) = x_1$, then

$$\mathbb{E}[u(X_1, X_2)] = \mathbb{E}[X_1] = \sum_{(x_1,x_2)\in S} x_1 f(x_1, x_2) = \sum_{x_1} x_1 f_{X_1}(x_1).$$

Ex $\quad u(X_1, X_2) = X_1 \rightarrow \mathbb{E}[X_1] = \sum_{x_1, x_2} x_1 f(x_1, x_2)$

" $\quad = X_2 \rightarrow \mathbb{E}[X_2] = \sum_{x_1, x_2} x_2 f(x_1, x_2)$

" $\quad = X_1 + X_2 \rightarrow \mathbb{E}[X_1 + X_2] = \sum_{x_1, x_2} (x_1 + x_2) f(x_1, x_2)$

" $\quad = X_1 \cdot X_2$

$\rightarrow \mathbb{E}[X_1 \cdot X_2] = \sum_{x_1, x_2} x_1 \cdot x_2 \cdot f(x_1, x_2)$

Expectations

Example

There are eight similar chips in a bowl: three marked $(0,0)$, two marked $(1,0)$, two marked $(0,1)$, and one marked $(1,1)$.

A player selects a chip at random.

Let X_1 and X_2 represent those two coordinates.

Find the joint PMF.

Compute $\mathbb{E}[X_1 + X_2]$.

$$f(x_1, x_2) = \begin{cases} 3/8, & (x_1, x_2) = (0,0) \\ 2/8, & (1,0) \\ 2/8, & (0,1) \\ 1/8, & (1,1) \end{cases}$$

$$\begin{aligned} \mathbb{E}[X_1 + X_2] &= \sum (x_1 + x_2) f(x_1, x_2) \\ &= (0+0)\cdot\frac{3}{8} + (1+0)\cdot\frac{2}{8} + (0+1)\cdot\frac{2}{8} + (1+1)\frac{1}{8} \\ &= 0 + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{3}{4}. \end{aligned}$$

Trinomial distribution

Example

In manufacturing a certain item, it is found that in normal production about 95% of the items are good ones, 4% are "seconds," and 1% are defective.

A company has a program of quality control by statistical methods, and each hour an online inspector observes 20 items selected at random, counting the number X of seconds and the number Y of defectives.

Suppose that the production is normal.

Find the probability that, in this sample of size $n = 20$, at least two seconds or at least two defective items are discovered.

Exercise

Roll a pair of four-sided dice, one red and one black.

Let X equal the outcome of the red die and let Y equal the sum of the two dice.

Find the joint PMF.

Are they independent?

Section 2.
The Correlation Coefficient

Covariance and Correlation coefficient

$\mathbb{E}[X] = \mu_X$, $\mathbb{E}[Y] = \mu_Y$

$Var(X) = \sigma_X^2$, $Var(Y) = \sigma_Y^2$

Definition

The covariance of X and Y is

X, Y discrete

$$\text{Cov}(X, Y) = \mathbb{E}[(X - \mu_X)(Y - \mu_Y)] = \sum_{x,y} (x-\mu_X)(y-\mu_Y) f(x,y)$$

The correlation coefficient of X and Y is

$$\rho = \frac{\text{Cov}(X, Y)}{\sigma_X \sigma_Y} = \frac{Cov(X,Y)}{\sqrt{Var(X)\, Var(Y)}}$$

Special cases

(i) $X = Y$: $Cov(X, X) = \mathbb{E}[(X-\mu_X)^2] = Var(X) = \sigma_X^2$

$$\rho = \frac{Cov(X,X)}{\sigma_X \cdot \sigma_X} = 1$$

(ii) $X = -Y$: $Cov(X, -X) = -\mathbb{E}[(X-\mu_X)^2] = -\sigma_X^2$

$$\rho = -1$$

(iii) X, Y indep.

$f(x,y) = f_X(x) f_Y(y)$

$$Cov(X,Y) = \sum_{x,y} (x-\mu_X)\cdot(y-\mu_Y)\cdot \underbrace{f(x,y)}_{f_X(x)\cdot f_Y(y)}$$

$$= \left(\sum_{x} (x-\mu_X) f_X(x)\right) \cdot \left(\sum_{y} (y-\mu_Y) f_Y(y)\right)$$

$$= \underbrace{\mathbb{E}[X-\mu_X]}_{=0} \cdot \underbrace{\mathbb{E}[Y-\mu_Y]}_{=0} = 0$$

Covariance and Correlation coefficient

Properties

1. If X and Y are independent, then $\mathrm{Cov}(X, Y) = 0$.
2. $\mathrm{Cov}(X, Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$.
3. $-1 \leq \rho \leq 1$.

② $$\mathrm{Cov}(X,Y) = \mathbb{E}[\underbrace{(X-\mu_X)(Y-\mu_Y)}]$$

$$= XY - \mu_X \cdot Y - \mu_Y \cdot X + \mu_X \cdot \mu_Y$$

$$= \mathbb{E}[XY] - \mu_X \underbrace{\mathbb{E}[Y]}_{=\mu_Y} \cancel{- \mu_Y \underbrace{\mathbb{E}[X]}_{=\mu_X}} \cancel{+ \mu_X \cdot \mu_Y}$$

$$= \mathbb{E}[XY] - \mu_X \cdot \mu_Y$$

③ $-1 \leq \rho \leq 1 \iff \rho^2 \leq 1 \qquad \rho = \dfrac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X) \cdot \mathrm{Var}(Y)}}$

$$\iff \mathrm{Cov}(X,Y) \leq \mathrm{Var}(X) \cdot \mathrm{Var}(Y)$$

$$\iff \left(\mathbb{E}[(X-\mu_X)\cdot(Y-\mu_Y)]\right)^2 \leq \mathbb{E}[(X-\mu_X)^2]\,\mathbb{E}[(Y-\mu_Y)^2]$$

↑ Cauchy – Schwartz Inequality.

Covariance and Correlation coefficient

Example

Let the joint PMF of X and Y be defined by

$$f(x,y) = \frac{x+2y}{18}$$

for $x = 1, 2$ and $y = 1, 2$.

Compute $\mathrm{Cov}(X, Y)$ and ρ.

$$\mathrm{Cov}(X,Y) = E[XY] - E[X]\cdot E[Y]$$

$$E[X\cdot Y] = \sum_{x,y} x\cdot y\cdot f(x,y)$$

$$= 1\cdot 1\cdot f(1,1) + 1\cdot 2\, f(1,2) + 2\cdot 1\cdot f(2,1) + 2\cdot 2\, f(2,2)$$

$$= 1\cdot\frac{(1+2)}{18} + 2\cdot\frac{(1+4)}{18} + 2\cdot\frac{(2+2)}{18} + 4\cdot\frac{(2+4)}{18}$$

$$= \frac{1}{18}(3 + 10 + 8 + 24) = \frac{45}{18} = \frac{5}{2}$$

$$E[X] = \sum_{x,y} x\cdot f(x,y)$$

$$= 1\cdot\frac{(1+2)}{18} + 1\cdot\frac{(1+4)}{18} + 2\cdot\frac{(2+2)}{18} + 2\cdot\frac{(2+4)}{18}$$

$$= \frac{1}{18}\cdot(3+5+8+12) = \frac{28}{18} = \frac{14}{9}$$

$$E[Y] = \sum_{x,y} y\, f(x,y)$$

$$= 1\cdot\frac{(1+2)}{18} + 2\cdot\frac{(1+4)}{18} + 1\cdot\frac{(2+2)}{18} + 2\cdot\frac{(2+4)}{18}$$

$$= \frac{1}{18}(3+10+4+12) = \frac{29}{18}$$

$$\mathrm{Cov}(X,Y) = \frac{5}{2} - \frac{14}{9}\cdot\frac{29}{18} \,''$$

The Least Squares Regression Line

Suppose we are trying to see if there is a pattern or a certain relation between two random variables X and Y.

