Midterm 2 Lecture Review Activity, Math 1554

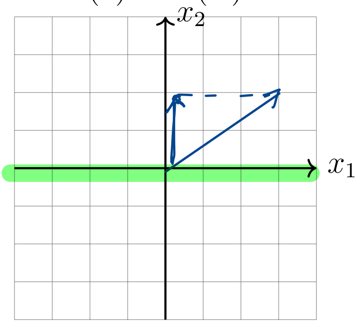
1. (3 points) T_A is the linear transform $x \to Ax$, $A \in \mathbb{R}^{2\times 2}$, that projects points in $\mathbb{R}^2$ onto the x_2-axis. Sketch the nullspace of A, the range of the transform, and the column space of A. How are the range and column space related to each other?

Image of e_1

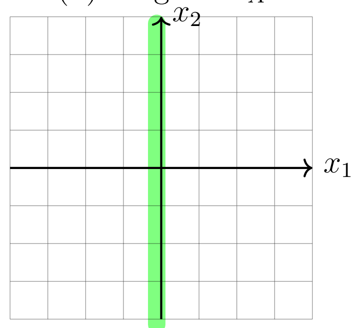
$A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

Image of e_2

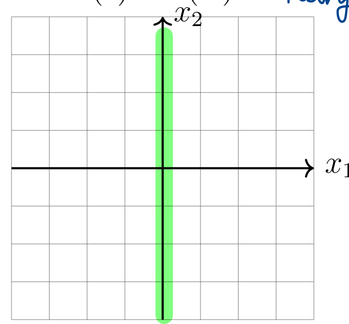
(a) Null(A)



(b) range of T_A



(c) Col(A) = Range (T_A)



2. Indicate **true** if the statement is true, otherwise, indicate **false**.

		true	false
a)	$S = \{\vec{x} \in \mathbb{R}^3 \mid x_1 = a, x_2 = 4a, x_3 = x_1x_2\}$ is a subspace for any $a \in \mathbb{R}$.	○	⊗
b)	If A is square and non-zero, and $A\vec{x} = A\vec{y}$ for some $\vec{x} \neq \vec{y}$, then $\det(A) \neq 0$.	○	⊗

$\begin{bmatrix} a \\ 4a \\ a \cdot 4a \end{bmatrix} = \vec{0}$

$A(\underbrace{\vec{x} - \vec{y}}_{\neq \vec{0}}) = \vec{0}$ $\qquad \det(A) = 0$

3. If possible, write down an example of a matrix or quantity with the given properties. If it is not possible to do so, write *not possible*.

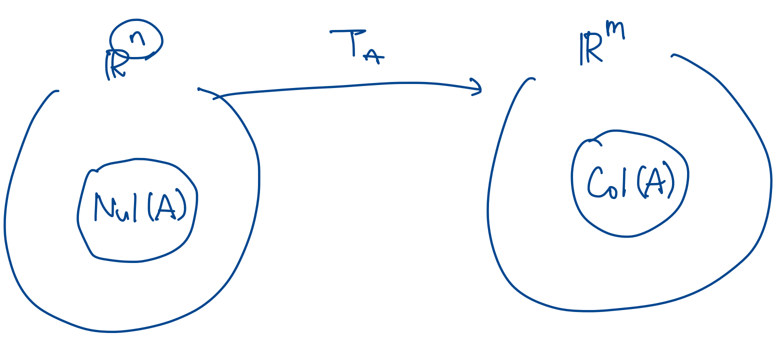
   (a) A is 2×2, Col$A$ is spanned by the vector $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and $\dim(\text{Null}(A)) = 1$. $A = \begin{pmatrix} 2 & -2 \\ 3 & -3 \end{pmatrix}$

   (b) A is $2 \times \overset{2}{2}$, Col$A$ is $\overset{\neq}{\text{spanned}}$ by $\overset{1}{\text{the}}$ vector $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ $\overset{+}{\text{and}}$ $\dim(\text{Null}(A)) = \overset{0}{0}$. $A = \begin{pmatrix} NP \end{pmatrix}$

   (c) A is in RREF and $T_A : \mathbb{R}^3 \to \mathbb{R}^3$. The vectors u and v are a basis for the range of T. $= \text{Col}(A)$.

