$$\begin{pmatrix} 1 & 4 \\ 2 & 8 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -2 \\ -4 \end{pmatrix} \longrightarrow \left[\begin{array}{cc|c} 1 & 4 & -2 \\ 2 & 8 & -4 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 4 & -2 \\ 0 & 0 & 0 \end{array}\right] \text{ RREF.}$$

Non pivot, 1 free.

Non pivot $\Rightarrow$ Consistent

Straight line : $\boxed{1 \cdot x_1 + 4 \cdot x_2 = -2}$

$\begin{cases} x_1 = 0, & x_2 = -\frac{1}{2} \\ x_1 = -2, & x_2 = 0 \end{cases}$

In-Class Midterm 1 Review, Math 1554

1. Consider the matrix A and vectors $\vec{b}_1$ and $\vec{b}_2$.

$$A = \begin{pmatrix} 1 & 4 \\ 2 & 8 \end{pmatrix}, \quad \vec{b}_1 = \begin{pmatrix} -2 \\ -4 \end{pmatrix}, \quad \vec{b}_2 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

If possible, on the grids below, draw

(i) the two vectors and the span of the columns of A,

(ii) the solution set of $A\vec{x} = \vec{b}_1$.

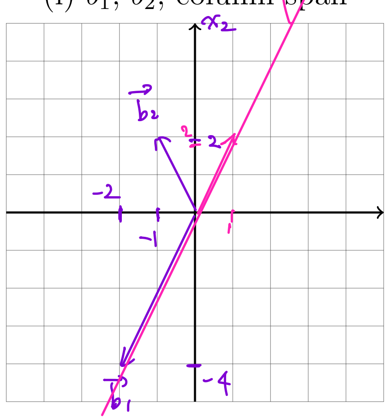
(iii) the solution set of $A\vec{x} = \vec{b}_2$.

$\text{Span}\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 8 \end{bmatrix}\right)$ $\quad 4\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$

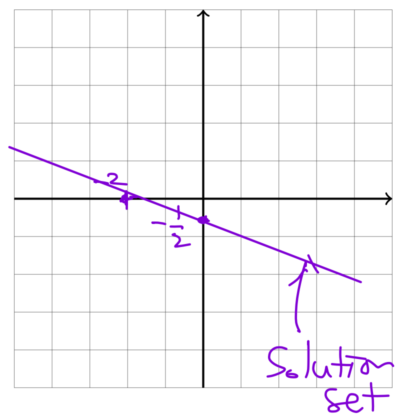
$= \left\{ a \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} + b \cdot \begin{bmatrix} 4 \\ 8 \end{bmatrix} : a, b \in \mathbb{R} \right\}$

$= \left\{ \underbrace{\cancel{(a+4b)}}_{c} \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} : \underbrace{\cancel{a, b}}_{c} \in \mathbb{R} \right\}$

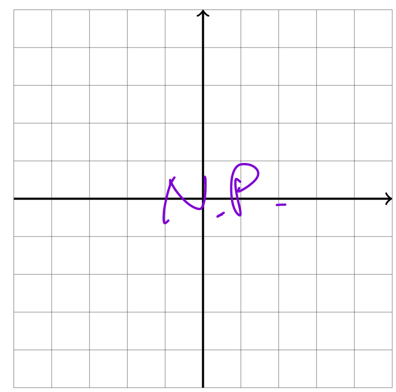
(i) $\vec{b}_1$, $\vec{b}_2$, column span



ii) solution set $Ax = \vec{b}_1$



iii) solution set $Ax = \vec{b}_2$

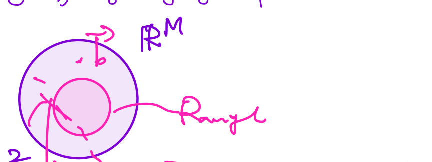


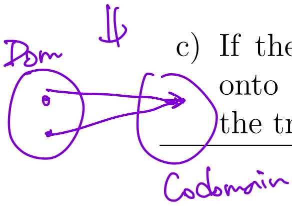
2. Indicate **true** if the statement is true, otherwise, indicate **false**. For the statements that are false, give a counterexample.