One of natural ways is to consider a linear relation between X and Y, that is, to figure out the best possible slope b such that $Y - \mu_Y = b(X - \mu_X)$ has small errors.

We measure the error by $\underbrace{\mathbb{E}[((Y-\mu_Y) - b(X-\mu_X))^2]}_{\text{minimize in } b}$.

$Y = b X + c$: Linear relation

$Y \approx b X + c$

best approx = minimize "error"

Choose

At least, $E[Y] = E[bX + c]$

$\mu_Y = b\mu_X + c$

$c = -b\mu_X + \mu_Y$

$$Y - \mu_Y \approx b(X - \mu_X)$$

$$\rho = \frac{Cov(X,Y)}{\sigma_X \sigma_Y}$$

The Least Squares Regression Line

One can see by some calculus that the error is minimized when

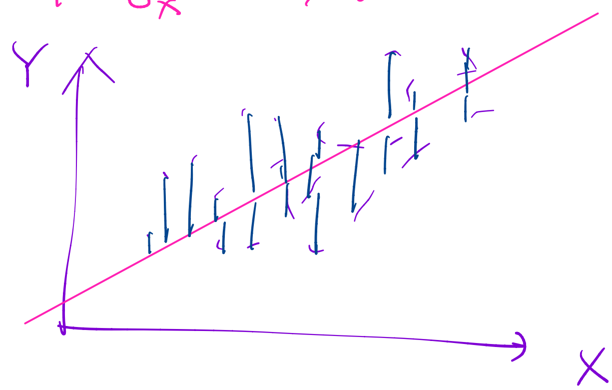
$$b = \rho\frac{\sigma_Y}{\sigma_X} = \frac{Cov(X,Y)}{\sigma_X \sigma_Y} \cdot \frac{\sigma_Y}{\sigma_X} = \frac{Cov(X,Y)}{Var(X)}$$

and the minimum error is $\sigma_Y^2(1-\rho^2)$.

The line $Y - \mu_Y = \rho\frac{\sigma_Y}{\sigma_X}(X-\mu_X)$ is called **the line of best fit**, or **the least squares regression line**.

$$Y - \mu_Y \approx \rho\frac{\sigma_Y}{\sigma_X}(X - \mu_X)$$

$$y - \mu_Y = \rho\frac{\sigma_Y}{\sigma_X}(x - \mu_X)$$



Trinomial

The Least Squares Regression Line

Example

Let X equal the number of ones and Y the number of twos and threes when a pair of fair four-sided dice is rolled.

Then X and Y have a trinomial distribution.

Find the least squares regression line.

$y = \rho \cdot \frac{\sigma_Y}{\sigma_X}(x - \mu_X) + \mu_Y$

$X = \begin{cases} 0 \\ 1 \\ 2 \end{cases}$ $\qquad Y = \begin{cases} 0 \\ 1 \\ 2 \end{cases}$

$\mu_X =$ $\quad \sigma_X =$ $\qquad \mu_Y =$ $\quad \sigma_Y =$

$X \sim Bin(2, \frac{1}{4})$ $\qquad Y \sim Bin(2, \frac{1}{2})$

Exercise

$f(x,y) = \begin{cases} (\frac{1}{4})^2 \\ 2 \cdot (\frac{1}{4})^1 (\frac{1}{2})^0 (\frac{1}{4})^1 \end{cases}$

$x=0, \; y=0$

$x=1, \; y=0$

$x=2, \; y=0$

$x=0, \; y=1$

$\vdots$

$\Rightarrow Cov(X,Y)$

ρ

Trinomial distribution

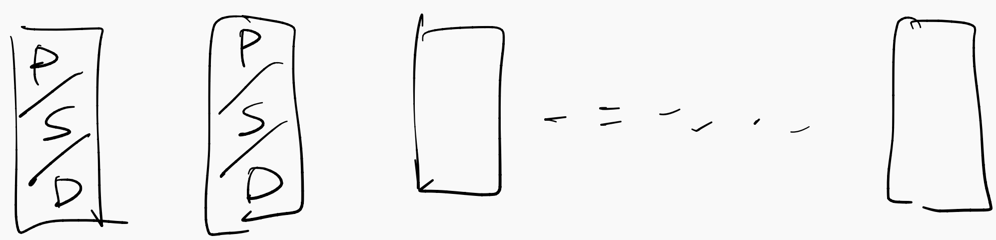
$p_1 + p_2 + p_3 = 1$

Consider an experiment with three outcomes, say perfect, seconds, and defective.

Let p_1, p_2, p_3 be the corresponding probabilities.

Repeat the experiment n times and let X, Y be the numbers of perfect and seconds.

We say (X, Y) has **the trinomial distribution**.



n times

$\binom{n}{x} \cdot \binom{n-x}{y} =$

$$f(x,y) = \boxed{\frac{n!}{x!\,y!\,(n-x-y)!}}\, p_1^{x} \cdot p_2^{y} \cdot p_3^{n-(x+y)}$$

Among n trials

x many perfect, y many seconds

$n-x-y$ many def.

Uncorrelated

Note X, Y indep $\Rightarrow$ $\rho = 0 = \text{Cov}(X, Y)$

We say X, Y are uncorrelated if $\rho = 0$.

If X, Y are independent then they are uncorrelated.

However, the converse is not true.

There exist X, Y such that

$\text{Cov}(X, Y) = 0$ but

X, Y dependent.

Exercise: Find an example!

$f_X(0) = \frac{1}{3} \qquad f_Y(1) = \frac{2}{3}$

Uncorrelated

Indep $\iff$ $f(x,y) = f_X(x) \cdot f_Y(y)$

$f(0,1) = \frac{1}{3} \neq \frac{1}{3} \cdot \frac{2}{3}$ Not indep.

Example

Let X and Y have the joint pmf $f(x,y) = \frac{1}{3}$ for $(x,y) = (0,1), (1,0), (2,1)$.

$$E[XY] = 0 \cdot 1 \cdot f(0,1) + 1 \cdot 0 \cdot f(1,0) + \underbrace{2 \cdot 1 \cdot f(2,1)}$$

$$= \frac{2}{3}$$

$$E[X] = 0\, f(0,1) + 1\, f(1,0) + 2 \cdot f(2,1) = 1$$

$$E[Y] = 0\, f(1,0) + 1\, f(0,1) + 1\, f(2,1) = \frac{2}{3}$$

$$Cov(X,Y) = E[XY] - E[X]E[Y]$$

$$= \frac{2}{3} - 1 \cdot \frac{2}{3} = 0.$$

X, Y uncorrelated.

Exercise

The joint pmf of X and Y is $f(x, y) = \frac{1}{6}$, $0 < x + y < 2$, where x and y are nonnegative integers.

Find the covariance and the correlation coefficient.

Section 3.
Conditional Distributions

Conditional distribution

Definition

The conditional probability mass function of X, given that $Y = y$, is defined by

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)}.$$

$$f_{X|Y}(x|y) = \mathbb{P}(X = x \mid Y = y)$$

$$= \frac{\mathbb{P}(X = x, Y = y)}{\mathbb{P}(Y = y)}$$

$$= \frac{f(x,y)}{f_Y(y)}$$

$f(x,y) \leftarrow$ joint PMF

$f_Y(y) \leftarrow$ marginal of Y

Conditional distribution

Example

Let the joint pmf of X and Y be defined by

$$f(x,y) = \frac{x+y}{21}$$

for $x = 1,2,3$ and $y = 1,2$. We have shown that

$$f_X(x) = \frac{2x+3}{21}, \qquad f_Y(y) = \frac{3y+6}{21}.$$

Find the conditional PMFs.