   $u = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, v = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, A = \begin{pmatrix} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & 0 \end{pmatrix}$

Dimension Thm $\quad A \in \mathbb{R}^{m \times n}$



$n = \dim(\text{Nul}(A)) + \underbrace{\dim(\text{Col}(A))}_{= \text{rank}(A)}$

rank$(A) \leq n$

rank$(A) = n \Rightarrow \text{Nul}(A) = \{\vec{0}\}$

$\Rightarrow T_A$ is one to one

Def T is $1-1$ if $A\vec{x} = A\vec{y}$ implies $\vec{x} = \vec{y}$

$\vec{z} = \vec{x} - \vec{y}$ $\quad A(\vec{x}-\vec{y}) = 0$ implies $\vec{x} - \vec{y} = 0$

$\boxed{A\vec{z} = 0}$ implies $\vec{z} = 0$.

$\mathrm{Nul}(A) = \{ \vec{z} : A\vec{z} = \vec{0} \}$

$= \{ \vec{0} \}$

4. Indicate whether the situations are possible or impossible by filling in the appropriate circle.

$A\vec{u} = \lambda \vec{u}$

$A\vec{v} = \mu \vec{v}$

	possible	impossible
4.i) Vectors $\vec{u}$ and $\vec{v}$ are eigenvectors of square matrix A, and $\vec{w} = \vec{u} + \vec{v}$ is also an eigenvector of A.	X	○
4.ii) $T_A = A\vec{x}$ is one-to-one, $\dim(\mathrm{Col}(A)) = \overset{3}{\cancel{4}}$, and $T_A : \mathbb{R}^3 \to \mathbb{R}^4$.	○	X

$A(\vec{u}+\vec{v}) = A\vec{u} + A\vec{v} = \lambda\vec{u} + \mu\vec{v}$

$= \bigcirc \cdot (\vec{u} + \vec{v})$ only when $\lambda = \mu$

5. (2 points) Fill in the blanks.

   (a) If A is a 6×4 matrix in RREF and $\mathrm{rank}(A) = 4$, what is the rank of A^T? $\boxed{}$

   (b) $T_A = A\vec{x}$, where $A \in \mathbb{R}^{2\times 2}$, is a linear transform that first rotates vectors in $\mathbb{R}^2$ clockwise by π radians about the origin, then scales their x-component by a factor of 3, then projects them onto the x_1-axis. What is the value of $\det(A)$? $\boxed{}$

6. (3 points) A virus is spreading in a lake. Every week,

   - 20% of the healthy fish get sick with the virus, while the other healthy fish remain healthy but could get sick at a later time.
   - 10% of the sick fish recover and can no longer get sick from the virus, 80% of the sick fish remain sick, and 10% of the sick fish die.

   Initially there are exactly 1000 fish in the lake.

   a) What is the stochastic matrix, P, for this situation? Is P regular?

   b) Write down any steady-state vector for the corresponding Markov-chain.

Midterm 2. Your initials: ______________

You do not need to justify your reasoning for questions on this page.

5. (3 points) Find the value of h such that the matrix

$$A = \begin{pmatrix} 5 & h \\ 1 & 3 \end{pmatrix}$$

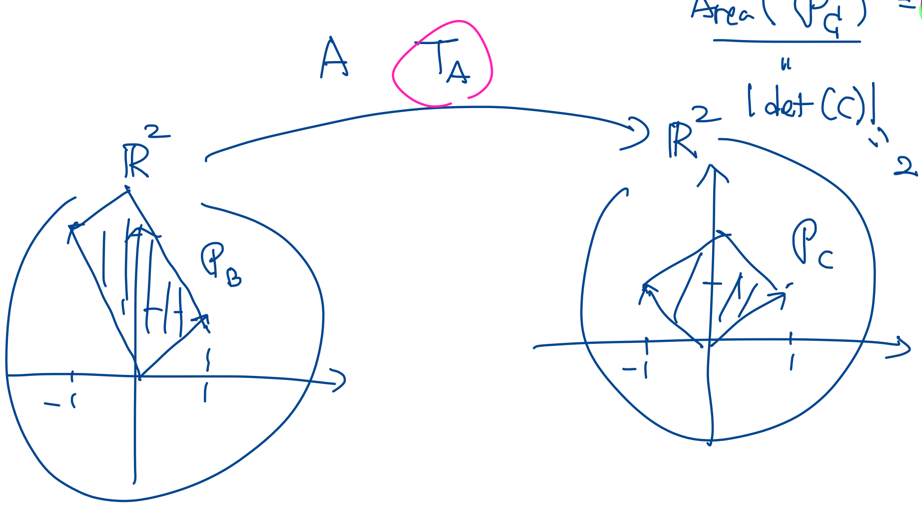
has an eigenvalue with algebraic multiplicity 2.