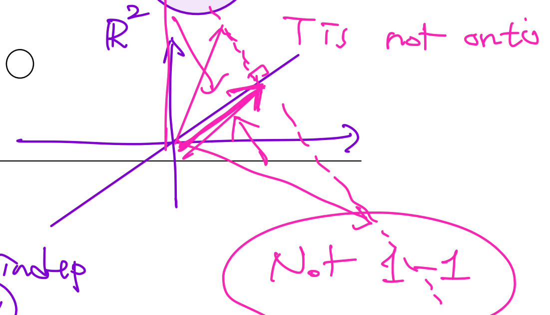
	true	false	counterexample
a) If $A \in \mathbb{R}^{M \times N}$ has linearly dependent columns, then the columns of A cannot span $\mathbb{R}^M$.	○	●	$\begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$
b) If there are some vectors $\vec{b} \in \mathbb{R}^M$ that are not in the range of $T(\vec{x}) = A\vec{x}$, then there cannot be a pivot in every row of A.	●	○	
c) If the transform $\vec{x} \to A\vec{x}$ projects points in $\mathbb{R}^2$ onto a line that passes through the origin, then the transform cannot be one-to-one.	●	○	

Not 1-1 $\Downarrow$

$\Leftrightarrow T$ is onto $\Leftrightarrow$ Columns span $\mathbb{R}^M$







Note $\{v_1, v_2, \cdots, v_5\}$: lin. dep. $\Uparrow$ $\{v_1, v_2, v_3\}$: lin. dep. | lin. indep $\Downarrow$ lin. indep.

3. If possible, write down an example of a matrix with the following properties. If it is not possible to do so, write *not possible*.

(a) A linear system that is homogeneous and has ~~no solutions~~.

$A\vec{x}=\vec{0}$ has always a trivial solution $\vec{x}=\vec{0}$

N.P.

(b) A standard matrix A associated to a linear transform, T. Matrix A is in RREF, and $T_A : \mathbb{R}^3 \to \mathbb{R}^4$ is one-to-one.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

(c) A 3×7 matrix A, in RREF, with exactly 2 pivot columns, such that $\underbrace{A\vec{x}=\vec{b}}_{=0}$ has exactly 5 free variables.

$$3\left\{\left[\begin{array}{ccccccc|c} 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]\right.$$

7

Non pivot

$\vec{b}$ can be anything.

4. Consider the linear system $A\vec{x}=\vec{b}$, where

$$A=\begin{pmatrix} 1 & 0 & 7 & 0 & -5 \\ 0 & 1 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \; \vec{b}=\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$$

(a) Express the augmented matrix $(\,A\,|\,\vec{b}\,)$ in RREF.

$$\left[\begin{array}{ccccc|c} 1 & 0 & 7 & 0 & -5 & 1 \\ 0 & 1 & 1 & 0 & 3 & 0 \\ 0 & 0 & 1 & 0 & 0 & 2 \end{array}\right] \to \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & -5 & -13 \\ 0 & 1 & 0 & 0 & 3 & -2 \\ 0 & 0 & 1 & 0 & 0 & 2 \end{array}\right]$$

$1-7\cdot 2$

$0-2$

Note Suppose $\vec{u},\vec{v}$ are solutions to $A\vec{x}=\vec{b}$

$\vec{u}-\vec{v}$ is a solution to $A\vec{x}=\vec{0}$

$A\vec{u}=\vec{b}$

$A\vec{v}=\vec{b}$ $\vec{b}$, $\vec{b}$

$A(\vec{u}-\vec{v})=A\vec{u}-A\vec{v}=0$

(b) Write the set of solutions to $A\vec{x}=\vec{b}$ in parametric vector form. Your answer must be expressed as a vector equation.

$$\begin{cases} 1\cdot x_1 - 5x_5 = -13 \\ 1\cdot x_2 + 3x_5 = -2 \\ 1\cdot x_3 = 2 \end{cases}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 5\cdot x_5 - 13 \\ -3x_5 - 2 \\ 2 \\ x_4 \\ x_5 \end{bmatrix} = x_4\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_5\cdot\begin{bmatrix} 5 \\ -3 \\ 0 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -13 \\ -2 \\ 2 \\ 0 \\ 0 \end{bmatrix}$$

$A\vec{x} = \vec{0}$: Solution? $\left\{ x_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 5 \\ -3 \\ 0 \\ 0 \\ 1 \end{bmatrix} : x_4, x_5 \in \mathbb{R} \right\}$

3. If possible, write down an example of a matrix with the following properties. If it is not possible to do so, write *not possible*.

   (a) A linear system that is homogeneous and has no solutions.

   (b) A standard matrix A associated to a linear transform, T. Matrix A is in RREF, and $T_A : \mathbb{R}^3 \to \mathbb{R}^4$ is one-to-one.

   (c) A 3×7 matrix A, in RREF, with exactly 2 pivot columns, such that $A\vec{x} = \vec{b}$ has exactly 5 free variables.

4. Consider the linear system $A\vec{x} = \vec{b}$, where

$$A = \begin{pmatrix} 1 & 0 & 7 & 0 & -5 \\ 0 & 1 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$$

   (a) Express the augmented matrix $(\, A \,|\, \vec{b} \,)$ in RREF.

   (b) Write the set of solutions to $A\vec{x} = \vec{b}$ in parametric vector form. Your answer must be expressed as a vector equation.