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{(x+y)/21}{(3y+6)/21} = \frac{x+y}{3y+6}$$

$$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{(x+y)/21}{(2x+3)/21} = \frac{x+y}{2x+3}.$$

In general $\mathbb{E}[u(Y) \mid X = x] = \sum_{y} u(y) \cdot f_{Y|X}(y|x)$

Conditional distribution

Definition

The conditional expectation of Y given $X = x$ is defined by

$$\mathbb{E}[Y|X = x] = \sum_{y} y f_{Y|X}(y|x).$$

The conditional variance of Y given $X = x$ is defined by

$$\begin{aligned} \mathrm{Var}(Y|X = x) &= \mathbb{E}[(Y - \mathbb{E}[Y|X = x])^2|X = x] \\ &= \mathbb{E}[Y^2|X = x] - (\mathbb{E}[Y|X = x])^2. \end{aligned}$$

Let X, Y be RVs with joint PMF.

$X \mid Y=y$ Conditional PMF $f_{X|Y}(x|y) = P(X=x \mid Y=y) = \dfrac{f(x,y)}{f_Y(y)}$

$Y \mid X=x$ " $f_{Y|X}(y|x) = \dfrac{f(x,y)}{f_X(x)}$

Conditional Expectation

$$\mathbb{E}[u(X) \mid Y=y] = \sum_x u(x) \cdot f_{X|Y}(x|y)$$

$$\mathrm{Var}(X \mid Y=y) = \mathbb{E}[(X - \mathbb{E}[X \mid Y=y])^2 \mid Y=y]$$

Conditional distribution

$$= \mathbb{E}[X^2 \mid Y=y] - (\mathbb{E}[X \mid Y=y])^2$$

Example

Let the joint PMF of X and Y be defined by

$$f(x,y) = \frac{x+y}{21} \qquad f_{Y|X}(y|x) = \frac{x+y}{2x+3}$$

for $x = 1,2,3$ and $y = 1,2$.

Find $\mathbb{E}[Y|X=3]$ and $\mathrm{Var}(Y|X=3)$.

$$\mathbb{E}[Y \mid X=3] = \sum_{y=1}^{2} y \cdot f_{Y|X}(y|3)$$

$$= 1 \cdot \frac{3+1}{6+3} + 2 \cdot \frac{3+2}{6+3} = \frac{4+10}{9}$$

$$= \frac{14}{9}.$$

$$\mathbb{E}[Y^2 \mid X=3] = \sum_{y=1}^{2} y^2 \cdot f_{Y|X}(y|3)$$

$$= 1^2 \cdot \frac{3+1}{6+3} + 2^2 \cdot \frac{3+2}{6+3} = \frac{4+20}{9} = \frac{24}{9}$$

$$\mathrm{Var}(Y \mid X=3) = \mathbb{E}[Y^2 \mid X=3] - (\mathbb{E}[Y \mid X=3])^2 = \frac{24}{9} - \left(\frac{14}{9}\right)^2$$

$$\mathbb{E}[Y \mid X = x] = \underbrace{\sum_y y \cdot f_{Y|X}(y|x)}_{\text{no } y, \text{ still have } x} \leftarrow \text{a function of } x$$

$$= h(x)$$

Consider $h(X) \leftarrow$ a new random variable

$$= \underbrace{\mathbb{E}[Y \mid X]}_{} \text{ notation}$$

Contional expectation as a function and a random variable

One can consider $\mathbb{E}[Y|X = x]$ as a function of x.

Say $h(x) = \mathbb{E}[Y|X = x]$

We define a random variable $\mathbb{E}[Y|X] = h(X)$.

$$\mathbb{E}[Y] = \sum_y y \cdot f_Y(y) \qquad f_Y(y) = \frac{3y+6}{21} = \frac{y+2}{7}$$

$$= 1 \cdot f_Y(1) + 2 \cdot f_Y(y)$$

$$= 1 \cdot \frac{3}{7} + 2 \cdot \frac{4}{7} = \frac{11}{7}$$

$$\mathbb{E}[\mathbb{E}[Y|X]] = \frac{33}{21} = \frac{11}{7}.$$

Contional expectation as a function and a random variable

$$f_X(x) = \frac{2x+3}{21} \qquad f_{Y|X}(y|x) = \frac{x+y}{2x+3}$$

Example

Let the joint pmf of X and Y be defined by

$$f(x,y) = \frac{x+y}{21}$$

for $x = 1, 2, 3$ and $y = 1, 2$. One can see that $\mathbb{E}[Y|X=1] = \frac{8}{5}$, $\mathbb{E}[Y|X=2] = \frac{11}{7}$ $\mathbb{E}[Y|X=3] = \frac{14}{9}$

Find the PMF of $\mathbb{E}[Y|X]$ and $\mathbb{E}[\mathbb{E}[Y|X]]$.

$Z = \mathbb{E}[Y|X]$ as a RV.

What is the PMF of $\mathbb{E}[Y|X]$?

$$h(x) = \mathbb{E}[Y|X=x]$$

$$h(x) = \begin{cases} \frac{8}{5} & \text{if } x=1 \\ \frac{11}{7} & \text{if } x=2 \\ \frac{14}{9} & \text{if } x=3 \end{cases}$$

$$Z = h(X)$$

$$f_Z(z) = \mathbb{P}(\mathbb{E}[Y|X] = z) =$$

$z = \cancel{1, 2, 3}$

or

$$\boxed{\frac{8}{5}, \frac{11}{7}, \frac{14}{9}}$$

$$\mathbb{P}(\mathbb{E}[Y|X] = \tfrac{8}{5}) = \mathbb{P}(h(X) = \tfrac{8}{5}) = \mathbb{P}(X=1) = \frac{5}{21}$$

$$\mathbb{P}(\mathbb{E}[Y|X] = \tfrac{11}{7}) = \mathbb{P}(X=2) = \frac{7}{21}$$

$$\mathbb{P}(\mathbb{E}[Y|X] = \tfrac{14}{9}) = \mathbb{P}(X=3) = \frac{9}{21}$$

$$f_Z(z) = \begin{cases} \frac{5}{21}, & z = \frac{8}{5} \\ \frac{7}{21}, & z = \frac{11}{7} \\ \frac{9}{21}, & z = \frac{14}{9} \end{cases}$$

$$\mathbb{E}[\underbrace{\mathbb{E}[Y|X]}] = \sum_x \mathbb{E}[Y|X=x] \cdot f_Z(x)$$

$$= \frac{8}{\cancel{5}} \cdot \frac{\cancel{5}}{21} + \frac{11}{\cancel{7}} \cdot \frac{\cancel{7}}{21} + \frac{14}{\cancel{9}} \cdot \frac{\cancel{9}}{21} = \frac{33}{21}$$

Contional expectation as a function and a random variable

Theorem

1. $\mathbb{E}[\mathbb{E}[Y|X]] = \mathbb{E}[Y]$ "Conditioning"
2. $\text{Var}(Y) = \mathbb{E}[\text{Var}(Y|X)] + \text{Var}(\mathbb{E}[Y|X])$

$$\mathbb{E}[\mathbb{E}[Y|X]] = \sum_x \underbrace{\mathbb{E}[Y|X=x]} \cdot \underbrace{\mathbb{P}(X=x)}$$

$$= \sum_x \left(\sum_y y \cdot \underbrace{f_{Y|X}(y|x)} \right) \cdot f_X(x)$$

$$= \sum_{x,y} y \cdot \frac{f(x,y)}{\cancel{f_X(x)}} \cdot \cancel{f_X(x)}$$

$$= \sum_{x,y} y \cdot f(x,y) = \mathbb{E}[Y].$$

Contional expectation as a function and a random variable

(Exp. RV mean $=4$ implies $\lambda = \frac{1}{4}$)

Example

$\lambda = 4$

Let X have a Poisson distribution with mean 4, and let Y be a random variable whose conditional distribution, given that $X = x$, is binomial with sample size $n = x + 1$ and probability of success p.

