Chapter 2. Discrete Distributions

Math 3215 Summer 2023

Georgia Institute of Technology

Section 1.
Random Variables of the Discrete Type

$S(X) = \{0, 1\} \Leftarrow \quad X : S \to \mathbb{R}$ s.t. $X = \begin{cases} 0, & \text{Heads} \\ 1, & \text{Tails} \end{cases}$

Ex Tossing a fair coin $\quad S = \{H, T\}$

Events $= \phi, \{H\}, \{T\}, \{H, T\} = S$

$\mathbb{P}$: $\phi \to 0$, $\{H\} \to \frac{1}{2}$, $\{T\} \to \frac{1}{2}$, $\{H,T\} \to 1$

$H \to 0$, $T \to 1$ $\leftarrow$ Random variable.

$\{\text{Outcomes}\} = \text{Sample space}$. $\quad S = [0, 1]$

$\cup$

Subsets $\Leftarrow$ Events.

$\mathbb{P} : \underbrace{\{\text{Events}\}}_{S} \rightarrow [0, 1]$

- $\mathbb{P}(A) \geq 0$
- $\mathbb{P}(S) = 1$
- $\forall$ $A_1, A_2, \dots, A_k$ mutually exclusive

$\mathbb{P}(A_1 \cup A_2 \cup \dots \cup A_k)$
$= \mathbb{P}(A_1) + \mathbb{P}(A_2) + \dots + \mathbb{P}(A_k)$

Random variables

$$X : S \longrightarrow \mathbb{R}$$

Definition

Given a random experiment with a sample space S, a function X that assigns one and only one real number $X(s) = r$ to each elements in S is called a random variable.

elements = outcome

The space of X is the set of real numbers $\{x : X(s) = x, s \in S\}$ and denoted by $S(X)$.

Random variables

Example

A rat is selected at random from a cage and its sex is determined.

The set of possible outcomes is female and male. Thus, the sample space is $S = \{\text{female, male}\}$. $= \{F, M\}$

$\{\text{Events}\} = \{ \phi, \{F\}, \{M\}, S\}$

$\mathbb{P}(\phi) = 0, \quad \mathbb{P}(S) = 1$

$\mathbb{P}(\{F\}) = \frac{1}{2} = \mathbb{P}(\{M\})$

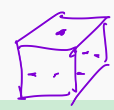
Define a random variable $X : \underset{\{F,M\}}{\overset{\shortparallel}{S}} \to \mathbb{R}$

$X(F) = \frac{1}{2}, \quad X(M) = -\frac{1}{2}$

$$X = \begin{cases} \frac{1}{2}, & \text{Female} \\ -\frac{1}{2}, & \text{Male}. \end{cases}$$

$S(X) = \{ \frac{1}{2}, -\frac{1}{2} \}$

Random variables



Example

Consider a random experiment in which we roll a six-sided die.

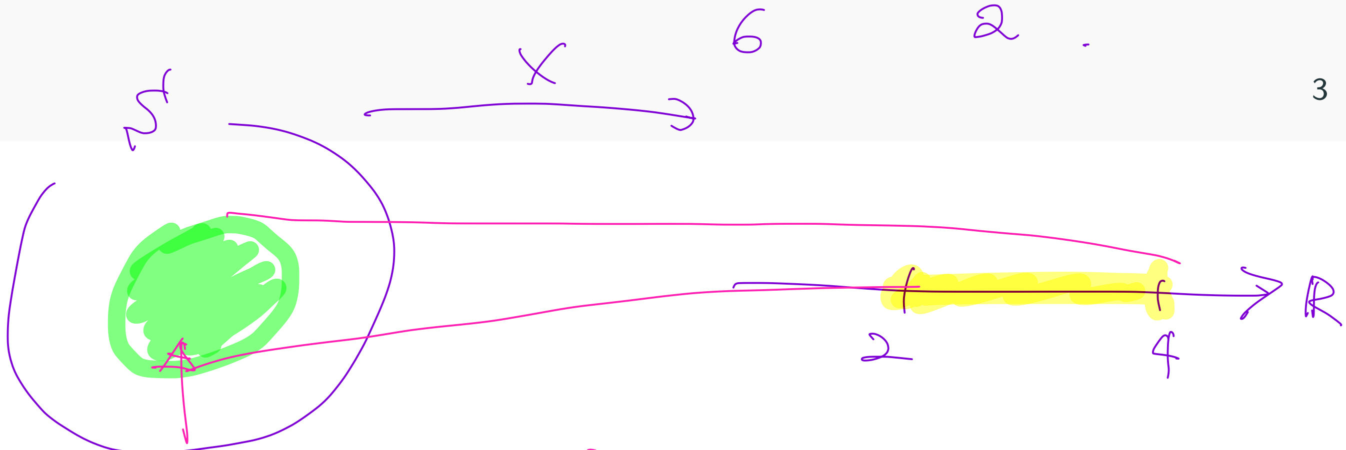
How many? $2^6 = 64$.

The sample space associated with this experiment is
$S = \{1, , 2, 3, 4, 5, 6\}$.

$\{\text{Events}\} = \{\emptyset, \{1\}, \ldots\}$

Let $\boxed{X(s) = s.}$ Compute $\mathbb{P}(2 \leq X \leq 4)$.

$$\mathbb{P}(X = 2, 3, 4) = \mathbb{P}(\{2, 3, 4\})$$
$$= \frac{3}{6} = \frac{1}{2}.$$



$\mathbb{P}(\{2,3,4\}) = \frac{1}{2}$

$\mathbb{P}(2 \leq X \leq 4)$

Discrete random variables

Definition

Let X be a random variable defined on a sample space S.

If S consists of finite outcomes or coutable outcomes, then X is called a discrete random variable.

{Natural numbers}

{ Integers }

{ rational number }

The probability mass function of X is

(pmf)

$$f(x) = \mathbb{P}(X = x)$$

$f(x) = \mathbb{P}(X = x)$

Discrete random variables

$S(x) =$ the space of X

Properties of PMF

The pmf $f(x)$ of a discrete random variable X is a function that satisfies the following properties:

- $f(x) \geq 0$ for all x, $x \in \mathbb{R}$, $\in S(X)$ $\leftarrow$ $\mathbb{P}(A) \geq 0$
- $\sum_{x \in S(X)} f(x) = 1$, and $\leftarrow$ $\mathbb{P}(S) = 1$
- $\mathbb{P}(X \in A) = \sum_{x \in A} f(x)$.

$\mathbb{P}(S) =$

$X(a_1)\ X(a_2)\ \ X(a_n)$

$\{x_1, x_2, \cdots, x_n\} \subseteq S(x)$

$$\mathbb{P}(X \in A) = \mathbb{P}(\{a_1, \cdots, a_n\})$$
$$= \mathbb{P}(\{a_1\}) + \mathbb{P}(\{a_2\}) + \cdots + \mathbb{P}(\{a_n\})$$
$$= \mathbb{P}(X = x_1) + \cdots + \mathbb{P}(X = x_n)$$
$$= \sum f(x_i)$$

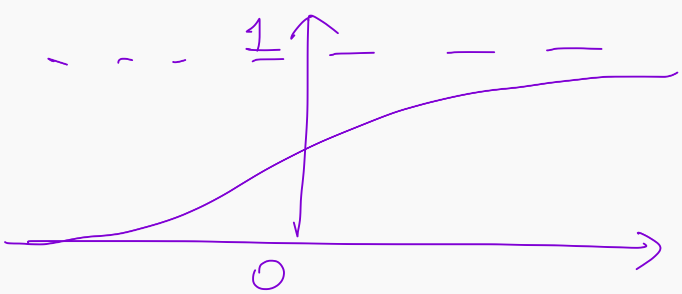
$f(x) = P(X = x)$: pmf

Discrete random variables

(d.f.)

The cumulative distribution function of X is

$$F(x) = P(X \le x)$$



$\lim_{x \to -\infty} F(x) = 0$

$\lim_{x \to \infty} F(x) = 1$

F : increasing function.

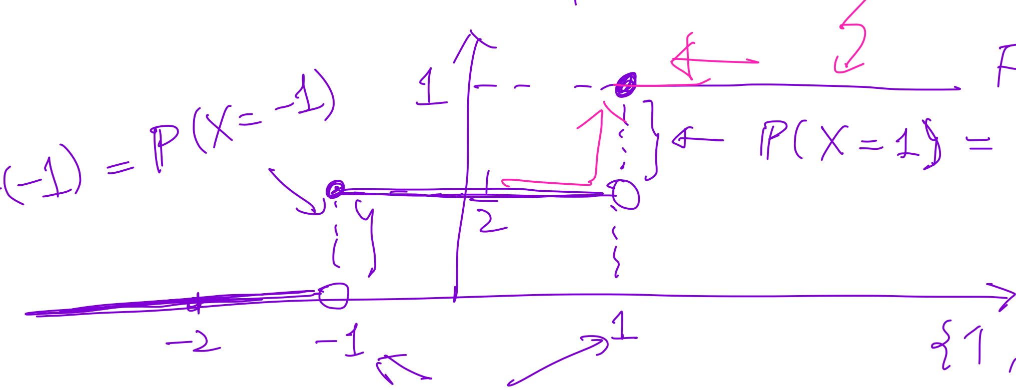
Ex $X = \begin{cases} 1, & H \\ -1, & T \end{cases}$

right continuous

$F(-2) = P(X \le -2)$

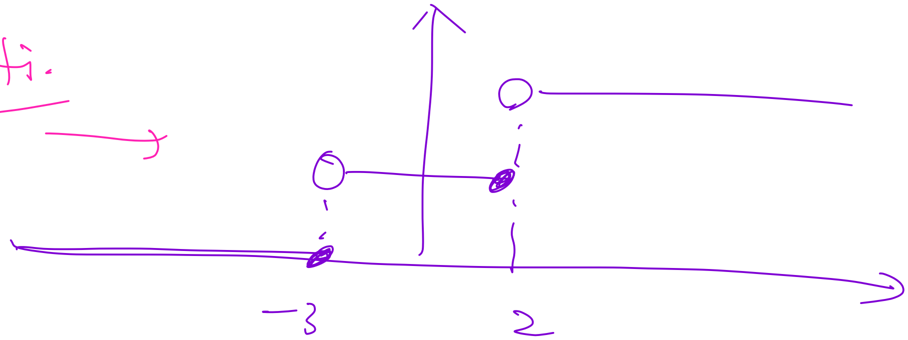
$P(X = 1) = f(1)$

$f(-1) = P(X = -1)$



$\{1, -1\}$ = Space of X

left conti.



