In-Class Final Exam Review Set A, Math 1554, Fall 2019

1. Indicate whether the statements are true or false.

$A : m \times n$ $\quad n > m \not\Rightarrow$

true	false	
⊠	○	If a linear system has more unknowns than equations, then the system has either no solutions or infinitely many solutions.
○	⊠	A $n \times n$ matrix A and its echelon form E will always have the same eigenvalues. $\lambda^2 - \lambda - 1 = 0$ $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$ $\lambda = 1, -1$
⊠	○	$x^2 - 2xy + 4y^2 \geq 0$ for all real values of x and y.
⊠	○	If matrix A has linearly dependent columns, then $\dim((\text{Row}A)^\perp) > 0$.
⊠	○	If λ is an eigenvalue of A, then $\dim(\text{Null}(A - \lambda I)) > 0$. geom. multi.
⊠	○	If A has QR decomposition $A = QR$, then Col$A$ = Col$Q$.
⊠	○	If A has LU decomposition $A = LU$, then rank(A) = rank(U).
⊠	○	If A has LU decomposition $A = LU$, then $\dim(\text{Null}\,A) = \dim(\text{Null}\,U)$.

2. Give an example of the following.

   i) A 4×3 lower triangular matrix, A. such that $\text{Col}(A)^\perp$ is spanned by $\text{Nul}(A^T)$

   the vector $\vec{v} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 1 \end{pmatrix}$. $\quad A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{pmatrix}$ $\quad A^T = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -3 \end{bmatrix}$

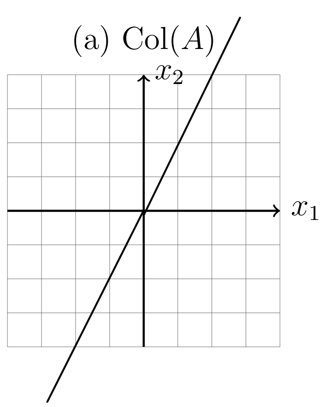
   ii) A 3×4 matrix A, that is in RREF, and satisfies $\dim\left((\text{Row } A)^\perp\right) = 2$ and $\dim\left((\text{Col } A)^\perp\right) = 2$

   $(\text{Row } A)^\perp = \text{Nul}(A)$ $\quad \dim(\text{Col}(A)) = 1$

   $A = \begin{pmatrix} \text{N.P.} \end{pmatrix}$ $\quad \text{Col}(A) \subseteq \mathbb{R}^3$

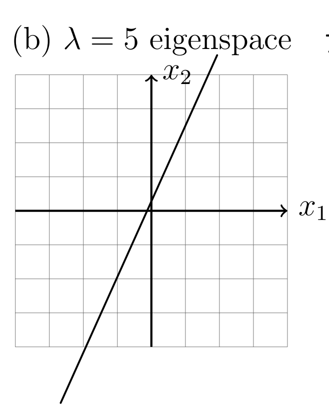
3. (3 points) Suppose $A = \begin{pmatrix} 3 & 1 \\ 6 & 2 \end{pmatrix}$. On the grid below, sketch a) Col(A), and b) the eigenspace corresponding to eigenvalue $\lambda = 5$.

(a) Col(A)



(b) $\lambda = 5$ eigenspace $= \text{Nul}(A - 5I)$

$= \text{Nul}\begin{bmatrix} -2 & 1 \\ 6 & -3 \end{bmatrix}$



$$Q(x) := {}^{1}x^2 - 2xy + 4y^2 \; \boxed{\geq 0} \quad \text{for every } x, y$$

$$= [x \;\; y] \underbrace{\begin{bmatrix} 1 & -1 \\ -1 & 4 \end{bmatrix}}_{\substack{\| \\ A}} \begin{bmatrix} x \\ y \end{bmatrix}$$

Q : P.S.D. $\Leftrightarrow$ $\lambda_i \geq 0$

$$(\lambda - \lambda_1)(\lambda - \lambda_2) = \lambda^2 - \underbrace{\boxed{5}}_{\substack{\| \\ \lambda_1 + \lambda_2}} \lambda + \overbrace{\boxed{3}}^{\lambda_1 \cdot \lambda_2} = 0$$

$$\lambda = \frac{5 \pm \sqrt{25 - 12}}{2} \geq 0$$

$$= (x^2 - 2xy + y^2) + 3 \cdot y^2$$

$$= (x-y)^2 + 3y^2 \geq 0$$

A has lin. dep. columns

$A \longrightarrow$ RREF $\nwarrow$ has $\underset{}{\overset{\text{Free var.} \\ \|}{\text{NONPIVOT}}}$ Columns