Find $\mathbb{E}[Y]$ and $\text{Var}(Y)$.

$$Y \mid X = x \sim \text{Bin}(x+1, p)$$

$$\mathbb{E}[Y] = \mathbb{E}[\underbrace{\mathbb{E}[Y \mid X]}] = \mathbb{E}[(X+1) \cdot p]$$

$$= p \cdot (\mathbb{E}[X] + 1) = 5p.$$

$$\text{Var}(Y) = \mathbb{E}[\text{Var}(Y \mid X)] + \text{Var}(\mathbb{E}[Y \mid X])$$

$$= \mathbb{E}[(X+1) \cdot p \cdot (1-p)] + \text{Var}((X+1) \cdot p)$$

$$= p \cdot (1-p)(\underbrace{\mathbb{E}[X]}_{=4} + 1) + p^2 \cdot \underbrace{\text{Var}(X)}_{=4}$$

$$= 5p(1-p) + 4p^2$$

If $\mathbb{E}[Y \mid X=x] = a + bx$

$\mathbb{E}[Y \mid X] = a + bX$, $\quad X \cdot \mathbb{E}[Y \mid X] = aX + bX^2$

$\mu_Y = \mathbb{E}[Y] = \mathbb{E}[\mathbb{E}[Y \mid X]] = \mathbb{E}[a + bX] = a + b \cdot \mathbb{E}[X] = a + b\mu_X$

$\mathbb{E}[XY] = \mathbb{E}[\mathbb{E}[XY \mid X]] = \mathbb{E}[\boxed{X}\, \mathbb{E}[Y \mid X]] = \mathbb{E}[aX + bX^2] = a\mathbb{E}[X] + b\mathbb{E}[X^2]$

Linear case

a function of x

??

Suppose $\mathbb{E}[Y|X=x]$ is linear in x, that is, $\mathbb{E}[Y|X=x] = a + bx$.

Then we have $\mu_Y = a + b\mu_X$ and $\mathbb{E}[XY] = a\mu_X + b\mathbb{E}[X^2]$.

Solving for a,, we have

$$a = \mu_Y - \rho\frac{\sigma_Y}{\sigma_X}\mu_X, \qquad b = \rho\frac{\sigma_Y}{\sigma_X}.$$

what is this?

Thus,

$$\mathbb{E}[Y|X=x] = \mu_Y + \rho\frac{\sigma_Y}{\sigma_X}(x - \mu_X).$$

$\mathbb{E}[X^2] = \mathrm{Var}(X) + (\mathbb{E}[X])^2 = \sigma_X^2 + \mu_X^2$

$\mathrm{Cov}(X, Y) = \mathbb{E}[X \cdot Y] - \mathbb{E}[X] \cdot \mathbb{E}[Y]$

$\mathbb{E}[XY] = \mathrm{Cov}(X,Y) + \mu_X \cdot \mu_Y = \rho \cdot \sigma_X \cdot \sigma_Y + \mu_X \mu_Y$

$Y \approx a + bX \rightarrow$ minimize errors

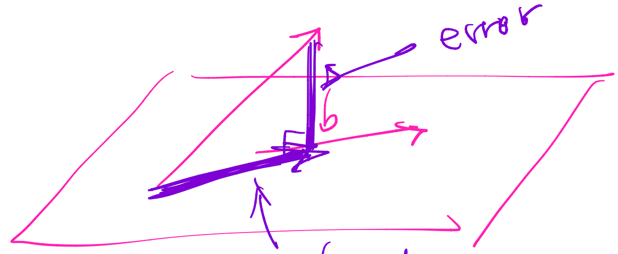
$\rightarrow (Y - \mu_Y) \approx \rho \cdot \frac{\sigma_Y}{\sigma_X}(X - \mu_X)$

$y - \mu_Y = \rho \cdot \frac{\sigma_Y}{\sigma_X}(x - \mu_X)$: line of best fit

least square regression

Linear Algebra :

$Ax = b$



Consistent

least square solution
= projection

Conditional Expectation = 'Projection'

Linear case

$$\binom{n}{x,y} = \frac{n!}{x!\,y!\,(n-x-y)!} = \binom{n}{x}\cdot\binom{n-x}{y}$$

Example

Let X and Y have the trinomial distribution with parameters n, p_X, p_Y, that is, the joint pmf is given by

$X \sim Bin(n, p_X)$

$$f(x,y) = \binom{n}{x,y} p_X^x p_Y^y (1-p_X-p_Y)^{n-x-y}.$$

Find $\mathbb{E}[Y|X=x]$.

$$f_X(x) = \sum_y f(x,y) = \binom{n}{x}\cdot p_x^x (1-p_x)^{n-x}$$

Each experiment has three results A, B, C

$p_X \quad p_Y \quad p_Z$

$(p_x + p_Y + p_Z = 1)$

Repeat n times

X = # of A , Y = # of B

$$E[Y|X=x] = \sum_y y\cdot f_{Y|X}(y|x) = \sum_y y\cdot\frac{f(x,y)}{f_X(x)}$$

$$f_{Y|X}(y|x) = \frac{\cancel{\binom{n}{x}}\binom{n-x}{y}\cancel{p_X^x}\cdot p_Y^y(1-p_X-p_Y)^{n-x-y}}{\cancel{\binom{n}{x}}\cancel{p_X^x}\cdot(1-p_X)^{(n-x-y)+y}}$$

$$= \binom{n-x}{y}\cdot\left(\frac{p_Y}{1-p_X}\right)^y\left(1-\frac{p_Y}{1-p_X}\right)^{(n-x)-y}$$

$$Y \mid X = x \sim \text{Bin}\left(n - x, \frac{p_Y}{1-p_X}\right) \qquad \frac{p_Y}{1-p_X} = \frac{p_Y}{p_X + p_Z}$$

$$E[Y \mid X = x] = (n - x)\cdot \frac{p_Y}{1-p_X}$$

$$E[X] = E[\,E[X \mid Y]\,]$$

$$= \underbrace{E[X \mid Y=1]}_{=3} \cdot \underbrace{P(Y=1)}_{=\frac{1}{3}} + \underbrace{E[X \mid Y=2]}_{=5+E[X]} \cdot \underbrace{P(Y=2)}_{=\frac{1}{3}} + \underbrace{E[X \mid Y=3]}_{=7+E[X]} \cdot \underbrace{P(Y=3)}_{=\frac{1}{3}}$$

Exercise

$$E[X] = 3\cdot\frac{1}{3} + (5 + E[X])\cdot\frac{1}{3} + (7 + E[X])\cdot\frac{1}{3} \Rightarrow E[X] = \boxed{}$$

A miner is trapped in a mine containing 3 doors.

The first door leads to a tunnel that will take him to safety after 3 hours of travel.

