MATH 461 LECTURE NOTE
WEEK 13

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1. COVARIANCE (SEC 7.4)

Covariance
The covariance between X and Y is defined by

$$\mathrm{Cov}(X,Y) = \mathbb{E}[(X-\mathbb{E}[X])(Y-\mathbb{E}[Y])] = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y].$$

Proposition 1. (i) $\mathrm{Cov}(X,Y) = \mathrm{Cov}(Y,X)$, $\mathrm{Cov}(X,X) = \mathrm{Var}(X)$, *and* $\mathrm{Cov}(aX+b,Y) = a\,\mathrm{Cov}(X,Y)$.
(ii) *We have*

$$\mathrm{Cov}(\sum_{i=1}^n X_i, \sum_{j=1}^m Y_j) = \sum_{i=1}^n \sum_{j=1}^m \mathrm{Cov}(X_i, Y_j).$$

In particular,

$$\mathrm{Var}(\sum_{i=1}^n X_i) = \sum_{i=1}^n \mathrm{Var}(X_i) + 2\sum_{i<j} \mathrm{Cov}(X_i, X_j).$$

Correlation coefficient
The correlation between X and Y is defined by

$$\rho(X,Y) = \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)\,\mathrm{Var}(Y)}}.$$

We say X and Y are uncorrelated if $\rho(X,Y) = 0$.

Remark 2. If X and Y are independent, then $\rho(X,Y) = 0$. Does the converse hold?

Proposition 3. (i) $-1 \le \rho(X,Y) \le 1$.
(ii) *For* $a > 0$, $\rho(aX+b,Y) = \rho(X,Y)$.
(iii) *If* $\rho(X,Y) = \pm 1$, *then* $Y = aX + b$.

Example 4. Toss a fair coin 3 times. Let X be the number of heads, and Y be the number of tails. Find $\mathrm{Cov}(X,Y)$.

Example 5. Let $X_1, X_2, \cdots$ be independent random variables with common mean μ and common variance σ_2. Set $Y_n = X_n + 2X_{n+1}$ for $n \ge 1$. For $j \ge 0$, find $\mathrm{Cov}(Y_n, Y_{n+j})$ and $\rho(Y_n, Y_{n+j})$.

REFERENCES

[SR] Sheldon Ross, *A First Course in Probability*, 9th Edition, Pearson

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