$Ax = 0$ $\Rightarrow$ has Nontrivial solution

Infinitely many

$\Rightarrow$ $\text{Nul}(A) \neq \{0\}$

$\Rightarrow$ $\dim(\underbrace{\text{Nul}(A)}_{\substack{\| \\ \text{Row}(A)^\perp}}) > 0$

QR decomposition (By product of Gram-Schmidt)

$$A = [\underbrace{x_1 \quad x_2 \quad \cdots \quad x_n}_{}] \quad \text{indep.}$$

$$\Big\downarrow \text{GS}$$

$$Q = [\underbrace{u_1 \quad u_2 \quad \cdots \quad u_n}_{}] \quad \text{orthonormal}$$

$\text{Span}\{x_1\} = \text{Span}\{u_1\}$

$\text{Span}\{x_1, x_2\} = \text{Span}\{u_1, u_2\}$

$\vdots$

$\underset{\substack{\| \\ \text{Col}(A)}}{\text{Span}\{x_1, \cdots x_n\}} = \underset{\substack{\| \\ \text{Col}(Q)}}{\text{Span}\{u_1, \cdots, u_n\}}$

$$Q^T A = \underbrace{Q^T Q}_{= I} R = R = \begin{bmatrix} x_1 \cdot u_1 & x_2 \cdot u_1 & \cdots \\ & \ddots & \\ 0 & & \end{bmatrix}$$

LU decomp.

$$\underbrace{A \xrightarrow{E_1} \cdots \xrightarrow{E_2} \cdots \cdots \xrightarrow{E_k}}_{\text{replacement}} \text{REF} = U$$

$$E_k \cdots E_2 E_1 A = U$$

$$A = \underbrace{E_1^{-1} \cdots E_{k-1}^{-1} E_k^{-1}}_{= L} U$$

4. Fill in the blanks.

Colums are lin. indep.

(a) If $A \in \mathbb{R}^{M\times N}$, $M < N$, and $A\vec{x} = 0$ does not have a non-trivial solution, how many pivot columns does A have? $\boxed{}$

(b) Consider the following linear transformation.

$T: \mathbb{R}^2 \to \mathbb{R}^3$

Domain Codomain

$$T(x_1, x_2) = (2x_1 - x_2,\ 4x_1 - 2x_2,\ x_2 - 2x_1)\,.$$

The domain of T is $\boxed{}$. The image of $\vec{x} = \begin{pmatrix}1\\0\end{pmatrix}$ under $T(\vec{x})$ is $\begin{pmatrix}2\\4\\-2\end{pmatrix}$. The co-domain of T is $\boxed{}$. The range of T is:

$$\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}-1\\-2\\1\end{bmatrix} \qquad = \begin{bmatrix}2 & -1\\4 & -2\\-2 & 1\end{bmatrix}$$

The range of T = Col(A), $A = \begin{bmatrix}| & |\\ T(e_1) & T(e_2)\\ | & |\end{bmatrix}$

5. Four points in $\mathbb{R}^2$ with coordinates (t, y) are $(0, 1)$, $(\frac{1}{4}, \frac{1}{2})$, $(\frac{1}{2}, -\frac{1}{2})$, and $(\frac{3}{4}, -\frac{1}{2})$. Determine the values of c_1 and c_2 for the curve $y = c_1 \cos(2\pi t) + c_2 \sin(2\pi t)$ that best fits the points. Write the values you obtain for c_1 and c_2 in the boxes below.