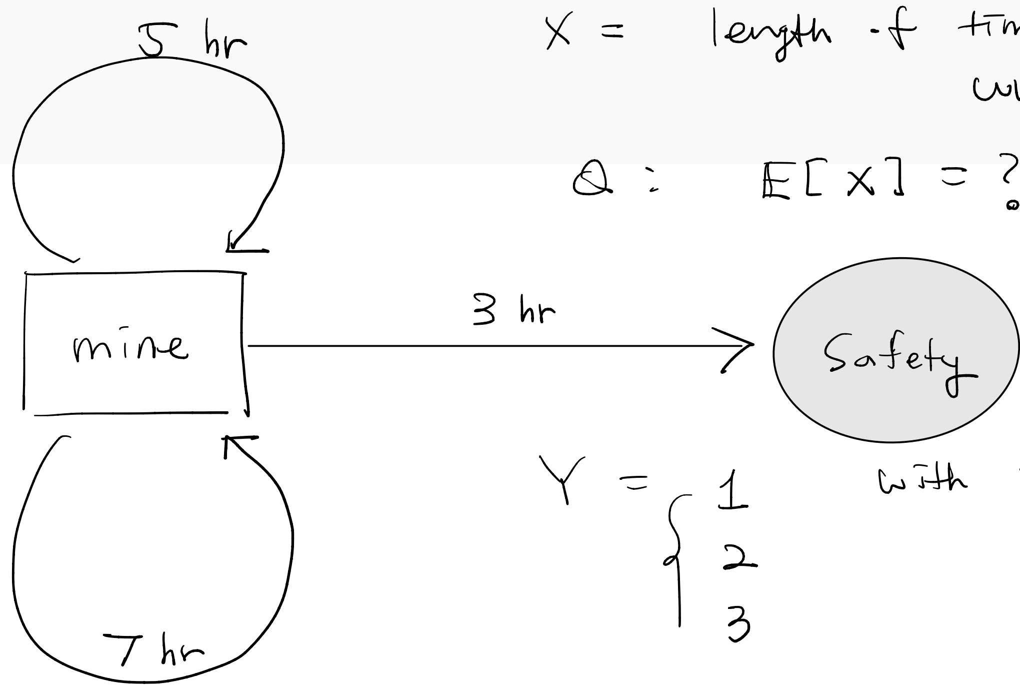
The second door leads to a tunnel that will return him to the mine after 5 hours of travel.

The third door leads to a tunnel that will return him to the mine after 7 hours.

If we assume that the miner is at all times equally likely to choose any one of the doors, what is the expected length of time until he reaches safety?

X = length of time until safety

Q: $E[X] = ?$



$$Y = \begin{cases} 1 & \text{with prob. } \frac{1}{3} \\ 2 & \text{"} \\ 3 & \text{"} \end{cases}$$

$$h(y) = \mathbb{E}[X \mid Y = y] = \sum_x x \cdot \underbrace{f_{X|Y}(x|y)}_{= \frac{f(x,y)}{f_Y(y)}} = \text{a function of } y$$

$$\mathbb{E}[X \mid Y] = h(Y) \quad : \text{a RV.}$$

$$\mathbb{E}[X] = \mathbb{E}[\,\mathbb{E}[X \mid Y]\,]$$

$$\mathbb{E}[X \cdot Y] = \mathbb{E}[\,Y \cdot \mathbb{E}[X \mid Y]\,]$$

hold for any RVs

$$\mathrm{Var}(X) = \mathbb{E}[\,\mathrm{Var}(X \mid Y)\,] + \mathrm{Var}(\,\mathbb{E}[X \mid Y]\,)$$

$$\mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y]$$

$$\mathbb{E}[X \cdot Y] = \mathbb{E}[X]\,\mathbb{E}[Y] \quad \text{when } X, Y \text{ indep}$$

$$\begin{aligned}
\mathrm{Var}(X+Y) &= \mathbb{E}[\underbrace{(X+Y)^2}] - (\mathbb{E}[X+Y])^2 \\
&= \mathbb{E}[X^2 + 2XY + Y^2] - (\mathbb{E}[X])^2 - 2\,\mathbb{E}[X]\,\mathbb{E}[Y] - (\mathbb{E}[Y])^2 \\
&= \mathrm{Var}(X) + \mathrm{Var}(Y) + 2\,\mathrm{Cov}(X,Y)
\end{aligned}$$

If X, Y indep. then $\mathrm{Cov}(X,Y) = 0$,

$$\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y)$$

Section 4.
Bivariate Distributions of the Continuous Type

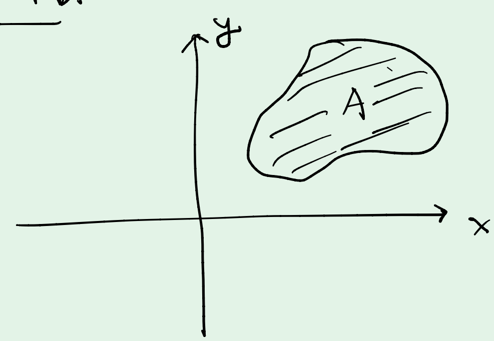
Joint PDF

Definition

An integrable function $f(x, y)$ is **the joint probability density function** of two random variables X, Y if

joint PDF

- $f(x, y) \geq 0$
- $\iint f(x, y)\, dxdy = 1$
- $\mathbb{P}((X, Y) \in A) = \iint_A f(x, y)\, dxdy$



The marginal density functions for X, Y are

$$f_X(x) = \int f(x, y)\, dy, \qquad f_Y(y) = \int f(x, y)\, dx.$$

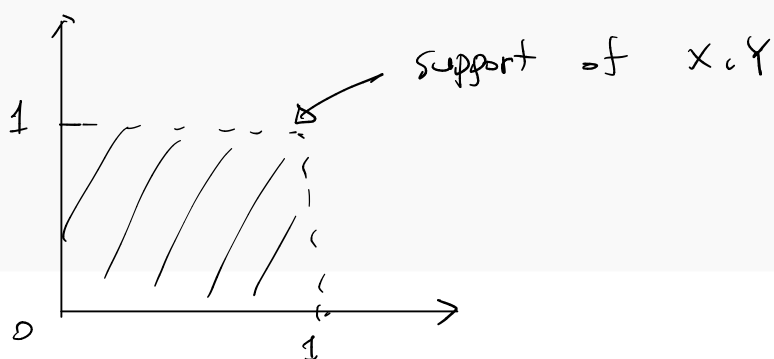
Joint PDF

Example

Let X and Y have the joint PDF

$$f(x,y) = \frac{4}{3}(1 - xy)$$

for $0 < x, y < 1$. Find f_X, f_Y, and $\mathbb{P}(Y \leq \frac{X}{2})$.



$$f_X(x) = \int f(x,y)\,dy = \int_0^1 \frac{4}{3}(1 - xy)\,dy$$

$$= \left[\frac{4}{3}\left(y - \frac{1}{2}xy^2\right)\right]_0^1 = \frac{4}{3}\left(1 - \frac{1}{2}x\right)$$

$$f_Y(y) = \int f(x,y)\,dx = \int_0^1 \frac{4}{3}(1 - xy)\,dx$$

$$= \left[\frac{4}{3}\left(x - \frac{1}{2}yx^2\right)\right] = \frac{4}{3}\left(1 - \frac{1}{2}y\right)$$

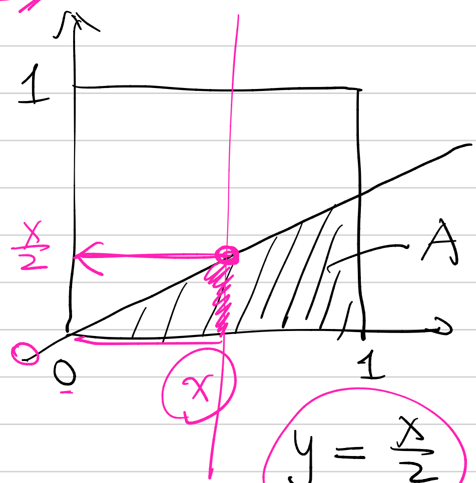
$$P\left(Y \le \frac{X}{2}\right) = \int_0^1 \int_0^{\frac{x}{2}} f(x,y)\,dy\,dx$$