N.P.

$F(x) = \mathbb{P}(X \leq x)$

Discrete random variables

$S = \{1, 2, 3, 4, 5, 6\}$

$f: \mathbb{R} \to \mathbb{R}$

$X = \begin{cases} 1 \\ 2 \\ \vdots \\ 6 \end{cases}$

Example

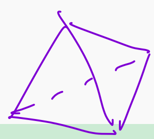
Roll a die, let X be the outcome.

Find the pmf and the cdf of X.

$$f(x) = \mathbb{P}(X = x) = \begin{cases} \frac{1}{6} & \text{when } x = 1, 2, \cdots, 6 \\ 0 & \text{otherwise} \end{cases}$$

$$F(x) = \mathbb{P}(X \leq x) = \begin{cases} 0, & x \in (-\infty, 1) \\ \frac{1}{6}, & x \in [1, 2) \\ \frac{2}{6}, & x \in [2, 3) \\ \vdots \end{cases}$$

Discrete random variables



$S = \{(1,1), (1,2), \ldots\}$

$X : S \to \mathbb{R}$

$(1,1) \longmapsto 1$

$(1,2) \longmapsto 2$

Example

Roll a fair four-sided die twice.

Let X equal the larger of the two outcomes if they are different and the common value if they are the same.

Find the pmf and the cdf of X.

① $S(X) = \{1, 2, 3, 4\}$

② $$f(x) = \mathbb{P}(X = x) = \begin{cases} 1/16 & x = 1 \\ 3/16 & x = 2 \\ 5/16 & x = 3 \\ 7/16 & x = 4 \\ 0 & \text{otherwise} \end{cases}$$

③ $F(x) = \ldots$

Exercise

Let X be a discrete random variable with pmf $f(x) = \log_{10}(\frac{x+1}{x})$ for $x = 1, 2, \cdots, 9$. (a) Verify that $f(x)$ satisfies the conditions of a pmf. (b) Find the cdf of X.

$$f(x) = \begin{cases} \log_{10}\frac{2}{1} \geq 0 & x=1 \\ \log_{10}\frac{3}{2} & x=2 \\ \vdots & \end{cases}$$

$(\log_{10} t \geq 0 \text{ when } t \geq 1)$

$$\log_{10}\frac{2}{1} + \log_{10}\frac{3}{2} + \log_{10}\frac{4}{3} + \cdots + \log_{10}\frac{10}{9}$$

$$= \log_{10}\left(\frac{2}{1}\cdot\frac{3}{2}\cdots\frac{10}{9}\right)$$

$$= \log_{10} 10 = 1$$

(b) $$F(x) = \mathbb{P}(X \leq x) = \begin{cases} 0 & x < 1 \\ \log_{10} 2 & 1 \leq x < 2 \\ \log_{10} 3 & 2 \leq x < 3 \\ \vdots & \\ \log_{10} k & k-1 \leq x < k, \quad k = 1, \cdots, 10 \\ 1 & x \geq 10 \end{cases}$$

Recall $X : S \to \mathbb{R}$: a random variable.

S = {outcomes}

X is a discrete RV if S is finite or countable

pmf : $f(x) = P(X = x)$ — • $f(x) \geq 0$ • $\sum f(x) = 1$ • $P(X \in A) = \sum_{x \in A} f(x)$

cdf : $F(x) = P(X \leq x)$ — $0 \leq F(x) \leq 1$, non-decreasing

Bar graph, Probability histogram, relative frequency histogram

$S = \{(1,1), (1,2), (1,3), (1,4), (2,1), (2,2), \dots\}$ → 2, 3, 4, ...

Example

A fair four-sided die with outcomes 1, 2, 3, and 4 is rolled twice.

Let X equal the sum of the two outcomes.

$S(X)$ = the space of X = $\{2, 3, 4, 5, \dots, 8\}$

PMF : $f(x) = P(X = x) = \begin{cases} 1/16 & x = 2, 8 \\ 2/16 & x = 3, 7 \\ 3/16 & x = 4, 6 \\ 4/16 & x = 5 \\ 0 & \text{otherwise} \end{cases}$

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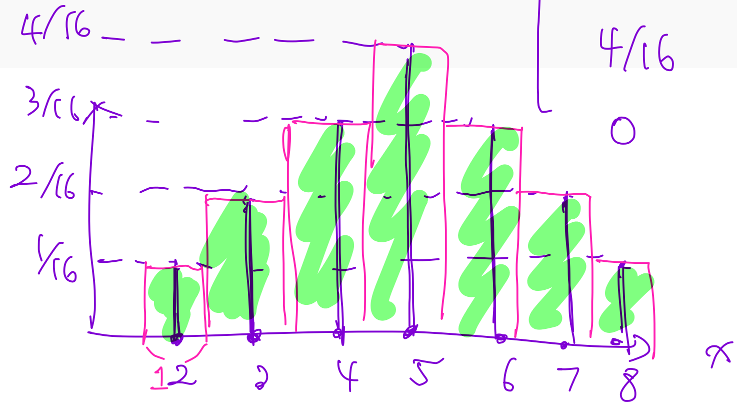
Bar Graph

Prob. Histogram.



probability

$\uparrow$ $n \to \infty$

relative frequency $= \dfrac{\text{\# of occurrences}}{\text{total \# of Exp}}$

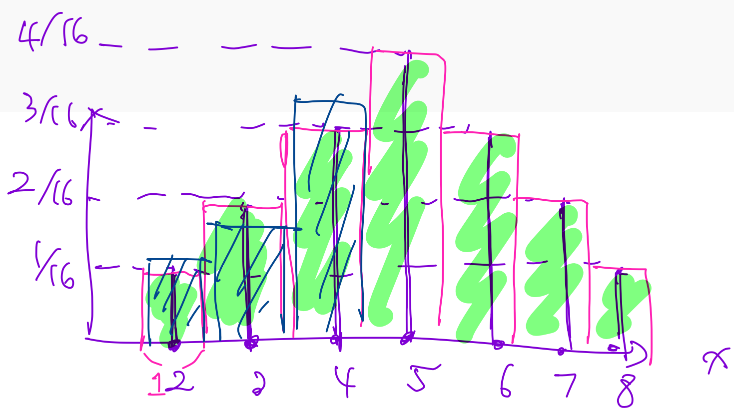
Bar graph, Probability histogram, relative frequency histogram

Example

Two fair four-sided dice are rolled. Write down the sum of the two outcomes. Repeat this 1000 times.

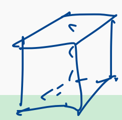
The sum of two outcomes	2	3	4	5	6	7	8
Number of Observations	71	124	194	258	177	122	54

$\frac{71}{1000} \to P(X=2)$ $\quad$ $\frac{124}{1000} \to P(X=3)$ ...



Section 2.
Mathematical Expectation

Definition of Expectation



Example

Consider the following game. A player roll a fair die.

If the event $A = \{1, 2, 3\}$ occurs, he receives one dollar; if $B = \{4, 5\}$ occurs, he receives two dollars; and if $C = \{6\}$ occurs, he receives three dollars.

$X = \text{Payment}$

If the game is repeated a large number n of times, what is the average payment?

Repeat n times $\quad n = a + b + c$

A happened a times $\longrightarrow$ \$ $a \cdot 1$

B " b times $\longrightarrow$ \$ $b \cdot 2$

C " c times $\longrightarrow$ \$ $c \cdot 3$

Total Payment $= 1 \cdot a + 2 \cdot b + 3 \cdot c$

$$\text{Average} = \frac{1 \cdot a + 2 \cdot b + 3 \cdot c}{n}$$

relative frequency

$$= 1 \cdot \underbrace{\frac{a}{n}}_{\mathbb{P}(X=1)} + 2 \cdot \underbrace{\frac{b}{n}}_{\mathbb{P}(X=2)} + 3 \cdot \underbrace{\frac{c}{n}}_{\mathbb{P}(X=3)}$$

Average of X = Expectation of X = $\mathbb{E}[X]$

$= 1 \cdot \mathbb{P}(X=1) + 2 \cdot \mathbb{P}(X=2) + 3 \cdot \mathbb{P}(X=3)$

$$= \sum_{x \in S(X)} x \cdot \underbrace{\mathbb{P}(X=x)}_{\text{pmf}} = \sum_{x \in S(X)} x \cdot f(x)$$

Definition of Expectation

Definition

If $f(x)$ is the pmf of a discrete random variable X with the space $S(X)$, and if the summation

$$\sum_{x \in S(X)} u(x) f(x)$$

exists, then the sum is called the mathematical expectation or the expected value of $u(X)$, and denoted by $\mathbb{E}[u(X)]$.

$$\boxed{\mathbb{E}[u(X)] = \sum_{x \in S(X)} u(x) \cdot f(x)}$$

Ex) • $u(x) = x$

$u(X) = X$

• $u(x) = x^2$

$u(X) = X^2$

$\mathbb{E}[X] = \sum x f(x)$

$\mathbb{E}[X^2] = \sum x^2 \cdot f(x)$

$\mathbb{E}[e^X] = \sum e^x \cdot f(x)$ . . .

$$\boxed{f(k) = \frac{c}{k^2}} = \mathbb{P}(X=k), \quad \underline{k = 1, 2, 3, \ldots}$$

How to determine c?

$$1 = \sum_{k=1}^{\infty} f(k) = c \cdot \underbrace{\sum_{k=1}^{\infty} \frac{1}{k^2}}_{= \frac{\pi^2}{6}}$$

$$\mathbb{E}[X] = \sum_{k=1}^{\infty} k \cdot f(k) = \sum_{k=1}^{\infty} \cancel{k} \cdot \frac{c}{k^{\cancel{2}}} = c \cdot \underbrace{\sum_{k=1}^{\infty} \frac{1}{k}}_{= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots \to \infty}$$

Definition of Expectation

Example

Let the random variable X have the pmf $f(x) = \frac{1}{3}$ for $x \in \{-1, 0, 1\} = S(X)$.

