MATH 461 LECTURE NOTE
WEEK 3

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1. Sample spaces having equally likely (Sec 2.5)

Let S be a sample space with finitely many outcomes. For convenience, let $S = \{1, 2, 3, \cdots, N\}$. In many cases, it is natural to assume that all outcomes in S are equally likey to occur. In other words, we assume

$$\mathbb{P}(\{1\}) = \mathbb{P}(\{2\}) = \cdots = \mathbb{P}(\{N\}).$$

By the axioms (ii) and (iii), we have

$$1 = \mathbb{P}(S) = \mathbb{P}(\{1\}) + \mathbb{P}(\{2\}) + \cdots + \mathbb{P}(\{N\}).$$

Therefore, $\mathbb{P}(\{i\}) = \frac{1}{N}$ for each $i = 1, 2, \cdots, N$. Define the probability of an event E by

$$\mathbb{P}(E) = \sum_{i \in E} \mathbb{P}(\{i\}) = \frac{\text{Number of Outcomes in } E}{\text{Number of Outcomes in } S}.$$

Then one can see that $(S, \mathbb{P})$ is a probability space.

Example 1. An urn contains 5 red, 6 blue, and 8 green balls. If a set of 3 balls is randomly selected, what is the probability that each of the balls will be

(i) of the same color?
(ii) of different colors?

Suppose we draw a ball, note its color, and replace it into the urn. If we draw 3 balls in this way, what is the probability that each of the balls are of the same color? or of different colors?

Example 2. If n people are present in a room, what is the probability that no two of them celebrate their birthday on the same day of the year? How large need n be so that this probability is less than $\frac{1}{2}$?

Example 3. A football team consists of 20 offensive and 20 defensive players. The players are to be paired in groups of 2 for the purpose of determining roommates. If the pairing is done at random, what is the probability that there are no offensive–defensive roommate pairs? What is the probability that there are 4 offensive–defensive roommate pairs?

Recall: Inclusion-Exclusion Principle
For events $E_1, E_2, \cdots, E_n$,

$$\begin{aligned}\mathbb{P}(E_1 \cup E_2 \cup \cdots E_n) = &(\mathbb{P}(E_1) + \mathbb{P}(E_2) + \cdots + \mathbb{P}(E_n)) \\ &- (\mathbb{P}(E_1E_2) + \mathbb{P}(E_1E_3) + \cdots + \mathbb{P}(E_iE_j) + \cdots) + \cdots \\ &+ (-1)^{r+1} \sum \mathbb{P}(E_{i_1}E_{i_2} \cdots E_{i_r}) + \cdots \\ &+ (-1)^{n+1}\mathbb{P}(E_1E_2 \cdots E_n).\end{aligned}$$

Example 4. If 4 married couples are arranged in a row, find the probability that no husband sits next to his wife.

Example 5. A closet contains 10 pairs of shoes. If 8 shoes are randomly selected, what is the probability that there will be (a) no complete pair? (b) exactly 1 complete pair?

2. Conditional Probabilities (Sec 3.2)

Let S be a sample space and E, F events. We consider the probability that E occurs given that F has given. This probability is called the conditional probability of E given F and denoted by $\mathbb{P}(E|F)$.

Suppose $S = \{(i,j) : 1 \leq i,j \leq 6\}$, $E = \{(3,j) : 1 \leq j \leq 6\}$, and $F = \{(i,j) : i+j = 8\}$. Assume that each outcome has the same probability. Since $F = \{(2,6),(3,5),(4,4),(5,3),(6,2)\}$ and only $(3,5)$ belongs to E, it is natural to think

$$\mathbb{P}(E|F) = \frac{1}{5} = \frac{\text{Number of outcomes in } E \text{ and } F}{\text{Number of outcomes in } F}.$$

In general, we can define the conditional probability as follows.

Definition: Conditional Probability
If $\mathbb{P}(F) > 0$, then

$$\mathbb{P}(E|F) = \frac{\mathbb{P}(EF)}{\mathbb{P}(F)}.$$

Example 6. A coin is flipped twice. Assuming that all four points in the sample space

$$S = (H,H),(H,T),(T,H),(T,T)$$

are equally likely, what is the conditional probability that both flips land on heads, given that (a) the first flip lands on heads? (b) at least one flip lands on heads?

Example 7. Two cards are randomly chosen without replacement from an ordinary deck of 52 cards. Let B be the event that both cards are aces and let A be the event that at least one ace is chosen. Find $\mathbb{P}(B|A)$.