$Y \le \frac{X}{2}$

$$= P\big((X,Y) \in A\big)$$

$$= \iint_A f(x,y)\,dx\,dy$$

$$= \int_0^1 \int_0^{\frac{x}{2}} \frac{4}{3}(1 - xy)\,dy\,dx$$



$y = \frac{x}{2}$

$$= \int_0^1 \left[\frac{4}{3}\left(y - \frac{x}{2}\cdot y^2\right)\right]_0^{\frac{x}{2}} dx$$

$$= \int_0^1 \left[\frac{4}{3}\left(\frac{x}{2} - \frac{x^3}{8}\right)\right] dx = \frac{4}{3}\cdot\left(\frac{x^2}{4} - \frac{x^4}{32}\right)\Big]_0^1$$

$$= \frac{4}{3}\cdot\left(\frac{1}{4} - \frac{1}{32}\right)$$

X, Y have the joint PDF $f(x,y)$

$$\mathbb{E}[u(X,Y)] = \iint u(x,y) f(x,y)\, dx\, dy$$

$$\mathbb{E}[X] = \iint x \cdot f(x,y)\, dx\, dy$$

$$\mathbb{E}[Y] = \iint y \cdot f(x,y)\, dx\, dy .$$

Joint PDF

Example

Let X and Y have the joint PDF

$$f(x,y) = \frac{3}{2}x^2(1-|y|)$$

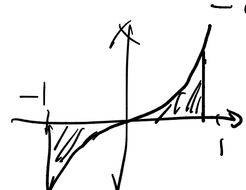
for $\boxed{-1 < x, y < 1.}$

Find $\mathbb{E}[X]$ and $\mathbb{E}[Y]$.

$$\mathbb{E}[X] = \iint x \cdot f(x,y)\, dx\, dy$$

$$= \int_{-1}^{1}\int_{-1}^{1} x \cdot \frac{3}{2} \cdot x^2 \cdot (1-|y|)\, dx\, dy$$

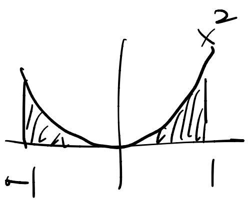
$$= \frac{3}{2}\int_{-1}^{1} (1-|y|)\left(\underbrace{\int_{-1}^{1} x^3\, dx}_{=0}\right) dy \quad = 0$$



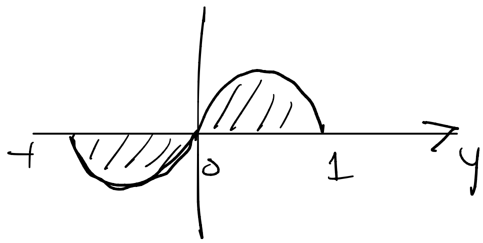
$$\mathbb{E}[Y] = \int_{-1}^{1}\int_{-1}^{1} y \cdot \frac{3}{2} x^2 (1-|y|)\, dx\, dy$$

$$= \frac{3}{2}\left(\int_{-1}^{1} x^2\, dx\right)\left(\int_{-1}^{1} y(1-|y|)\, dy\right)$$

$$= \frac{3}{2} \cdot 2 \cdot \left(\int_0^1 x^2 \, dx\right) \cdot 0 = 0$$



$$y \cdot (1-|y|) = \begin{cases} y(1-y) & \text{if } y \geq 0 \\ y(1+y) & \text{if } y < 0 \end{cases}$$



Independent random variables

Definition

Two random variables X, Y with joint pdf are independent if and only if $f(x, y) = f_X(x) f_Y(y)$.

Note

Each RV has PDF

① If X, Y Indep. conti. RVs, then there is a joint PDF $f(x,y) = f_X(x) \cdot f_Y(y)$.

② In general, there is a case that X, Y are continuous RV (we have $f_X(x)$, $f_Y(y)$) but there is no joint PDF.

Independent random variables

Example

Let X and Y have the joint pdf $f(x, y) = 2$ for $\boxed{0 < x < y < 1.}$

3 Inequalities → Define a region

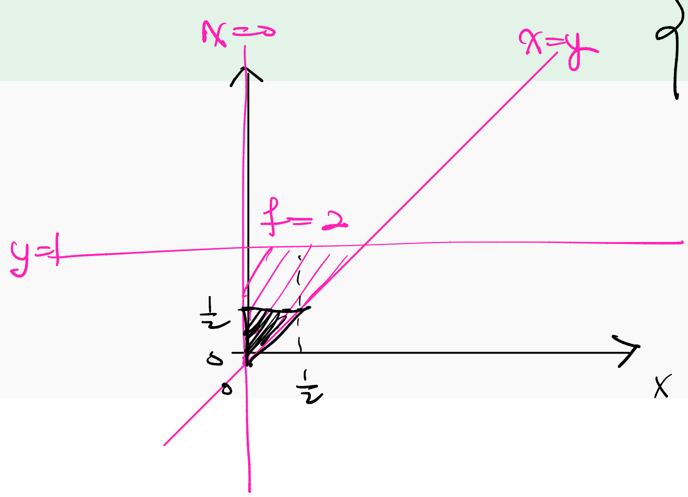
Compute $\mathbb{P}(0 < X, Y < \frac{1}{2})$.

Are they independent?

$$\begin{cases} x = 0 \\ x = y \\ y = 1 \end{cases}$$

↓ give you bdry

$\mathbb{P}(0 < X, Y < \frac{1}{2}) = \frac{1}{4}$



Conditional densities and Conditional Expectation

Definition

The conditional density of Y given $X = x$ is defined by

$$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)}.$$

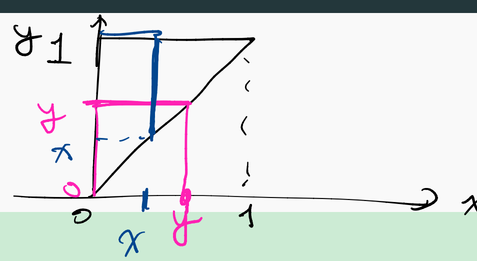
As in the discrete case, the conditional expectation and the conditional variance are defined by

$$\mathbb{E}[Y|X = x] = \int y f_{Y|X}(y|x)\, dy,$$

$$\mathrm{Var}(Y|X = x) = \mathbb{E}[(Y - \mathbb{E}[Y|X = x])^2|X = x]. = \mathbb{E}[Y^2|X=x] - (\mathbb{E}[Y|X=x])^2$$

$$\mathbb{E}[u(Y) \mid X = x] = \int u(y) f_{Y|X}(y|x)\, dy$$

Conditional densities and Conditional Expectation



Example

Let X and Y have the joint PDF $f(x, y) = 2$ for $0 < x < y < 1$.

Then, $f_X(x) = 2(1 - x)$ for $0 < x < 1$ and $f_Y(y) = 2y$ for $0 < y < 1$.

Find $\mathbb{E}[X|Y = y]$ and $\mathbb{E}[Y|X = x]$.

$$\mathbb{E}[X \mid Y=y] = \int x \cdot \underline{f_{X|Y}(x|y)}\, dx$$

$$= \int_0^y x \cdot \frac{\cancel{2}}{\cancel{2} \cdot y}\, dx = \frac{1}{y} \cdot \int_0^y x\, dx = \frac{1}{y} \cdot \left[\frac{x^2}{2}\right]_0^y$$

$$= \frac{y}{2}.$$

$$\mathbb{E}[Y \mid X=x] = \int y \cdot f_{Y|X}(y|x)\, dy$$

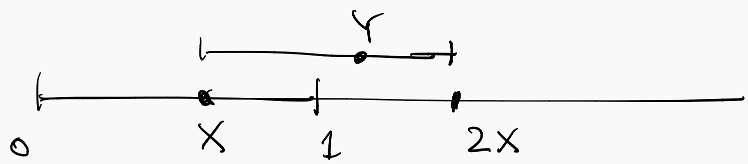
$$= \int_x^1 y \cdot \frac{\cancel{2}}{\cancel{2}(1-x)}\, dy$$

$$= \frac{1}{1-x} \left[\frac{y^2}{2}\right] = \frac{1}{1-x} \cdot \frac{1}{2} \cdot (1 - x^2)$$

$$= \frac{1+x}{2}$$

$\mathbb{E}[X \mid Y] = \frac{Y}{2}$, $\mathbb{E}[Y \mid X] = \frac{X+1}{2}$

Conditional densities and Conditional Expectation



Example

Let X be $U(0,1)$, and let the conditional distribution of Y, given $X = x$ be $U(x, 2x)$.