Let $Y = u(X) = X^2$. Find the pmf of Y and $\mathbb{E}[Y] = \mathbb{E}[X^2]$.

① $\mathbb{E}[Y] = \mathbb{E}[\underbrace{u(X)}_{= X^2}] = \sum_{x=-1,0,1} x^2 \cdot f(x)$

$= (-1)^2 \cdot \underbrace{f(-1)}_{=\frac{1}{3}} + 0^2 \cdot \underbrace{f(0)}_{=\frac{1}{3}} + 1^2 \cdot \underbrace{f(1)}_{=\frac{1}{3}}$

$= \frac{2}{3}$.

② $Y = X^2$ a RV $\longrightarrow$ $f_Y(y) = \mathbb{P}(Y=y) = \begin{cases} \frac{1}{3}, & y=0 \\ \frac{2}{3}, & y=1 \end{cases}$

$S(Y) = \{0, 1\}$

$\mathbb{E}[Y] = \sum y \cdot f_Y(y) = 0 \cdot \frac{1}{3} + 1 \cdot \frac{2}{3} = \frac{2}{3}$.

Linearity.

Properties of Expectation

In particular

$$\mathbb{E}[aX + b] = a\,\mathbb{E}[X] + b.$$

Theorem

1. If c is a constant, then $\mathbb{E}[c] = c$.
2. If c is a constant and u is a function, then $\mathbb{E}[cu(X)] = c\mathbb{E}[u(X)]$.
3. If c_1 and c_2 are constants and u_1 and u_2 are functions. then

$$\mathbb{E}[c_1 u_1(X) + c_2 u_2(X)] = c_1\mathbb{E}[u_1(X)] + c_2\mathbb{E}[u_2(X)].$$

② $$\mathbb{E}[c \cdot u(X)] = \sum c\, u(x) \cdot f(x)$$

$$= c \cdot \sum u(x) \cdot f(x) = c \cdot \mathbb{E}[u(X)]$$

③ $$\mathbb{E}[c_1 \cdot u_1(X) + c_2 u_2(X)]$$

$$= \sum (c_1 \cdot u_1(x) + c_2 \cdot u_2(x))\, f(x)$$

$$= \sum c_1\, u_1(x)\, f(x) + \sum c_2\, u_2(x)\, f(x)$$

$$= c_1 \sum u_1(x) \cdot f(x) + c_2 \sum u_2(x)\, f(x) = c_1\mathbb{E}[u_1(X)]$$

$$+ c_2\,\mathbb{E}[u_2(X)]$$

Properties of Expectation

Example

Let X have the pmf $f(x) = \frac{x}{10}$ for $x = 1, 2, 3, 4$.

Find $\mathbb{E}[X]$, $\mathbb{E}[X^2]$ and $\mathbb{E}[X(5-X)]$.

$$\mathbb{E}[X] = \sum_{x=1}^{4} x \cdot f(x) = 1\cdot f(1) + 2\cdot f(2) + 3\cdot f(3) + 4\, f(4)$$

$$= 1\cdot\frac{1}{10} + 2\cdot\frac{2}{10} + 3\cdot\frac{3}{10} + 4\cdot\frac{4}{10}$$

$$= 3$$

$$\mathbb{E}[X^2] = \sum_{x=1}^{4} x^2\cdot f(x) = 1^2\cdot f(1) + 2^2\cdot f(2) + 3^2\cdot f(3) + 4^2 f(4)$$

$$= 1^2\cdot\frac{1}{10} + 2^2\cdot\frac{2}{10} + 3^2\cdot\frac{3}{10} + 4^2\cdot\frac{4}{10}$$

$$= \frac{1 + 8 + 27 + 64}{10} = 10.$$

$$\mathbb{E}[X(5-X)] = \sum_{x=1}^{4} (x(5-x))\cdot f(x) = \cdots$$

$$= \mathbb{E}[5X - X^2] = 5\,\mathbb{E}[X] - \mathbb{E}[X^2] = 5.$$

Note : $\mathbb{E}[X \cdot (X-1)] \neq \mathbb{E}[X] \cdot \mathbb{E}[(X-1)]$.

Properties of Expectation

Example

An experiment has probability of success $p \in (0,1)$ and probability of failure $q = 1 - p$.

This experiment is repeated independently until the first success occurs.

Let X be the number of trials. Find $\mathbb{E}[X]$.

Exercise

An insurance compan sells an automobile policy with a deductible of one unit. Let X be the amount ofthe loss having pmf

$$f(x) = \begin{cases} 0.9 & x = 0, \\ \frac{c}{x} & x = 1, 2, 3, 4, 5, 6, \end{cases}$$

where c is a constant. Determine c and the expected value of the amount the insurance company must pay.

① $1 = \underline{0.9} + \frac{c}{1} + \frac{c}{2} + \cdots + \frac{c}{6}$

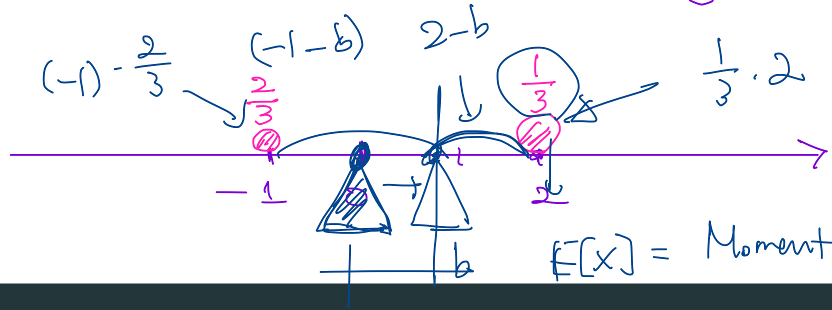
$\frac{1}{10} = c \cdot \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{6}\right), \quad c = \frac{1}{10 \cdot \left(\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{6}\right)}$

② $E[X] = \sum \underline{x} \cdot f(x)$

$= \sum_{x=1}^{6} \cancel{x} \cdot \frac{c}{\cancel{x}} = 6 \cdot c = \frac{6}{10 \cdot \left(1 + \frac{1}{2} + \cdots + \frac{1}{6}\right)}$

Section 3.
Special Mathematical Expectations

$$X = \begin{cases} 2 & , \text{ with prob. } \frac{1}{3} \\ -1 & , \quad " \quad \frac{2}{3} \end{cases}$$



$\mathbb{E}[X] = \text{Moment} = \frac{1}{3}\cdot 2 + \frac{2}{3}\cdot(-1) = 0$

Moments

First moment about $x = b$ $= \mathbb{E}[(X-b)]$

$\overset{(Ex)}{=} (2-b)\cdot\frac{1}{3} + (-1-b)\cdot\frac{2}{3}$.

The expectation or mean of a random variable X is

$$\mu = \mathbb{E}[X] = \sum x f(x).$$

This is also called the first moment about the origin.

The first moment about the mean μ is $\mathbb{E}[X-\mu] = \mathbb{E}[X - \mathbb{E}[X]]$

If $b = \mu = \mathbb{E}[X]$, $\quad = \mathbb{E}[X] - \mathbb{E}[X] = 0$.

$X_1 \longrightarrow Y_1 = X_1 - b_1$

$X_2 \qquad\quad Y_2 = X_2 - b_2$

$\vdots$

Moments

1st moment of X about b = $\mathbb{E}[(X-b)]$

The second moment of X about b is $\mathbb{E}[(X-b)^2]$.

If $b = \mu$, it is also called the variance of X and denoted by $\text{Var}(X) = \sigma^2$.

μ = $\mathbb{E}[X]$

Its positive square root is the standard deviation of X and denoted by $\text{Std}(X) = \sigma$.

$$\mathbb{E}[X] = \mu$$

$$\text{Var}(X) = \mathbb{E}[(X-\mu)^2] = \sigma^2$$

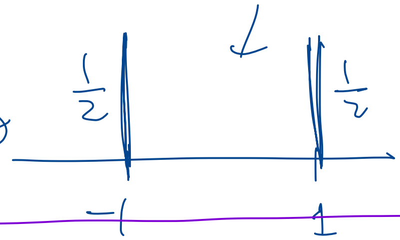
$$\text{Std}(X) = \sqrt{\text{Var}(X)} = \sigma$$

Ex

$\text{Var}(X) = 1$

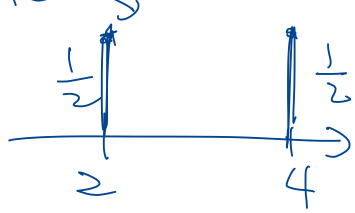
$X = \begin{cases} 1 & \text{w/ prob. } \frac{1}{2} \\ -1 & \text{" } \frac{1}{2} \end{cases}$

$\mathbb{E}[X] = 0$



$\mathbb{E}[Y] = 3$ $\text{Var}(Y) = 1$

$Y = \begin{cases} 2 & \text{, w.p. } \frac{1}{2} \\ 4 & \text{w.p. } \frac{1}{2} \end{cases}$



$\mathbb{E}[Z] = 10000$

$Z = \begin{cases} 100 \\ -100 \end{cases}$

Moments

Example

Roll a fair die and let X be the outcome.

