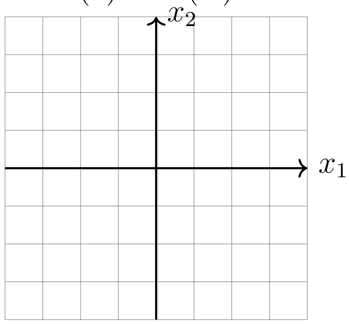
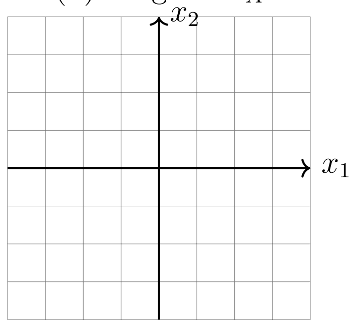
Midterm 2 Lecture Review Activity, Math 1554

1. (3 points) T_A is the linear transform $x \to Ax$, $A \in \mathbb{R}^{2\times 2}$, that projects points in $\mathbb{R}^2$ onto the x_2-axis. Sketch the nullspace of A, the range of the transform, and the column space of A. How are the range and column space related to each other?

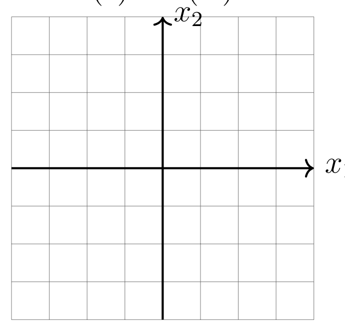
(a) Null(A)



(b) range of T_A



(c) Col(A)



2. Indicate **true** if the statement is true, otherwise, indicate **false**.

	true	false
a) $S = \{\vec{x} \in \mathbb{R}^3 \mid x_1 = a, x_2 = 4a, x_3 = x_1x_2\}$ is a subspace for any $a \in \mathbb{R}$.	○	X
b) If A is square and non-zero, and $A\vec{x} = A\vec{y}$ for some $\vec{x} \neq \vec{y}$, then $\det(A) \neq 0$.	○	X

3. If possible, write down an example of a matrix or quantity with the given properties. If it is not possible to do so, write *not possible*.

   (a) A is 2×2, Col$A$ is spanned by the vector $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and $\dim(\text{Null}(A)) = 1$. $A = \begin{pmatrix} 2 & 0 \\ 3 & 0 \end{pmatrix}$

   (b) A is 2×2, Col$A$ is spanned by the vector $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and $\dim(\text{Null}(A)) = 0$. $A = \begin{pmatrix} NP \end{pmatrix}$

   (c) A is in RREF and $T_A: \mathbb{R}^3 \to \mathbb{R}^3$. The vectors u and v are a basis for the range of T.
   $u = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, v = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, A = \begin{pmatrix} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & 0 \end{pmatrix}$

$\underline{A\vec{x} = A\vec{y} \quad \text{for} \quad \vec{x} \neq \vec{y}}$ $\iff$ T Not 1-1

$\underbrace{A(\vec{x}-\vec{y}) = 0}_{\neq 0}$ $\overset{\text{IMT}}{\iff}$ A Not invertible

has non trivial solution $\iff$ det (A) = 0

$\dim(\mathrm{Null}(A)) + \dim(\mathrm{Col}(A)) = n = \#$ of Col.

4. Indicate whether the situations are possible or impossible by filling in the appropriate circle.

		possible	impossible
4.i)	Vectors $\vec{u}$ and $\vec{v}$ are eigenvectors of square matrix A, and $\vec{w} = \vec{u} + \vec{v}$ is also an eigenvector of A.	⊗	○
4.ii)	$T_A = A\vec{x}$ is one-to-one, $\dim(\mathrm{Col}(A)) = 4$, and $T_A : \mathbb{R}^3 \to \mathbb{R}^4$.	○	⊗

$A \in \mathbb{R}^{4\times 3}$

$\dim(\mathrm{Col}(A)) = \mathrm{rank}(A)$

$Au = \lambda_1 u$ $\qquad A(u+v) = \lambda_1 u + \lambda_2 v$

$Av = \lambda_2 v$

if $\lambda_1 = \lambda_2$ $\quad = \lambda_1 (u+v)$

if $u = v$

5. (2 points) Fill in the blanks.