$$c_1 = \boxed{} \qquad c_2 = \boxed{}$$

$$1 = c_1 \cos(0) + c_2 \sin(0) = 1\cdot c_1 + 0\cdot c_2$$

$$\frac{1}{2} = c_1 \cos\left(\frac{\pi}{2}\right) + c_2 \sin\left(\frac{\pi}{2}\right) = 0\cdot c_1 + 1\cdot c_2$$

$$-\frac{1}{2} = c_1 \cos(\pi) + c_2 \sin(\pi) = -1\cdot c_1 + 0\cdot c_2$$

$$-\frac{1}{2} = c_1 \cos\left(\frac{3\pi}{2}\right) + c_2 \sin\left(\frac{3\pi}{2}\right) = 0\cdot c_2 - 1\cdot c_2$$

$$\underbrace{\begin{bmatrix}1\\-\frac{1}{2}\\-\frac{1}{2}\\-\frac{1}{2}\end{bmatrix}}_{=b} = \underbrace{\begin{bmatrix}1 & 0\\0 & 1\\-1 & 0\\0 & -1\end{bmatrix}}_{A} \underbrace{\begin{bmatrix}c_1\\c_2\end{bmatrix}}_{x}$$

$Ax = b$

$A^TAx = A^Tb$

Normal Eqn.

In-Class Final Exam Review Set B, Math 1554, Fall 2019

1. Indicate whether the statements are true or false.

true	false	
X		For any vector $\vec{y} \in \mathbb{R}^2$ and subspace W, the vector $\vec{v} = \vec{y} - \text{proj}_W\vec{y}$ is orthogonal to W.
	X	If A is $m \times n$ and has linearly dependent columns, then the columns of A cannot span $\mathbb{R}^m$. $\begin{bmatrix} 1 & 1 \end{bmatrix}$
	X	If a matrix is invertible it is also diagonalizable. $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$
X		If E is an echelon form of A, then $\text{Null}\,A = \text{Null}\,E$.
	X	If the SVD of $n \times n$ singular matrix A is $A = U\Sigma V^T$, then $\text{Col}A = \text{Col}U$. ($\text{Col}U = \mathbb{R}^n$)
	X	If the SVD of $n \times n$ matrix A is $A = U\Sigma V^T$, $r = \text{rank}A$, then the first r columns of V give a basis for $(\text{Null}A.)^\perp$

2. Give an example of:

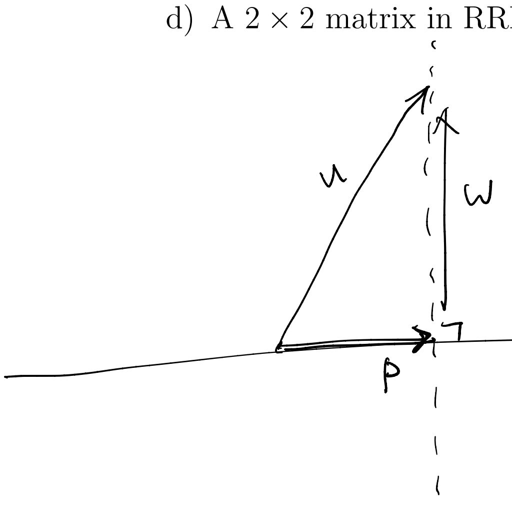
a) a vector $\vec{u} \in \mathbb{R}^3$ such that $\text{proj}_{\vec{p}}\vec{u} = \vec{p}$, where $\vec{u} \neq \vec{p}$, and $\vec{p} = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$: $\quad \vec{u} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

b) an upper triangular 4×4 matrix A that is in RREF, 0 is its only eigenvalue, and its corresponding eigenspace is 1-dimensional. $\quad A = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$= \text{Nul}(A)$ $\qquad$ dim = 1 $\qquad$ dim = 2

c) A 3×4 matrix, A, and $\text{Col}(A)^\perp$ is spanned by $\begin{pmatrix} 1 \\ -3 \\ -4 \end{pmatrix}$.

d) A 2×2 matrix in RREF that is diagonalizable and not invertible.



$u = p + w$

and $\quad p \cdot w = 0$.

$w = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$

$y, \quad W$

$y = \underbrace{\hat{y}}_{= \text{proj}_W(y)} + \underbrace{w^\perp}_{\in W^\perp}$ unique.