Find $\mathbb{E}[X]$ and $\mathrm{Var}(X)$.

$$\mathbb{E}[X] = \sum_{x} x \cdot \underbrace{f(x)}_{=\frac{1}{6}} = \frac{1}{6}\cdot(1+2+\cdots+6) = \frac{21}{6} = \frac{7}{2} = \mu$$

$$\mathrm{Var}(X) = \mathbb{E}\Big[\big(X - \overset{\mu}{\tfrac{7}{2}}\big)^2\Big]$$

$$= \sum_{x} \left(x - \frac{7}{2}\right)^2 \cdot f(x)$$

$$= \frac{1}{6}\cdot\Big(\big(\overbrace{1-\tfrac{7}{2}}^{-\frac{5}{2}}\big)^2 + \big(\overbrace{2-\tfrac{7}{2}}^{-\frac{3}{2}}\big)^2 + \big(\overbrace{3-\tfrac{7}{2}}^{-\frac{1}{2}}\big)^2 + \big(\underbrace{4-\tfrac{7}{2}}_{\frac{1}{2}}\big)^2 + \big(\underbrace{5-\tfrac{7}{2}}_{\frac{3}{2}}\big)^2 + \big(\underbrace{6-\tfrac{7}{2}}_{\frac{5}{2}}\big)^2 \Big)$$

$$= 2\cdot\frac{1}{6}\cdot\left(\frac{5^2}{4} + \frac{3^2}{4} + \frac{1^2}{4}\right) = \frac{35}{12}$$

Moments

Theorem

$\mathrm{Var}(X) = \sigma^2 = \mathbb{E}[(X-\mu)^2] = \mathbb{E}[X^2] - \mu^2 = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$

$(a-b)^2$

Proof) $\mathrm{Var}(X) = \mathbb{E}[\underbrace{(X - \mathbb{E}[X])^2}]$

$= \mathbb{E}[X^2 - 2X \cdot \mathbb{E}[X] + (\mathbb{E}[X])^2]$

constant

$= \mathbb{E}[X^2] - 2 \cdot \underbrace{\mathbb{E}[X \cdot \mathbb{E}[X]]} + (\mathbb{E}[X])^2$

$\mathbb{E}[X]\ \mathbb{E}[X]$

$= \mathbb{E}[X^2] - 2 \cdot (\mathbb{E}[X])^2 + (\mathbb{E}[X])^2$

$= \mathbb{E}[X^2] - (\mathbb{E}[X])^2$ $\blacksquare$

Properties

① $\mathrm{Var}(c \cdot X) = \mathbb{E}[(cX)^2] - (\mathbb{E}[cX])^2$

$= c^2 \cdot \mathbb{E}[X^2] - c^2 \cdot (\mathbb{E}[X])^2$

$= c^2 \cdot (\mathbb{E}[X^2] - (\mathbb{E}[X])^2) = c^2 \cdot \mathrm{Var}(X)$

Moments

$\mathrm{Var}(c) = \mathbb{E}[c^2] - (\mathbb{E}[c])^2 = c^2 - c^2 = 0$

constant

Theorem

$\sigma^2 = \mathbb{E}[(X - \mu)^2] = \mathbb{E}[X^2] - \mu^2$

② $\mathrm{Var}(aX + b) = a^2 \mathrm{Var}(X)$

Exercise.

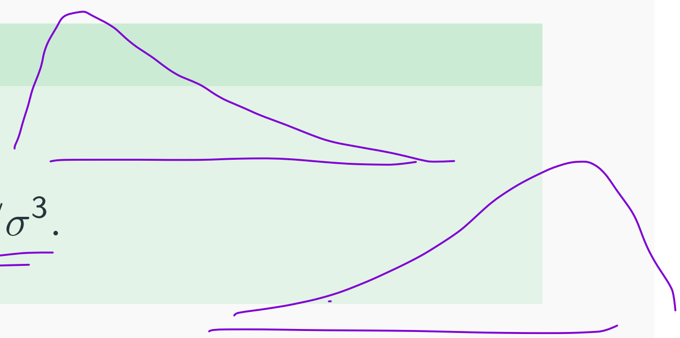
Moments

In general, the r-th moment of X about b is $\mathbb{E}[(X-b)^r]$.

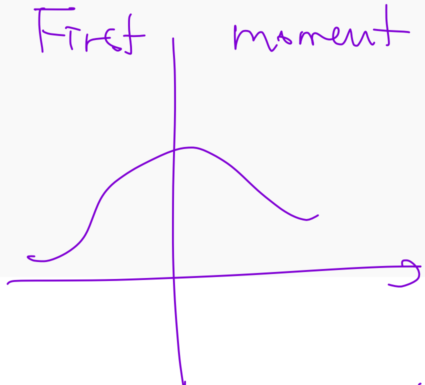
Definition

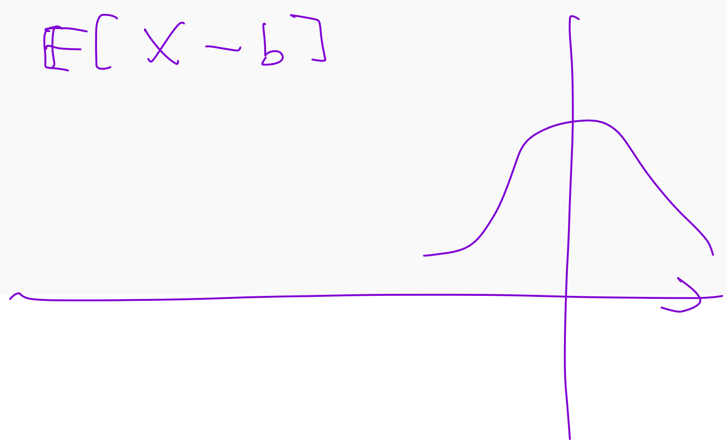
Index of skewness is defined by

$$\gamma = \mathbb{E}[(X-\mu)^3]/\sigma^3.$$

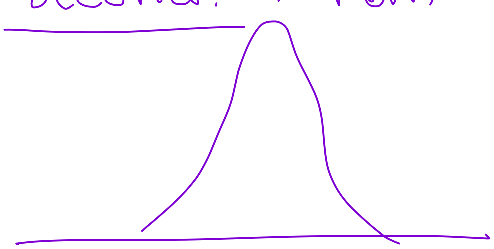


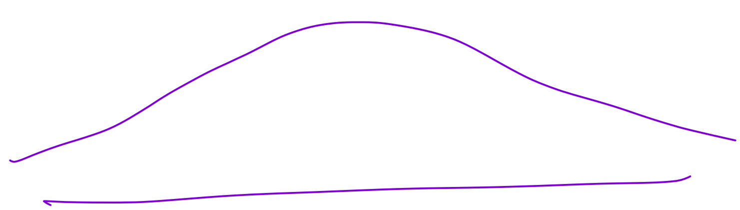
First moment $\mathbb{E}[X-b]$





Second moment



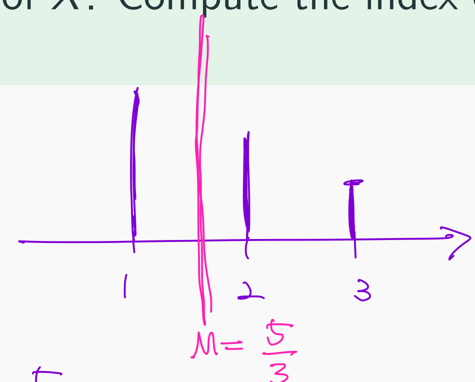


Moments

Example

Let $f(x) = \frac{4-x}{6}$ for $x = 1, 2, 3$ be the pmf of X. Compute the index of skewness.

$$f(x) = \begin{cases} \frac{1}{2} & x=1 \\ \frac{1}{3} & x=2 \\ \frac{1}{6} & x=3 \end{cases}$$



$$E[X] = 1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{3} + 3 \cdot \frac{1}{6} = \frac{5}{3}.$$

$$Var(X) = E[X^2] - (E[X])^2 = \left(1^2 \cdot \frac{1}{2} + 2^2 \cdot \frac{1}{3} + 3^2 \cdot \frac{1}{6}\right) - \left(\frac{5}{3}\right)^2$$

$$= \frac{10}{3} - \frac{25}{9} = \frac{5}{9} = \sigma^2$$

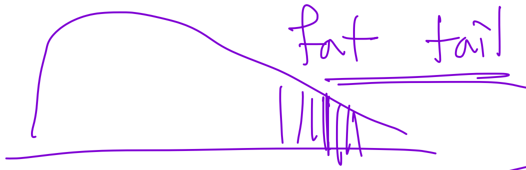
$$Std(X) = \sqrt{\frac{5}{9}} = \frac{\sqrt{5}}{3}.$$

$$\text{Index of Skewness} = \frac{E[(X-\mu)^3]}{\sigma^3}$$

$$= \frac{1}{\left(\frac{\sqrt{5}}{3}\right)^3}\left(\left(1-\frac{5}{3}\right)^3 \cdot \frac{1}{2} + \left(2-\frac{5}{3}\right)^3 \cdot \frac{1}{3} + \left(3-\frac{5}{3}\right)^3 \frac{1}{6} \right)$$

$$= \frac{27}{5\sqrt{5}} \cdot \left(\left(- \frac{\cancel{8}^{\,4}}{27} \right) \cdot \frac{1}{\cancel{8}} + \frac{1}{27} \cdot \frac{1}{3} + \frac{\cancel{64}^{\,32}}{27} \cdot \frac{1}{\cancel{6}3} \right)$$

$\underline{> 0}$



Skew to right.

If < 0

Skew to left

Moment generating functions

Definition

Let X be a discrete random variable and assume that there exists $h > 0$ such that

$$M(t) = \mathbb{E}[e^{tX}] = \sum e^{tx} f(x)$$

is finite for all $t \in (-h, h)$. Then, $M(t) = \mathbb{E}[e^{tX}]$ is called the moment generating function (mgf).

$M(0) = 1$

$$M'(0) = \frac{d}{dt} M(t) \Big|_{t=0} = \frac{d}{dt} \mathbb{E}[e^{tX}] \Big|_{t=0} = \mathbb{E}[X e^{tX}] \Big|_{t=0}$$

$$= \mathbb{E}[X]$$

$$M''(0) = \mathbb{E}[X^2]$$

MGF of $X = M(t) = \mathbb{E}[e^{tX}] = \sum_x e^{tx} \cdot \underbrace{f(x)}_{\text{pmf of } X}$

for $t \in (-h, h)$, $h > 0$.

$= f(x) = \mathbb{P}(X=x)$

① $M(0) = \mathbb{E}[e^{0 \cdot X}] = \mathbb{E}[1] = 1$.

② $M'(t)\Big|_{t=0} = \frac{d}{dt}\mathbb{E}[e^{tX}]\Big|_{t=0} = \frac{d}{dt}\sum e^{tx} f(x)\Big|_{t=0} = \sum \frac{d}{dt}(e^{tx} f(x))\Big|_{t=0}$

$= \sum x \cdot e^{tx} f(x)\Big|_{t=0} = \sum x f(x) = \mathbb{E}[X]$.