Find $\mathbb{E}[Y]$ and $\text{Var}(Y)$.

$Y \mid X \sim$ Unif

$$\mathbb{E}[Y] = \mathbb{E}[\,\mathbb{E}[Y \mid X]\,]$$

$$= \mathbb{E}\left[\frac{X + 2X}{2}\right] = \frac{3}{2} \cdot \mathbb{E}[X] = \frac{3}{2} \cdot \frac{1}{2} = \frac{3}{4}.$$

Recall

- X, Y have joint PDF if

$$\begin{cases} f(x,y) \geq 0 \\ \int_{\mathbb{R}}\int_{\mathbb{R}} f(x,y) \, dx\, dy = 1 \\ \mathbb{P}((X,Y) \in A) = \iint_A f(x,y)\, dx\, dy \end{cases}$$

- $\mathbb{E}[u(X,Y)]$ $\qquad u(x,y) = x$ or y or x^2, y^2, xy

$$= \iint u(x,y) f(x,y) \, dx\, dy$$

- Conditional density : $f_{Y|X}(y|x) \leftarrow$ PDF of $Y | X=x$

$$= \frac{f(x,y)}{f_X(x)}$$

$$f_X(x) = \int f(x,y)\, dy$$

$$\mathbb{E}[Y \mid X=x] = \int y \cdot f_{Y|X}(y|x)\, dy$$

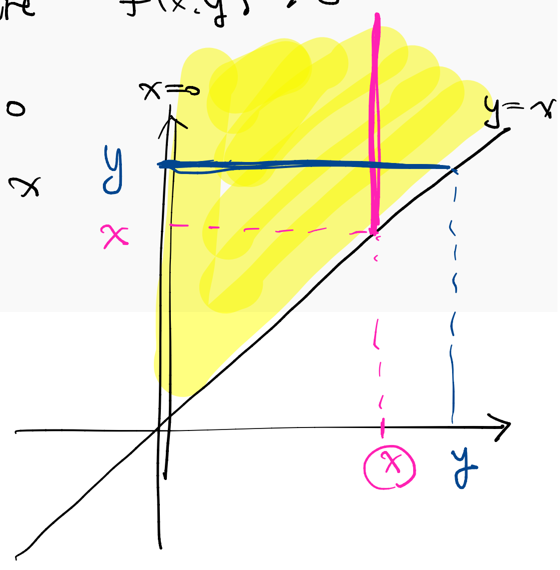
Exercise

Let $f(x, y) = 2e^{-x-y}$, $0 < x \leq y$ [redacted] , be the joint pdf of X and Y.

Find $f_X(x)$ and $f_Y(y)$. Are X and Y independent?

$\{0 < x \leq y\}$ defines a region where $f(x,y) > 0$

Consists of two ineq. $\begin{cases} x > 0 \\ y \geq x \end{cases}$



y-axis

$\begin{cases} \boxed{x = 0} \\ y = x \end{cases}$ define the boundary

$f_X(x) = \underbrace{\int_x^\infty f(x,y)\, dy}$

fixed

$= \int_x^\infty 2e^{-x-y}\, dy = 2e^{-x} \int_x^\infty e^{-y}\, dy$

$= \begin{cases} 2e^{-x} \left[-e^{-y}\right]_x^\infty = 2e^{-x} \cdot e^{-x} = 2e^{-2x} & \text{for } x > 0 \\ 0 & \text{for } x \leq 0 \end{cases}$

fixed

$$f_Y(y) = \int_0^y f(x,y)\,dx = \int_0^y 2e^{-x-y}\,dx = 2e^{-y}\left[-e^{-x}\right]_0^y$$

$$= \begin{cases} 2e^{-y}(1-e^{-y}) & \text{for } y > 0 \quad \cancel{[y \ge x]} \\ 0 & \text{for } y \le 0 \end{cases}$$

<u>Recall</u> X, Y Indep If and only If

$$f(x,y) = f_X(x)\cdot f_Y(y)$$

$$f(x,y) = 2e^{-x-y} \neq 2e^{-2x}\cdot 2e^{-y}(1-e^{-y}) = f_X(x)\, f_Y(y)$$

X, Y Not Indep.

<u>Recall</u> $X \sim \mathrm{Exp}(\lambda)$ $\qquad f(x) = \lambda e^{-\lambda x}, \quad x > 0$

Section 5.
The Bivariate Normal Distribution

$X \sim N(\mu, \sigma^2)$ (μ = mean = $\mathbb{E}[X]$, $\sigma^2 = Var(X)$)

$$f(x) = \frac{1}{\sqrt{2\pi}\cdot\sigma} \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad -\infty < x < \infty$$

X : Normal , Gaussian

Motivation

Let X be a random variable.

We construct a random variable Y in the following way:

The conditional distribution of Y given $X = x$ satisfies

1. it is normal for each x
2. $\mathbb{E}[Y|X = x]$ is linear in x $\Rightarrow$ $\mathbb{E}[Y|X=x] = bx + c = \rho \cdot \frac{\sigma_Y}{\sigma_X}(x-\mu_X) + \mu_Y$
3. $\mathrm{Var}(Y|X = x)$ is constant in x $\Rightarrow$ $Var(Y|X=x) = \sigma_Y^2(1-\rho^2)$

Use $Var(Y) = \mathbb{E}[Var(Y|X)] + Var(\mathbb{E}[Y|X])$

$$Y|X=x \sim N\left(\rho\frac{\sigma_Y}{\sigma_X}(x-\mu_X)+\mu_Y,\ \sigma_Y^2(1-\rho^2)\right)$$

$$f_{Y|X}(y|x) = \frac{1}{\sqrt{2\pi}\cdot\sigma_Y\sqrt{1-\rho^2}}\, e^{-\frac{\left(y-\left(\rho\frac{\sigma_Y}{\sigma_X}(x-\mu_X)+\mu_Y\right)\right)^2}{2\sigma_Y^2(1-\rho^2)}}$$

Motivation

Then, $Y|X = x$ is normal with mean $\mu_Y + \rho\frac{\sigma_Y}{\sigma_X}(x - \mu_X)$ and variance $\sigma_Y^2(1 - \rho^2)$.

The conditional density is

$$f_{Y|X}(y|x) = \frac{1}{\sigma_Y\sqrt{2\pi}\sqrt{1-\rho^2}} \exp\left(-\frac{(y - (\mu_Y + \rho\frac{\sigma_Y}{\sigma_X}(x - \mu_X)))^2}{2\sigma_Y^2(1-\rho^2)}\right)$$

$$+$$

$$X \sim N(\mu_X, \sigma_X^2)$$

$\Rightarrow$ (X, Y) Bivariate Normal.

$$f(x,y) = f_{Y|X}(y|x) \cdot \underbrace{f_X(x)}_{= \frac{1}{\sqrt{2\pi}\cdot\sigma_X} e^{-\frac{(x-\mu_X)^2}{2\sigma_X^2}}}$$

Bivariate normal distribution

$X \sim N(\mu_X, \sigma_X^2)$

If X itself has normal distribution, (X, Y) is called **a bivariate normal random variables**.