Computing Probabilities via Conditioning
If $\mathbb{P}(F) > 0$, then

$$\mathbb{P}(EF) = \mathbb{P}(F)\frac{\mathbb{P}(EF)}{\mathbb{P}(F)} = \mathbb{P}(E|F)\mathbb{P}(F).$$

In general,

$$\mathbb{P}(E_1E_2\cdots E_n) = \mathbb{P}(E_1)\mathbb{P}(E_2|E_1)\mathbb{P}(E_3|E_2E_1)\cdots\mathbb{P}(E_n|E_1E_2\cdots E_{n-1}).$$

Example 8. Suppose that an urn contains 8 red balls and 4 white balls. We draw 2 balls from the urn without replacement. If we assume that at each draw each ball in the urn is equally likely to be chosen, what is the probability that both balls drawn are red?

Example 9. An ordinary deck of 52 playing cards is randomly divided into 4 piles of 13 cards each. Compute the probability that each pile has exactly 1 ace.

3. Bayes's Formula (Sec 3.3)

Law of Total Probability. Let S be a sample space and E, F events. Since E can be decomposed into two disjoint events EF and EF^c, we have

$$\mathbb{P}(E) = \mathbb{P}(EF) + \mathbb{P}(EF^c) = \mathbb{P}(E|F)\mathbb{P}(F) + \mathbb{P}(E|F^c)\mathbb{P}(F^c)$$

Example 10. An insurance company believes that people can be divided into two classes: those who are accident prone and those who are not. The company's statistics show that an accident-prone person will have an accident at some time within a fixed 1-year period with probability .4, whereas this probability decreases to .2 for a person who is not accident prone. If we assume that 30 percent of the population is accident prone, what is the probability that a new policyholder will have an accident within a year of purchasing a policy?

LoTP: General version
If $S = \bigcup_{i=1}^n F_i$ and F_i's are mutually disjoint (exclusive), then

$$\mathbb{P}(E) = \sum_{i=1}^n \mathbb{P}(EF_i) = \sum_{i=1}^n \mathbb{P}(E|F_i)\mathbb{P}(F_i).$$

Example 11. There are 15 tennis balls in a box, of which 9 have not previously been used. Three of the balls are randomly chosen, played with, and then returned to the box. Later, another 3 balls are randomly chosen from the box. Find the probability that none of these balls has ever been used.

Bayes formula. When computing conditional probability $\mathbb{P}(F|E)$, sometimes it is the case that computing $\mathbb{P}(E|F)$ is easier. In that case, we use the following identity

$$\mathbb{P}(F|E) = \frac{\mathbb{P}(FE)}{\mathbb{P}(E)} = \frac{\mathbb{P}(E|F)\mathbb{P}(F)}{\mathbb{P}(E)} = \frac{\mathbb{P}(E|F)\mathbb{P}(F)}{\mathbb{P}(E|F)\mathbb{P}(F) + \mathbb{P}(E|F^c)\mathbb{P}(F^c)}.$$

Example 12. A laboratory blood test is 95 percent effective in detecting a certain disease when it is, in fact, present. However, the test also yields a "false positive" result for 1 percent of the healthy persons tested, i.e., if a healthy person is tested, then, with probability .01, the test result will imply that he or she has the disease. If .5 percent of the population actually has the disease, what is the probability that a person has the disease given that the test result is positive?

Bayes formula: General version
If $S = \bigcup_{i=1}^n F_i$ and F_i's are mutually disjoint (exclusive), then

$$\mathbb{P}(F_i|E) = \frac{\mathbb{P}(E|F_i)\mathbb{P}(F_i)}{\mathbb{P}(E)} = \frac{\mathbb{P}(E|F_i)\mathbb{P}(F_i)}{\sum_{i=1}^n \mathbb{P}(E|F_i)\mathbb{P}(F_i)}.$$

Example 13. A plane is missing, and it is presumed that it was equally likely to have gone down in any of 3 possible regions. Let $1 - \beta_i$, $i = 1, 2, 3$, denote the probability that the plane will be found upon a search of the ith region when the plane is, in fact, in that region. (The constants β_i are called overlook probabilities, because they represent the probability of overlooking the plane; they are generally attributable to the geographical and environmental conditions of the regions.) What is the conditional probability that the plane is in the i-th region given that a search of region 1 is unsuccessful?

Example 14. A worker has asked her supervisor for a letter of recommendation for a new job. She estimates that there is an 80 percent chance that she will get the job if she receives a strong recommendation, a 40 percent chance if she receives a moderately good recommendation, and a 10 percent chance if she receives a weak recommendation. She further estimates that the probabilities that the recommendation will be strong, moderate, and weak are .7, .2, and .1, respectively.

(i) How certain is she that she will receive the new job offer?
(ii) Given that she does receive the offer, how likely should she feel that she received a strong recommendation? a moderate recommendation? a weak recommendation?
(iii) Given that she does not receive the job offer, how likely should she feel that she received a strong recommendation? a moderate recommendation? a weak recommendation?

References

[SR] Sheldon Ross, *A First Course in Probability*, 9th Edition, Pearson

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