$M'(0) = \mathbb{E}[X] \leftarrow$ first moment of X about 0

Moment generating functions

Properties

1. $M(0) = 1$
2. $M'(0) = \mathbb{E}[X]$
3. $M''(0) = \mathbb{E}[X^2]$
4. In general, $M^{(r)}(0) = \mathbb{E}[X^r]$.

Moment generating functions

Geometric RV.

Example

$x = 1, 2, \ldots$

Let $f(x) = q^{x-1}p$ where $p \in (0,1)$ and $\boxed{q = 1 - p.} \Rightarrow$ $p = 1 - q$.

Compute $M(t)$.

① Is $f(x)$ a PMF?

$$\sum_{x=1}^{\infty} \underbrace{f(x)}_{\geq 0} = q^{1-1}\cdot p + q^{2-1}\cdot p + q^{3-1}\cdot p + \cdots$$

$$= p + q\cdot p + q^2 \cdot p + \cdots \quad \leftarrow \text{Geometric Series}$$

$\times q$ $\times q$

$$= \frac{\text{First term}}{1 - \text{Ratio}} = \frac{p}{1-q} = 1$$

② $M(t) = E[e^{tX}] = \sum_{x=1}^{\infty} e^{tx} \cdot q^{x-1} \cdot p$

$$= e^{t\cdot 1} \cdot q^{1-1} \cdot p + e^{t\cdot 2} \cdot q^{2-1} \cdot p + e^{t\cdot 3} \cdot q^{3-1} \cdot p + \cdots$$

$$= e^{t} \cdot p + e^{2t} \cdot q \cdot p + e^{3t} \cdot q^2 \cdot p + \cdots$$

$\times e^t \cdot q$ $\times e^t \cdot q$ $\times e^t q$

$$= \frac{e^t \cdot p}{1 - e^t \cdot q} \leftarrow$$

$-1 < e^t \cdot q < 1$

$$= \frac{e^t \cdot p}{1 - e^t \cdot (1-p)}$$

Ex $\mathbb{E}[X] = \sum_{x=1}^{\infty} x \cdot p \cdot q^{x-1} = ?$

why?

$$\underline{1} = \sum_{k=0}^{n} f(k) = \sum_{k=0}^{n} \binom{n}{k} \left(\tfrac{1}{2}\right)^n$$

Binomial Thm

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k \cdot b^{n-k}$$

$a = b = \frac{1}{2}$

Exercise

Find $\mathbb{E}[X]$ and $\mathbb{E}[X(X-1)]$ for a discrete random varialbe X with the pmf

$\mathbb{E}[X^2] - \mathbb{E}[X]$

$$f(k) = \binom{4}{k}\left(\frac{1}{2}\right)^4 \quad \leftarrow \quad n=4.$$

for $k = 0, 1, 2, 3, 4$.

$$\binom{4}{k} = \frac{4!}{k!\,(4-k)!}$$

$$f(k) = \binom{n}{k}\left(\tfrac{1}{2}\right)^n \qquad k = 0, 1, \cdots, n$$

$$\mathbb{E}[X] = \sum_{k=0}^{n} k \cdot \binom{n}{k} \cdot \left(\tfrac{1}{2}\right)^n = \sum_{k=1}^{n} \cancel{k} \cdot \frac{n\,(n-1)!}{\underbrace{\cancel{k!}}_{(k-1)!}\,\underbrace{(n-k)!}_{((n-1)-(k-1))!}} \left(\tfrac{1}{2}\right)^n$$

$k! = k \cdot (k-1)!$

$$= \sum_{k=1}^{n} n \cdot \binom{n-1}{k-1} \left(\tfrac{1}{2}\right)^n \qquad (k-1 = j)$$

$$= n \Bigg(\underbrace{\sum_{j=0}^{n-1} \binom{n-1}{j} \left(\tfrac{1}{2}\right)^{n-1}}_{=n} \Bigg) \cdot \left(\tfrac{1}{2}\right) = \frac{n}{2}$$

$$M(t) = \mathbb{E}[e^{tX}] = \sum_{k=0}^{n} \underline{e^{tk}} \binom{n}{k} \left(\tfrac{1}{2}\right)^n$$

$$= \sum_{k=0}^{n} \binom{n}{k} \cdot \left(\frac{e^t}{2}\right)^k \cdot \left(\frac{1}{2}\right)^{n-k} = \left(\frac{1}{2} + \frac{1}{2}e^t\right)^n$$

Section 4.
The Binomial Distribution

Bernoulli random variables

A Bernoulli experiment, more commonly called a Bernoulli trial, is a random experiment with two outcomes.

Say $S = \{\text{success, failure }\}$ and $\mathbb{P}(\text{sucess}) = p$ for some $p \in (0, 1)$. Then $\mathbb{P}(\text{failure}) = q = 1 - p$.

$p + q = 1$

A random variable X is a Bernoulli random variable with success probability p is $X = 1$ if success and 0 otherwise.

$$X = \begin{cases} 1, & \text{with prob. } p \\ 0, & \text{with prob. } 1-p = q \end{cases}$$

$$X = \begin{cases} 1 & \text{w.p. } p \\ 0 & \text{w.p } q = 1-p \end{cases} \qquad f(x) = \begin{cases} p & \text{if } x=1 \\ q & \text{if } x=0 \\ 0 & \text{otherwise.} \end{cases}$$

Bernoulli random variables

Theorem

Let X be a Bernoulli random variable with success probability p.

$\mathbb{E}[X] = \sum_x^1 x \cdot f(x) = 0 \cdot f(0) + 1 \cdot f(1) = 0 \cdot q + 1 \cdot p = p.$

$\mathrm{Var}[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = p - p^2 = p \cdot (1-p) = pq.$

$$\mathbb{E}[X^2] = \sum_x x^2 f(x) = 0^2 \cdot f(0) + 1^2 \cdot f(1) = p$$

$$M(t) = \mathbb{E}[e^{tX}] = \sum_x^1 e^{tx} f(x) = \underline{e^{t\cdot 0}} f(0) + e^{t\cdot 1} f(1)$$

$$= 1 \cdot q + e^t \cdot p = 1-p + e^t p$$

$$= 1 + (e^t - 1) \cdot p.$$

$f(k) = \mathbb{P}(X = k) = \mathbb{P}(k \text{ success in } n \text{ experiments}) = \underbrace{\binom{n}{k} p^k q^{n-k}}_{k = 0, \dots, n}$

S/F S/F

n times

success prob. p

Binomial random variables

Consider a sequence of independent Bernoulli experiments with success probability p.

Let X be the number of success trials in the first n experiments.

This is called a binomial random variable with the number of trials n and success probability p.

We use the notation $X \sim b(n, p) = \text{Bin}(n, p)$.

① If $n=1$, Binomial = Bernolli

② $X \sim Bin(n, p)$, $X = X_1 + X_2 + \cdots + X_n$

Indep. Ber(p)

Binomial random variables

\# of arrangement $\quad$ k success $\quad$ $(n-k)$ failure

Theorem

Let X a binomial random variable with the number of trials n and success probability p.

The pmf of X is $\quad f(k) = \binom{n}{k} p^k \cdot (1-p)^{n-k}, \quad k = 0, 1, \cdots, n$

$\mathbb{E}[X] =$ $n \cdot p$

$\mathrm{Var}[X] =$ $n \cdot p \cdot (1-p) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$

$$M(t) = \mathbb{E}[e^{tX}] = \sum_{k=0}^{n} e^{tk} \binom{n}{k} p^k \cdot q^{n-k}$$

$$= \sum_{k=0}^{n} \binom{n}{k} \cdot (pe^t)^k (q)^{n-k}$$

$$= (\underline{pe^t + q})^n =$$

$$M'(t) = n(pe^t+q)^{n-1} \cdot (pe^t+q)' = n(pe^t+q)^{n-1} \cdot p \cdot e^t$$

$$M'(0) = n \cdot (p+q)^{n-1} \cdot p \cdot e^0 = n \cdot p. = \mathbb{E}[X]$$

Binomial random variables

Example

Out of millions of instant lottery tickets, suppose that 20% are winners. If eight such tickets are purchased, what is the probability of purchasing two winning tickets?

# of trials = 8

success prob = 0.2

X = # of winning tickets $\sim Bin(8, 0.2)$

$P(X=2) = \binom{8}{2} (0.2)^2 (0.8)^6$

Binomial random variables

Example

H5N1 is a type of influenza virus that causes a severe respiratory disease in birds called avian influenza (or "bird flu").

Although human cases are rare, they are deadly; according to the World Health Organization the mortality rate among humans is 60%.

Let X equal the number of people, among the next 25 reported cases, who survive the disease.

# of trials = 25

success prob. = 0.4

Assuming independence, the distribution of X is $b(25, 0.4)$. What is the probability that ten or fewer of the cases survive?

$$\mathbb{P}(X \leq 10) = \sum_{k=0}^{10} \binom{25}{k} \cdot (0.4)^{k} \cdot (0.6)^{25-k}$$

$$\simeq 0.5858.$$

Binomial random variables

Theorem

The mgf of a binomial random variable X is

$$M(t) = (\underset{1-p}{\overset{\shortparallel}{q}} + pe^t)^n$$

$$= (1 + (e^t - 1)p)^n$$

Binomial random variables

Exercise

It is believed that approximately 75% of American youth now have insurance due to the health care law.

Suppose this is true, and let X equal the number of American youth in a random sample of $n = 15$ with private health insurance.