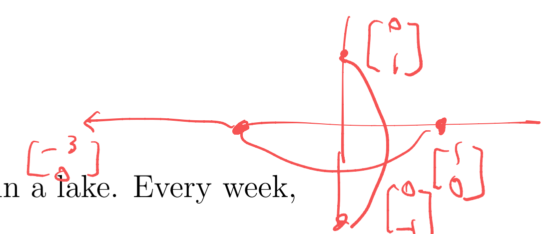
   (a) If A is a 6×4 matrix in RREF and $\mathrm{rank}(A) = 4$, what is the rank of A^T? $\boxed{4}$

   (b) $T_A = A\vec{x}$, where $A \in \mathbb{R}^{2\times 2}$, is a linear transform that first rotates vectors in $\mathbb{R}^2$ clockwise by π radians about the origin, then scales their x_1-component by a factor of 3, then projects them onto the x_1-axis. What is the value of $\det(A)$? $\boxed{0}$

   180°

$\det(3A) = 3^n \det(A)$

$$A = \begin{bmatrix} -3 & 0 \\ 0 & 0 \end{bmatrix}$$



6. (3 points) A virus is spreading in a lake. Every week,

   - 20% of the healthy fish get sick with the virus, while the other healthy fish remain healthy but could get sick at a later time.
   - 10% of the sick fish recover and can no longer get sick from the virus, 80% of the sick fish remain sick, and 10% of the sick fish die.

   Initially there are exactly 1000 fish in the lake.

   a) What is the stochastic matrix, P, for this situation? Is P regular?

   b) Write down any steady-state vector for the corresponding Markov-chain.

$A \longrightarrow \longrightarrow \longrightarrow RREF$

$$\begin{matrix} 1 & & & 1 \\ 0 & & & 0 \\ -1 & & & \end{matrix}$$

$A = LU$

$1 \leq \underbrace{\dim(\mathrm{Null}(A - 0 \cdot I))}_{= 1} =$ geom. multiplicity for $\lambda = 0 \leq$ alg. mult.

$$A^5 w = A^5(v_1 + 2 \cdot v_2 + 4 v_3)$$

$$= (\lambda_1)^5 v_1 + 2 \cdot (\lambda_2)^5 v_2 + 4 (\lambda_3)^5 v_3$$

$$= -v_1 + 2 \cdot v_2 = -\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -2 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} a & b & c \\ d+3a & e+3b & f+3c \\ d+3g & e+3h & f+3i \end{bmatrix} \longrightarrow \begin{bmatrix} a & b & c \\ d & e & f \\ \cancel{d}+3g & \cancel{e}+3h & \cancel{f}+3i \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} a & b & c \\ d & e & f \\ 3g & 3h & 3i \end{bmatrix}$$

$$c\begin{bmatrix}1\\0\end{bmatrix}$$

$$\begin{bmatrix}1 & a & b\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}-2\\1\\0\end{bmatrix}, \quad \begin{bmatrix}3\\0\\1\end{bmatrix}$$

$$= \begin{bmatrix}0\\0\end{bmatrix} = \begin{bmatrix}-2+a\\0\end{bmatrix}$$

$$a = 2, \quad b = -3$$

$$H = \mathrm{Null}\begin{pmatrix}1 & 3 & -3 & 2\\ -2 & -6 & 6 & -4\end{pmatrix} = \mathrm{Null}\begin{pmatrix}1 & 3 & -3 & 2\\ 0 & 0 & 0 & 0\end{pmatrix}$$

$$\begin{cases} x_1 + 3x_2 - 3x_3 + 2x_4 = 0\\ -2x_1 - 6x_2 + 6x_3 - 4x_2 = 0\end{cases}$$

$$\begin{bmatrix}1 & 3 & -3 & 2\\ -2 & -6 & 6 & -4\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = 0$$

$$A = \cancel{\begin{bmatrix} 2 & 0 & 0 \\ -4 & 0 & -6 \\ 0 & 0 & 2 \end{bmatrix}}$$

$$\mathcal{B} = \{v_1 , v_2\}$$

$$\lambda_1 = 2 \longrightarrow v_1 = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix}$$

$$\lambda_2 = 0 \longrightarrow v_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$[A\vec{x}]_{\mathcal{B}}$$

① $A\vec{x} = c_1 \cdot v_1 + c_2 v_2$

$$\vec{x} = \begin{bmatrix} 9 \\ 6 \\ -8 \end{bmatrix} \qquad \left[\begin{array}{cc|c} v_1 & v_2 & A\vec{x} \end{array}\right]$$

② $\underline{\underline{\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}}}$ //

$$\vec{x} = c_1 v_1 + c_2 v_2 + c_2 v_3$$

$$A\vec{x} = c_1 A v_1 + c_2 A v_2 + c_3 A v_3 \Longrightarrow \begin{bmatrix} 2c_1 \\ 2c_2 \end{bmatrix}$$

$$= \underline{2 \cdot c_1 v_1 + 2 \cdot c_2} v_2 + 0$$