$$h = \boxed{}$$

6. (3 points) Let $\mathcal{P}_B$ be a parallelogram that is determined by the columns of the matrix $B = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}$, and $\mathcal{P}_C$ be a parallelogram that is determined by the columns of the matrix $C = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$. Suppose A is the standard matrix of a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ that maps $\mathcal{P}_B$ to $\mathcal{P}_C$. What is the value of $|\det(A)|$?

$$|\det(A)| = \boxed{\frac{1}{2}}$$

$$\underbrace{\text{Area}(\mathcal{P}_C)}_{=\,|\det(C)|\,=\,2} = |\det(A)| \cdot \underbrace{\text{Area}(\mathcal{P}_B)}_{=\,|\det(B)|\,=\,4}$$



Midterm 2 Make-up. Your initials: _______________

$$\begin{cases} P v_1 = 0 \cdot v_1 = 0 \\ P \cdot v_2 = \frac{1}{2} v_2 \\ P \cdot v_3 = 1 \cdot v_3 \end{cases}$$

9. (6 points) **Show all work for problems on this page.**
Consider the Markov chain $\vec{x}_{k+1} = P\vec{x}_k, \ k = 0, 1, 2, \ldots$.
Suppose P has eigenvalues $\lambda_1 = 0$, $\lambda_2 = 1/2$ and $\lambda_3 = 1$. Let $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$ be eigenvectors corresponding to λ_1, λ_2, and λ_3, respectively:

$$\vec{v}_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, \quad \vec{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

Note: you may leave your answers as linear combinations of the vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$.
(i) If $\vec{x}_0 = \frac{1}{6}\vec{v}_1 + \frac{1}{3}\vec{v}_2 + \frac{1}{2}\vec{v}_3$, then what is $\vec{x}_2$?

$$\vec{x}_1 = P \cdot \vec{x}_0 = P \cdot \left(\frac{1}{6}\vec{v}_1 + \frac{1}{3}\vec{v}_2 + \frac{1}{2}\vec{v}_3 \right)$$
$$= \frac{1}{6} \cdot \underbrace{P\vec{v}_1}_{0} + \frac{1}{3} \underbrace{P\vec{v}_2}_{\frac{1}{2}v_2} + \frac{1}{2} \underbrace{P\vec{v}_3}_{v_3}$$
$$= \frac{1}{6} v_2 + \frac{1}{2} v_3$$

$\vec{x}_2 =$

$$\vec{x}_2 = P\vec{x}_1 = P \cdot \left(\frac{1}{6}\vec{v}_2 + \frac{1}{2} v_3 \right)$$
$$= \frac{1}{6} \cdot \frac{1}{2} \vec{v}_2 + \frac{1}{2} \cdot 1 \cdot \vec{v}_3$$

(ii) If $\vec{x}_0 = \begin{pmatrix} 1/4 \\ 1/2 \\ 1/4 \end{pmatrix}$, then what is $\vec{x}_1$?
Hint: write $\vec{x}_0$ as a linear combination of $\vec{v}_1, \vec{v}_2, \vec{v}_3$.

$$\vec{x}_0 = a \cdot \vec{v}_1 + b\vec{v}_2 + c \cdot \vec{v}_3$$ Find a, b, c.

$\vec{x}_1 =$

$$\begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ \frac{1}{4} \end{bmatrix} = \begin{bmatrix} -1 & 0 & 1 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} -1 & 0 & 1 & \frac{1}{4} \\ 1 & -1 & 1 & \frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{4} \end{array}\right] \longrightarrow \underline{a, b, c} = \cdots$$

(iii) If $\vec{x}_0 = \begin{pmatrix} 1/4 \\ 1/2 \\ 1/4 \end{pmatrix}$, then what is $\vec{x}_k$ as $k \to \infty$?

$\vec{x}$: prob. vector, $\Rightarrow$ $P \cdot \vec{x}$: a prob. vector

$\lim_{k\to\infty} \vec{x}_k =$ $\frac{1}{2}\vec{v}_3 = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{bmatrix}$.

$$x_0 = \underline{a \cdot v_1} + b \cdot v_2 + c \cdot v_3$$

$$\vec{x}_k = P^k x_0 = b \cdot \left(\tfrac{1}{2}\right)^k \cdot v_2 + c \cdot 1 \cdot v_3 \xrightarrow[k\to\infty]{} \boxed{c \cdot \vec{v}_3}$$

$c \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} c \\ c \\ 0 \end{bmatrix}$ prob. vector. $\Rightarrow$ $c = \frac{1}{2}$.