How is X distributed? Find the probability that X is at least 10. Find the mean, variance, and standard deviation of X.

$$X \sim \text{Bin}(15, 0.75)$$

$$\mathbb{E}[X] = n \cdot p = 15 \cdot \frac{3}{4} = \frac{45}{4}$$

$$\text{Var}[X] = n \cdot p \cdot q = n \cdot p \cdot (1-p) = 15 \cdot \frac{3}{4} \cdot \frac{1}{4} = \frac{45}{16}.$$

$$\text{Std}(X) = \sqrt{npq} = \sqrt{\frac{45}{16}} = \frac{3\sqrt{5}}{4}$$

$$\mathbb{P}(X \geq 10) = \sum_{k=10}^{15} \binom{15}{k} (0.75)^k \cdot (0.25)^{15-k}$$

Section 5.
The Hypergeometric Distribution

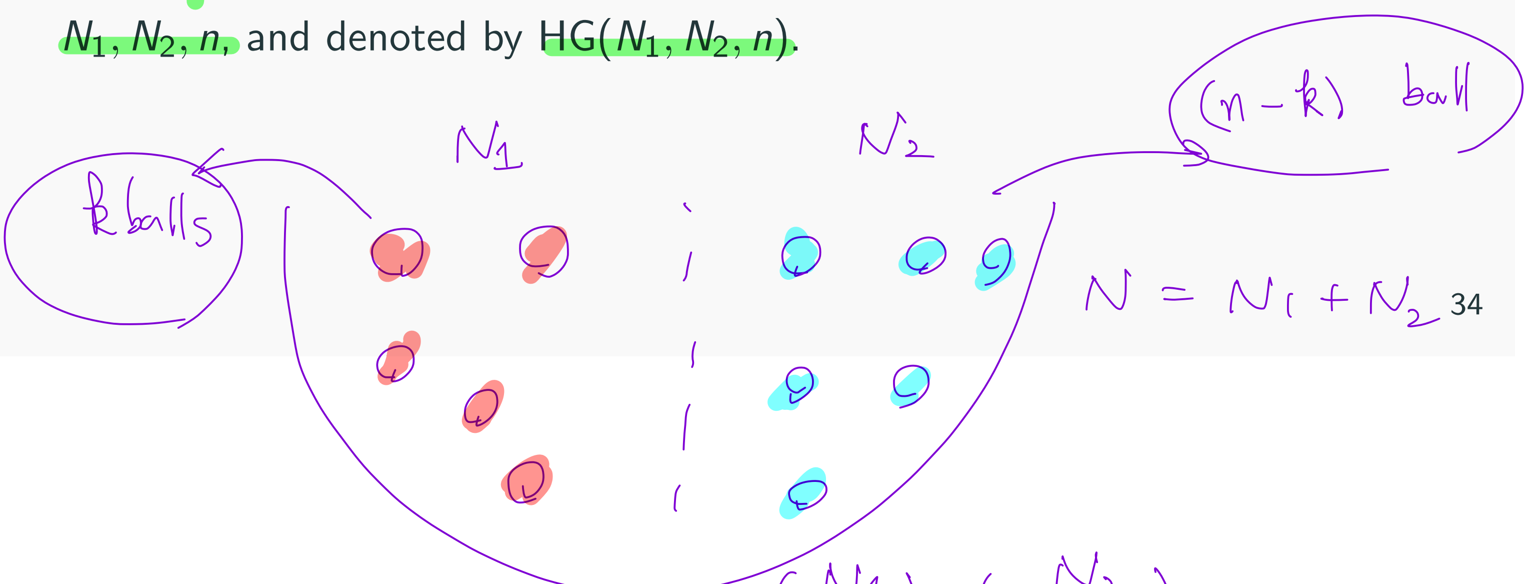
The Hypergeometric Distribution

There is a collection of N_1 red balls and N_2 blue balls.

Sample n balls at random without replacement ($n \leq N_1 + N_2$).

Let X be the number of red balls chosen. $\quad 0 \leq X \leq \min\{n, N_1\}$

Then, X is called a hypergeometric random variable with parameters N_1, N_2, n, and denoted by $\text{HG}(N_1, N_2, n)$.



$N = N_1 + N_2$

$$f(k) = \mathbb{P}(X = k) = \frac{\binom{N_1}{k}\binom{N_2}{n-k}}{\binom{N}{n}}$$

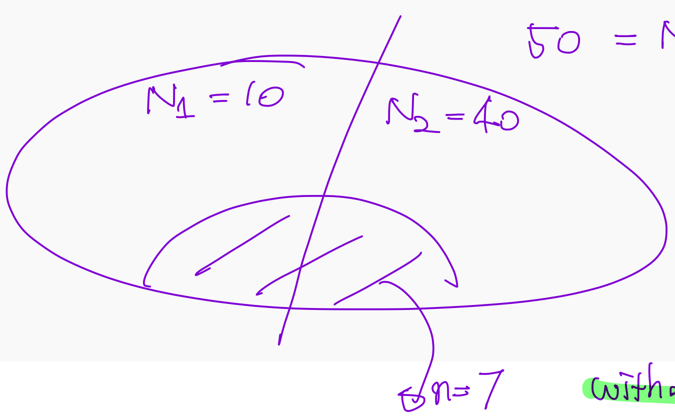
The Hypergeometric Distribution

Example

In a small pond there are 50 fish, ten of which have been tagged.

If a fisherman's catch consists of seven fish selected at random and without replacement, and X denotes the number of tagged fish, what is the probability that exactly two tagged fish are caught?

$50 = N$



$X = \#$ of tagged fish

$\sim HG(10, 40, 7)$

without replacement

$$P(X = 2) = \frac{\binom{10}{2}\binom{40}{5}}{\binom{50}{7}}$$

With replacement $\to$ # of trials $= 7$

Bin. success prob $= \frac{N_1}{N} = \frac{10}{50} = \frac{1}{5}$.

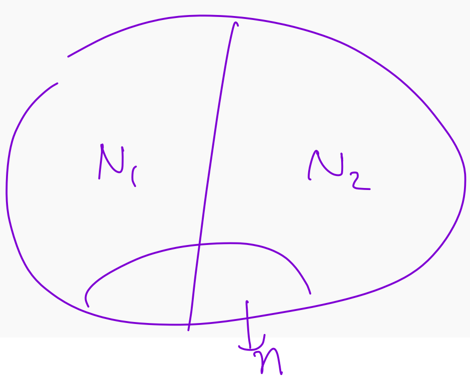
The Hypergeometric Distribution

Theorem

$$\mathbb{P}(X = k) = \frac{\binom{N_1}{k}\binom{N_2}{n-k}}{\binom{N}{n}}$$

$$\mathbb{E}[X] = n\frac{N_1}{N_1+N_2}$$

$$\mathrm{Var}[X] = n\frac{N_1}{N_1+N_2}\frac{N_2}{N_1+N_2}$$



without replacement $\longrightarrow HG(N_1, N_2, n)$

with replacement $\longrightarrow Bin\left(n, \frac{N_1}{N_1+N_2}\right)$

$\Downarrow$

$$\mathbb{E}[X] = n \cdot \frac{N_1}{N_1+N_2}$$

$$\mathrm{Var}[X] = n \cdot \frac{N_1}{N_1+N_2} \cdot \frac{N_2}{N_1+N_2}$$

1. Bernoulli RV : 2 outcomes $\{ \text{Success}, \text{Failure} \} = \Omega$

$$\mathbb{P}(\text{Success}) = p \in (0,1)$$

$$\mathbb{P}(\text{Failure}) = 1-p = q$$

$$X = \begin{cases} 1 & \text{if Success} \\ 0 & \text{otherwise} \end{cases} \leftarrow \text{Bernoulli RV.}$$

2. Binomial RV : Repeat Bernoulli Exp, n times

Count $\underbrace{\#\text{ of Success}}_{\substack{\| \\ X}}$: Binomial RV

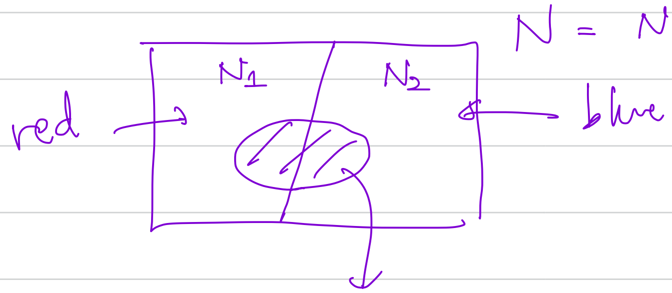
PMF $f(k) = \mathbb{P}(X = k)$ (k = # of success)

$$= \binom{n}{k} p^k \cdot (1-p)^{n-k}$$

$\mathbb{E}[X] = n \cdot p$ $\quad$ $Var(X) = n \cdot p \cdot q = n \cdot p(1-p)$.

3. Hyper Geometric RV :

$N = N_1 + N_2$



n chosen without replacement

count # of red $= X$: HG RV.

PMF $$f(k) = \frac{\binom{N_1}{k}\binom{N_2}{n-k}}{\binom{N}{n}}$$ $\leftarrow$ n balls out of N balls

$X = 0, 1, \cdots, \underline{\min\{n, N_1\}}$

$X \sim HG(N_1, N_2, n)$

The Hypergeometric Distribution

HG - total population $\langle$ two kinds $r/b.$

sample n without replacement.

with replacement $\to$ Bin.

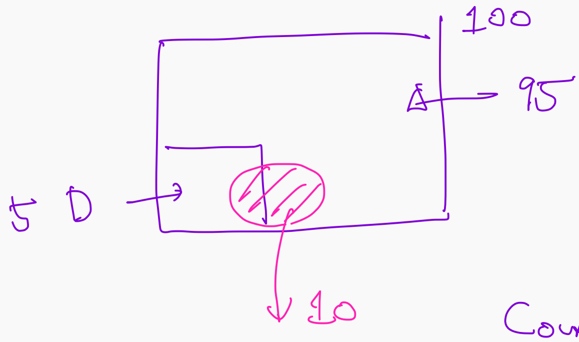
Exercise

In a lot (collection) of 100 light bulbs, there are five bad bulbs.

without replacement

An inspector inspects ten bulbs selected at random.

Find the probability of finding at least one defective bulb.



Count # of Defective $= X \sim HG(5, 95, 10)$

$$P(X \geq 1) = \sum_{k=1}^{\cancel{10}?\,5} P(X=k) = \sum_{k=1}^{5} \frac{\binom{5}{k}\cdot\binom{95}{10-k}}{\binom{100}{10}}$$

$$= 1 - P\left((X \geq 1)^c\right) = 1 - P(X=0)$$

$$= 1 - \frac{\binom{5}{0}\cdot\binom{95}{10}}{\binom{100}{10}}$$

Section 6.
The Negative Binomial Distribution

fix # of trial

Binomial : repeat n time → # of success.