Bivariate normal distribution

$\begin{pmatrix} E[X] \\ E[Y] \end{pmatrix}$

Definition

multivariate Normal

$(X_1, X_2, \cdots, X_n)$

We say (X, Y) has a bivariate normal distribution with mean vector $\begin{pmatrix} \mu_X \\ \mu_Y \end{pmatrix}$ and covariance matrix $\begin{pmatrix} \sigma_X^2 & \rho\sigma_X\sigma_Y \\ \rho\sigma_X\sigma_Y & \sigma_Y^2 \end{pmatrix}$ if its joint pdf is given by

$$f(x, y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)}\left(\underline{\frac{\bar{x}^2}{\sigma_X^2} - 2\frac{\rho\bar{x}\bar{y}}{\sigma_X\sigma_Y} + \frac{\bar{y}^2}{\sigma_Y^2}}\right)\right)$$

where $\bar{x} = x - \mu_X$ and $\bar{y} = y - \mu_Y$.

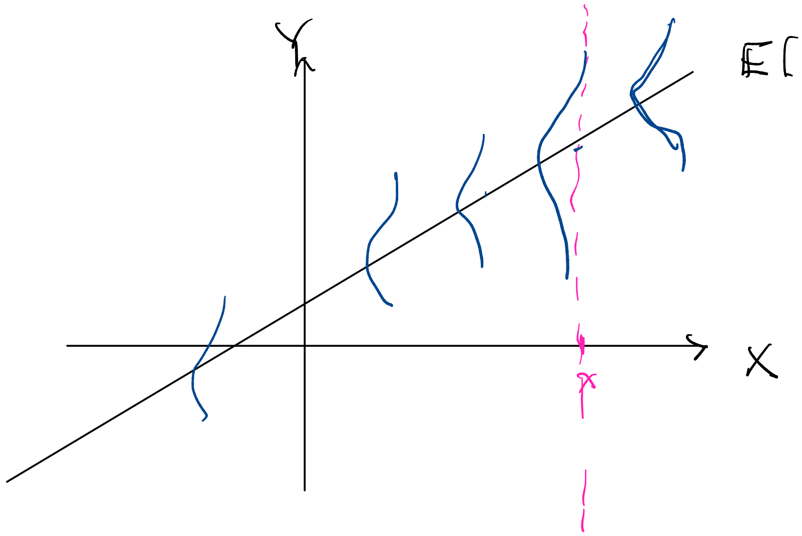
$$= \left(\frac{\bar{x}}{\sigma_x}\right)^2 - 2\rho\left(\frac{\bar{x}}{\sigma_x}\right)\left(\frac{\bar{y}}{\sigma_y}\right) + \left(\frac{\bar{y}}{\sigma_y}\right)^2$$

$$= \begin{bmatrix} \bar{x} & \bar{y} \end{bmatrix} \begin{bmatrix} \sigma_x^2 & \rho\sigma_x\sigma_y \\ \rho\sigma_x\sigma_y & \sigma_y^2 \end{bmatrix}^{-1} \begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix}$$

$$\begin{bmatrix} Cov(X,X) & Cov(X,Y) \\ Cov(Y,X) & Cov(Y,Y) \end{bmatrix}$$

(X, Y) : Bivariate Normal

$\Rightarrow \begin{cases} Y \mid X = x & : \text{Normal} \\ X & : \text{Normal} \end{cases}$



$\mathbb{E}[Y|X=x] = \rho\frac{\sigma_Y}{\sigma_X}(x-\mu_X)+\mu_Y$

Bivariate normal distribution

Example

Let us assume that in a certain population of college students, the respective grade point averages, say X and Y, in high school and the first year of college have a bivariate normal distribution with parameters $\mu_X = 2.9$, $\mu_Y = 2.4$, $\sigma_X = 0.4$, $\sigma_Y = 0.5$, and $\rho = 0.6$.

Find $\mathbb{P}(2.1 < Y < 3.3|X = 3.2)$.

$Y|X=x \sim N\left(\rho\cdot\frac{\sigma_Y}{\sigma_X}(x-\mu_X)+\mu_Y,\ \sigma_Y^2(1-\rho^2)\right)$

$Y|X=3.2 \sim N\left(\underbrace{0.6\,\frac{0.5}{0.4}(3.2-2.9)+2.4}_{m},\ \underbrace{(0.5)^2(1-0.6^2)}_{s^2}\right)$

s

W

$\mathbb{P}(2.1 < Y < 3.3 \mid X=3.2)$

$= \underline{\mathbb{P}(2.1 < W < 3.3)}$ $\qquad W \sim \underline{N(m, s^2)}$

$\frac{W-m}{s} \sim N(0,1)$

$= \mathbb{P}\left(\frac{2.1-m}{s} < Z < \frac{3.3-m}{s}\right)$

$Z \sim N(0,1)$

$= \Phi\left(\frac{3.3-m}{s}\right) - \Phi\left(\frac{2.1-m}{s}\right)$

$=$ Use the table.

Recall X, Y are uncorrelated if $\left\{ \begin{array}{l} Cov(X,Y) = 0 \\ \text{or} \\ \rho = 0 \end{array} \right.$

Fact . If X, Y indep $\Rightarrow$ X, Y uncorrelated

. The converse is not true in general

Bivariate normal distribution

Theorem

If X and Y have a bivariate normal distribution with correlation coefficient ρ, then X and Y are independent if and only if $\rho = 0$.

In other words,

(X, Y) independent $\Longleftrightarrow$ (X, Y) uncorrelated.

$$f(x,y) = \frac{1}{2\pi \sigma_X \sigma_Y \sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)}\left(\frac{\bar{X}^2}{\sigma_X^2} - 2\rho \frac{\bar{X}}{\sigma_X}\frac{\bar{Y}}{\sigma_Y} + \frac{\bar{Y}^2}{\sigma_Y^2}\right)\right)$$

if $\rho = 0$

$$= \frac{1}{\sqrt{2\pi} \cdot \sigma_X \cdot \sqrt{2\pi}\, \sigma_Y} \exp\left(-\frac{1}{2} \cdot \left(\frac{\bar{X}^2}{\sigma_X^2} + \frac{\bar{Y}^2}{\sigma_Y^2}\right)\right)$$

$$= f_X(x) \cdot f_Y(y)$$

Exercise

For a female freshman in a health fitness program, let X equal her percentage of body fat at the beginning of the program and Y equal the change in her percentage of body fat measured at the end of the program.

Assume that X and Y have a bivariate normal distribution with $\Rightarrow Y \sim N(\mu_Y, \sigma_Y^2)$

$\mu_X = 24.5$, $\mu_Y = -0.2$, $\sigma_X = 4.8$, $\sigma_Y = 3$, and $\rho = -0.32$.

Find $\mathbb{P}(1.3 < Y < 5.8)$, $\mathbb{E}[Y|X = x]$, and $\mathrm{Var}(Y|X = x)$.

$Y \sim N(-0.2, 3^2)$

$\frac{Y - (-0.2)}{3} \sim N(0,1)$

$$\mathbb{P}(1.3 < Y < 5.8)$$
$$= \mathbb{P}\left(\frac{1.3-(-0.2)}{3} < Z < \frac{5.8-(0.2)}{3}\right)$$
$$= \mathbb{P}\left(\frac{1.5}{3} < Z < \frac{6}{3}\right) = \mathbb{P}(0.5 < Z < 2)$$
$$= \Phi(2) - \Phi(0.5) \quad \text{(Use the table)}$$

$$\mathbb{E}[Y \mid X = x] = \rho \frac{\sigma_Y}{\sigma_X}(x - \mu_X) + \mu_Y$$

$$\mathrm{Var}(Y \mid X = x) = \underline{\sigma_Y^2(1-\rho^2)}$$

doesnot depend on x