Geometric : repeat until first success → # of trials

Geometric RV

Geometric random variables

Consider a sequence of independent Bernoulli trials (experiments) with success probability

Let X be the number of trials until the first success.

This random variable is called a geometric random variable.

H → Success

Repeat Bernoulli Exp. with success prob. $p \in (0,1)$ until first success.

PMF $f(k) = \mathbb{P}(\underline{X = k})$ 1^{st} success

$$= P(F\ F\ F\ F\ F \ldots \boxed{S})$$

$\underbrace{}_{k \text{ times}}$

$$= \underbrace{(1-p)^{k-1}}\cdot p$$

$k = 1, 2, \ldots$

Geometric random variables

Theorem

The pmf of X is $f(k) = (1-p)^{k-1}\cdot p$ for $k = 1, 2, 3, \ldots$

$\mathbb{E}[X] = \frac{1}{p}$

$\mathrm{Var}[X] = \frac{q}{p^2} = \frac{1-p}{p^2}$

$M(t) = \frac{pe^t}{1-(1-p)e^t}$ "

① $\sum_{k=1}^{\infty} \underline{f(k)} = 1$

Pf: $\sum_{k=1}^{\infty}(1-p)^{k-1}\cdot p = \frac{1^{st}\text{ term}}{1 - \text{Ratio}} = \frac{p}{1-(1-p)} = 1$ ∎

Geometric Series

In the same way

$$M(t) = \mathbb{E}[e^{tX}] = \sum_{k=1}^{\infty} e^{tk}(1-p)^{k-1}\cdot p$$

② $\mathbb{E}[X] = \sum_{k=1}^{\infty} k\cdot f(k) = \sum_{k=1}^{\infty} k\cdot(1-p)^{k-1}\cdot p$

$$E[X] = 1 \cdot (1-p)^0 \cdot p + 2 \cdot (1-p) \cdot p + 3 (1-p)^2 p + 4 (1-p)^3 \cdot p + \cdots$$

$$-\ (1-p)\, E[X] = 1 (1-p) \cdot p + 2 \cdot (1-p)^2 p + 3 \cdot (1-p)^3 \cdot p + \cdots$$

$$(1-(1-p))\, E[X] = \underbrace{(1-p)^0 \cdot p + (1-p) \cdot p + (1-p)^2 \cdot p + \cdots}_{\text{Geom. Series}}$$

$$p \cdot E[X] = \frac{p}{1-(1-p)} = 1$$

$$\therefore\ E[X] = \frac{1}{p}.$$

Geometric random variables

Example

Some biology students were checking eye color in a large number of fruit flies.

For the individual fly, suppose that the probability of white eyes is 1/4 and the probability of red eyes is 3/4, and that we may treat these observations as independent Bernoulli trials.

$\leftarrow$ # of trials $= X \sim \text{Geom}(\overset{p}{\tfrac{1}{4}})$

What is the probability that at least four flies have to be checked for eye color $\underbrace{\text{to observe a white-eyed fly}}_{\text{success}}$?

$p = \frac{1}{4}$

$p + (1-p)\cdot p + (1-p)^2 \cdot p$

$$\mathbb{P}(X \geq 4) = \underbrace{\sum_{k=4}^{\infty} (1-p)^{k-1} \cdot p}_{\text{Geometric series}} = \sum_{k=1}^{3} (1-p)^{k-1} \cdot p$$

$$= \frac{(1-p)^{4-1} \cdot \cancel{p}}{\cancel{1-(1-p)}} = (1-p)^3 = \left(\frac{3}{4}\right)^3.$$

$X \sim \text{Geom}(p)$

- $\mathbb{P}(X > k) = (1-p)^k$
- $\mathbb{P}(X > a+b \mid X > a) = \mathbb{P}(X > b)$ $\leftarrow$ memoryless property.

$X \sim \text{Geom}(p)$: # of trials until first success

X = # of trials until r^{th} success.

$\sim$ Negative Binomial (r, p)

Negative Binomial random variables

= Indep. Sum of Geometric RVs.

Consider a sequence of independent Bernoulli trials with success probability p

Let X be the number of trials until the r-th success.

This random variable is called a negative binomial random variable.

$$\underbrace{\overbrace{F\ F\ F\ S}^{X_1 \sim \text{Geom}(p)}\ \Big|\ \overbrace{F\ F\ S}^{X_2 \sim \text{Geom}(p)}\ \Big|\ \overbrace{F\ F\ F\ F\ S}^{X_3 \sim \text{Geom}(p)}}_{X \sim \text{NegBin}(3,p)}$$

4 : Geom. RV

NegBin(3, p)

$X \sim \text{NegBin}(3, p)$

$X = X_1 + X_2 + X_3$.

$(k-1)$ spots $<$ $r-1$ success, $k-r$ failure

$X \sim NegBin(r, p)$

r-th success

F F S F - - - - - - F $\boxed{S}$

$\underbrace{\qquad\qquad\qquad\qquad}_{k \text{ times}}$

$$\mathbb{P}(X=k) = \binom{k-1}{r-1} p^r (1-p)^{k-r}$$

Negative Binomial random variables

Theorem

The pmf of X is

$$f(k) = \binom{k-1}{r-1} p^r (1-p)^{k-r}$$

for $k = r, r+1, \cdots$ and otherwise zero.

$\mathbb{E}[X] = \frac{r}{p}$ $= r \cdot \underbrace{\frac{1}{p}}$ Expectation of Geometric RV

$\mathrm{Var}[X] = \frac{rq}{p^2}$ $= r \cdot \frac{q}{p^2}$

Variance of Geometric RV

$M(t) = \left(\frac{pe^t}{1-(1-p)e^t}\right)^r$

$X = X_1 + X_2 + \cdots + X_r \sim NegBin(r, p)$

$X_i \sim Geom(p)$

Negative Binomial random variables

A negative binomial random variable can be written as a sum of independent geometric random variables.

Negative Binomial random variables

Example

Suppose that during practice a basketball player can make a free throw 80% of the time.

Furthermore, assume that a sequence of free-throw shooting can be thought of as independent Bernoulli trials.

Let X equal the minimum number of free throws that this player must attempt to make a total of ten shots.

Find the mean of X.

$p = 0.8$ $\qquad X \sim \text{NegBin}(\overset{r}{10}, 0.8)$

$E[X] = r \cdot \frac{1}{p} = 10 \cdot \frac{1}{0.8} = 10 \cdot \frac{5}{4} = \frac{25}{2} = 12.5$

A B C D

Exercise Coupon Collection Problem.

One of four different prizes was randomly put into each box of a cereal.

If a family decided to buy this cereal until it obtained at least one of each of the four different prizes, what is the expected number of boxes of cereal that must be purchased?

collect 3 prizes = A, B, C

$X_1 \sim \text{Geom}(\frac{3}{4})$ $\quad$ $X_2 \sim \text{Geom}(\frac{2}{4})$ $\quad$ $X_3 \sim \text{Geom}(\frac{1}{4})$

1st: A $\quad$ A $\quad$ A $\quad$ A $\quad$ B/C/D $\quad$ A/B $\quad$ A/B $\quad$ C/D $\quad$ D

S.p = $\frac{1}{4}$

1 prize $\quad$ success prob = $\underline{3/4}$ $\quad$ 2 prizes = A, B $\quad$ $p = \frac{2}{4}$

$$E[1 + X_1 + X_2 + X_3] = 1 + E[X_1] + E[X_2] + E[X_3]$$

$$= 1 + \frac{4}{3} + \cancel{\frac{4}{2}}\,2 + \frac{4}{1} = \frac{25}{3}.$$

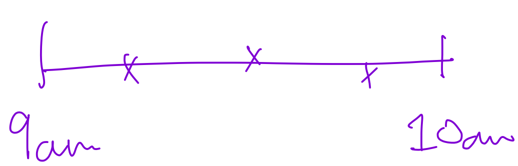
Section 7.
The Poisson Distribution

Definition

Some experiments result in counting the number of times particular events occur at given times or with given physical objects.

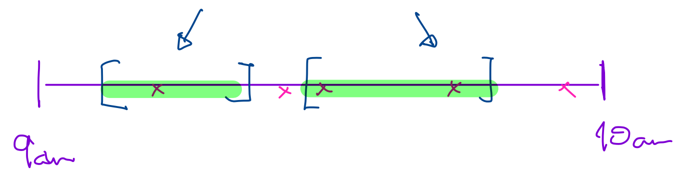
Example

- the number of cell phone calls passing through a relay tower between 9 and 10 A.M.
- the number of flaws in 100 feet of wire
- the number of customers that arrive at a ticket window between noon and 2P.M.
- the number of defects in a 100-foot roll of aluminum screen that is 2 feet wide.



Definition

Counting such events can be looked upon as observations of a random variable associated with an approximate Poisson process, provided that the conditions in the following definition are satisfied.

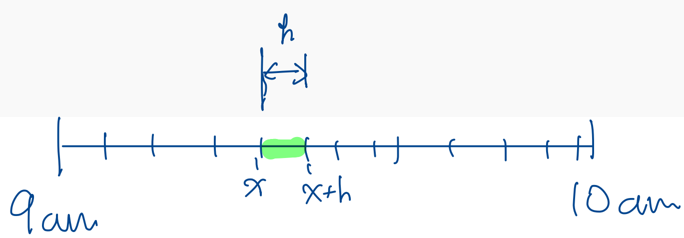


Definition

Let the number of occurrences of some event in a given continuous interval be counted. Then we have an approximate Poisson process with parameter $\lambda > 0$ if

- The numbers of occurrences in nonoverlapping subintervals are independent.
- The probability of exactly one occurrence in a sufficiently short subinterval of length h is approximately λh.
- The probability of two or more occurrences in a sufficiently short subinterval is essentially zero.

Under these assumption, consider the number of occurrences in a time interval $[0, 1]$.



$\mathbb{P}($ 1 event occurs in $[x, x+h]) \approx \lambda \cdot h$ $\quad \forall x$

$\mathbb{P}($ events in $[x, x+h] \geq 2) \approx 0$

if h is small

$0 \quad \frac{1}{n} \quad \frac{2}{n} \quad \cdots \quad \frac{k}{n} \quad \frac{k+1}{n} \quad \cdots \quad \frac{n-1}{n} \quad 1 = \frac{n}{n}$

Indep. $\left\{ \begin{array}{l} 0 \text{ event} \\ \text{or} \\ 1 \text{ event} \end{array} \right. \Rightarrow$ Indep. Bernoulli. with $p = \frac{\lambda}{n}$

repeat n $\sim X_n$

$E[X_n] = n \cdot p = n \cdot \frac{\lambda}{n} = \lambda$

$X = \#$ of event in $[0,1] \approx \boxed{\text{Bin}(n, \frac{\lambda}{n}) \text{ large } n}$.

Definition

Split $[0,1]$ into n subintervales $[0, \frac{1}{n}], [\frac{1}{n}, \frac{2}{n}], \cdots, [\frac{n-1}{n}, 1]$.

In each subinterval, at most one event occurs with probability $\frac{\lambda}{n}$.

Thus, the number of occurrences is a binomial random variable with n nad $\frac{\lambda}{n}$.

As $n \to \infty$, the random variable gets close to some random variable X.

We say X is a Poisson random variable with parameter λ if its pmf is

$X_n \sim \text{Bin}(n, \frac{\lambda}{n})$

$$\mathbb{P}(X = k) = \frac{e^{-\lambda}\lambda^k}{k!} = \lim_{n \to \infty} \mathbb{P}(X_n = k)$$

$$= \lim_{n \to \infty} \binom{n}{k} \left(\frac{\lambda}{n}\right)^k \cdot \left(\frac{n-\lambda}{n}\right)^{n-k}$$

for $k = 0, 1, 2, \cdots$.

What is λ? Occurrence rate

= mean.

$[0, 1]$

$$\frac{n!}{k!\,(n-k)!} \cdot \frac{\lambda^k}{n^k} \cdot \left(\frac{n-\lambda}{n}\right)^{n-k}$$

Stirling's formula.

$X \sim \text{Poisson}(\lambda) = \#$ of occurrences

PMF $\quad f(k) = e^{-\lambda} \dfrac{\lambda^k}{k!}, \quad k = 0, 1, 2, \ldots$

$$1 \overset{?}{=} \sum_{k=0}^{\infty} f(k) = \sum_{k=0}^{\infty} e^{-\lambda} \frac{\lambda^k}{k!} = e^{-\lambda} \underbrace{\boxed{\sum_{k=0}^{\infty} \frac{\lambda^k}{k!}}}_{= e^{\lambda}} = e^{-\lambda} \cdot e^{\lambda} = 1.$$

Definition

Fact: $\displaystyle\sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = e^x$

Theorem

$\mathbb{E}[X] = \lambda$

$\text{Var}[X] = \lambda$

$M(t) =$

$$\mathbb{E}[X] = \sum_{k=\cancel{0}\,1}^{\infty} k \cdot f(k) = \sum_{k=1}^{\infty} \cancel{k}\, e^{-\lambda} \cdot \frac{\lambda^k}{\cancel{k!}\,(k-1)!} \qquad k-1 = j$$

$$= \sum_{j=0}^{\infty} e^{-\lambda} \frac{\lambda^j \cdot \lambda}{j!} = \lambda e^{-\lambda} \underbrace{\boxed{\sum_{j=0}^{\infty} \frac{\lambda^j}{j!}}}_{= e^{\lambda}}$$

$$= \lambda e^{-\lambda} \cdot e^{\lambda} = \lambda$$

$$\text{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = \lambda + \lambda^2 - \lambda^2 = \lambda$$

$$\mathbb{E}[X^2] \overset{?}{=} \sum_{k=0}^{\infty} k^2 \cdot e^{-\lambda} \frac{\lambda^k}{k!}$$

$$k^2 \cdot \frac{1}{k!} = \frac{\cancel{k}^{\cancel{2}}}{\cancel{k}\cdot(k-1)\cdots 2\cdot 1}$$

$$E[X(X-1)] = \sum_{k=\cancel{0}\,2}^{\infty} \cancel{k\cdot(k-1)}\, e^{-\lambda} \frac{\lambda^k}{\cancel{k!}\,(k-2)!}$$

why?

$$= \cdots = \lambda^2$$

$$E[X^2 - X] = \lambda^2$$

$$E(X^2) - \underbrace{E[X]}_{=\lambda} = \lambda^2 \implies E[X^2] = \lambda + \lambda^2$$

$$M(t) = E[e^{tX}] = \sum_{k=0}^{\infty} (e^t)^k e^{-\lambda} \frac{\lambda^k}{k!}$$

$$= e^{-\lambda} \sum_{k=0}^{\infty} \frac{(\lambda e^t)^k}{k!}$$

$$= e^{-\lambda} \cdot e^{\lambda e^t} = e^{\lambda\cdot(e^t-1)}$$

Exercise

$$E[2^X] = ?$$

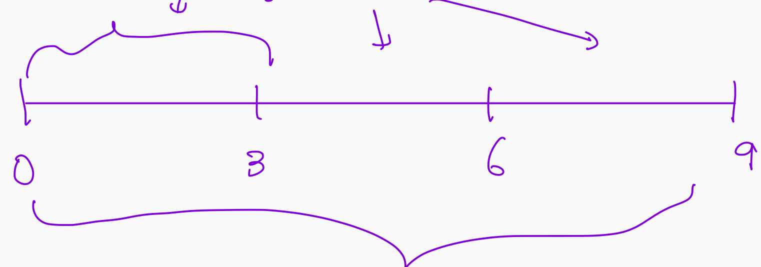
Definition

Example

In a large city, telephone calls to 911 come on the average of two every 3 minutes.

If one assumes an approximate Poisson distribution, what is the probability of five or more calls arriving in a 9 minute period?

average call = 2



Average # of calls $= 6 = \lambda$

X = # of calls in $[0, 9] \sim \text{Poisson}(6)$

$$P(X \geq 5) = \sum_{k=5}^{\infty} e^{-6} \frac{6^k}{k!}$$

Poisson Approximation to Binomial

\# of trials success prob.

Supose X is a binomial random variable $b(n, p)$, n is large, and p is small but np converges to some constant, say λ.

In this case, X can be approximated by a Poisson random variable with parameter λ.

This approximation is quite accurate if $n \geq 20$, $p \leq 0.05$ or $n \geq 100$, $p \leq 0.1$.

Ex $X \sim \mathrm{Bin}\left(100, \frac{1}{20}\right)$

$Y \sim \mathrm{Poisson}(\lambda)$ $\quad \lambda = \underbrace{\mathbb{E}[X]}_{} = n \cdot p = 100 \cdot \frac{1}{20} = 5.$ $\quad \mathbb{E}[Y]$

$P(X \leq 5) \approx P(Y \leq 5)$

$X \sim \text{Geom}(p)$

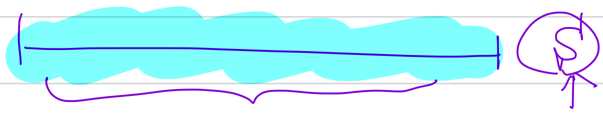
# of trials until first success.

$f(k) = (1-p)^{k-1} \cdot p$ for $k = 1, 2, 3, \cdots$.

$X \sim \text{NegBin}(r, p)$

# of trials until r^{th} success.

$f(k) = p \cdot p^{r-1} \cdot (1-p)^{k-r} \cdot \binom{k-1}{r-1}$ for $k = r, r+1, \cdots$

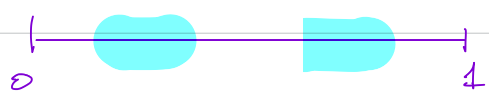


$k-1$ trials

k^{th} trial.

$r-1$ success

$k-r$ failure



← store // $X = $ # of customers during $[0, 1]$.

- no overlap → independence.
- small interval : 0 or 1 customer
- " prob $= \underline{\underline{\lambda}} \cdot$ (size of interval)

$X \sim \text{Poisson}(\lambda)$

$f(k) = e^{-\lambda} \cdot \frac{\lambda^k}{k!}$ for $k = 0, 1, 2, \cdots$

$\lambda = \mathbb{E}[X]$

Poisson Approximation to Binomial

Example

A manufacturer of Christmas tree light bulbs knows that 2% of its bulbs are defective.

$$X \sim \mathrm{Bin}(\underset{n \text{ large}}{100}, \underset{p \text{ small}}{0.02}) \qquad n \cdot p = 2 = \lambda$$

Assuming independence, the number of defective bulbs in a box of 100 bulbs has a binomial distribution with parameters n = 100 and p = 0.02.

Find the probability that a box of 100 of these bulbs contains at most three defective bulbs.

$$P(X \leq 3) = \sum_{k=0}^{3} \binom{100}{k} (0.02)^k (0.98)^{100-k}$$

$$\approx P(Y \leq 3) = \sum_{k=0}^{3} e^{-2} \frac{2^k}{k!}$$

$$Y \sim \mathrm{Poisson}(\lambda = 2)$$

$$= e^{-2} \cdot \left(1 + \frac{2}{1} + \frac{2^2}{2!} + \frac{2^3}{3!}\right)$$

Poisson Approximation to Binomial

Exercise

Suppose that the probability of suffering a side effect from a certain flu vaccine is 0.005. If 1000 persons are vaccinated, approximate the probability that (a) At most one person suffers. (b) Four, five, or six persons suffer.

X = # of people suffering a side ~~effect~~ $\sim \text{Bin}(1000, 0.005)$

$\uparrow$ large $\quad \uparrow$ small.

(a) P(At most 1 person suffers)

$$= \mathbb{P}(X \leq 1) = \sum_{k=0}^{1} \binom{1000}{k} (0.005)^k \cdot (1-0.005)^{1000-k}$$

$$= 1 \cdot 1 \cdot (0.995)^{1000} + 1000 \cdot (0.005) \cdot (0.995)^{999}$$

$$\approx \mathbb{P}(Y \leq 1) = \sum_{k=0}^{1} e^{-5} \frac{5^k}{k!} = e^{-5}(1+5) = 6 \cdot e^{-5}.$$

$Y \sim \text{Poisson}(\lambda)$

$\lambda = n \cdot p = 1000 \cdot 0.005 = 5$

Ex $\quad \mathbb{P}(Y \leq 4) \overset{\text{table.}}{=}$