SVD

$$\underset{m\times n}{A} = \underset{m\times m}{U} \cdot \underset{m\times n}{\Sigma'} \cdot \underset{n\times n}{V^T}$$

$$V = [\; \underbrace{v_1 \;\; v_2 \cdots v_r}_{\text{ONB for } Nul(A)^\perp = Col(A^T) = Row(A)} \;\; \underbrace{v_{r+1} \cdots v_n}_{\text{ONB for } \underline{Nul(A)}} \;]$$

Eigenvectors for A^TA

$\in Nul(A^TA) = Nul(A)$

$Nul(A)^\perp$
$\shortparallel$
$Col(A^T)$
$\shortparallel$
$Row(A)$

$$U = [\; \underbrace{u_1 \;\; u_2 \cdots u_r}_{\text{ONB for } Col(A)} \;\; \underbrace{u_{r+1} \cdots u_m}_{\text{ONB for } Col(A)^\perp} \;]$$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{1}{4} & -\frac{3}{4} & 0 & 0 \end{bmatrix} 3$$

$$Col(A)^\perp = \text{Span}\left\{\begin{bmatrix} 1 \\ -3 \\ -4 \end{bmatrix}\right\}$$

$\underline{Nul(A^T)} =$

$$A^T = \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} & '' & \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -3 \\ -4 \end{bmatrix} = 0 = \begin{bmatrix} 1-4a \\ -3-4b \\ 0 \\ 0 \end{bmatrix}$$

$a = \frac{1}{4}$

$b = -\frac{3}{4}$

$2x_1 - x_2 = 3$

$x_2 = 2x_1 - 3$

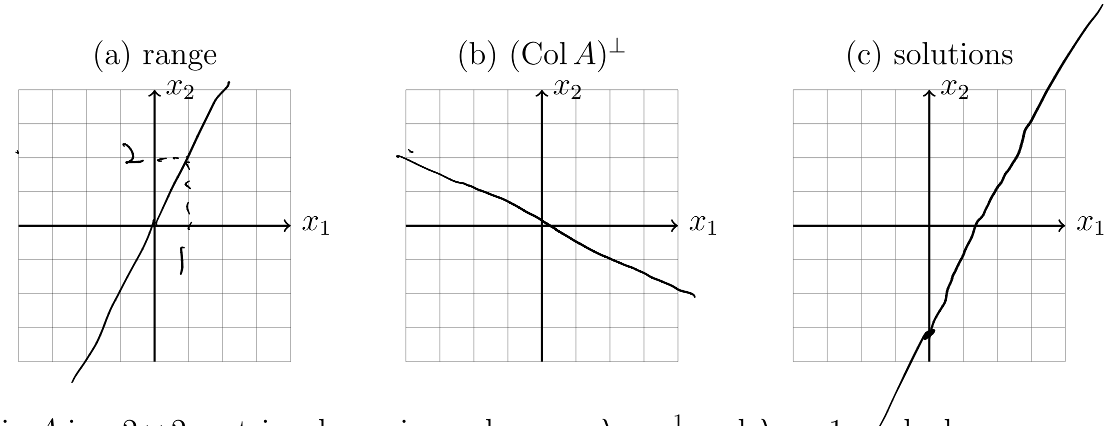
$$= (2x_1 - x_2)\begin{bmatrix} 1 \\ 2 \end{bmatrix} = 3 \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$\left[\begin{array}{cc|c} 2 & -1 & 3 \\ 4 & -2 & 6 \end{array}\right]$ // $Ax = \begin{bmatrix}2\\4\end{bmatrix}\cdot x_1 + \begin{bmatrix}-1\\-2\end{bmatrix}\cdot x_2 = \begin{bmatrix}3\\6\end{bmatrix}$

$T: x \mapsto Ax$

3. Suppose $A = \begin{pmatrix} 2 & -1 \\ 4 & -2 \end{pmatrix}$. On the grid below, sketch a) the range of $x \to Ax$, b) $(\text{Col } A)^{\perp}$, (c) set of solutions to $A\vec{x} = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$.

$\parallel$ Col(A)



Slope = 2

y-int = −3

4. Matrix A is a 2×2 matrix whose eigenvalues are $\lambda_1 = \frac{1}{2}$ and $\lambda_2 = 1$, and whose corresponding eigenvectors are $\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$. Calculate

1. $A(\vec{v}_1 + 4\vec{v}_2)$ $= Av_1 + 4\cdot Av_2 = \frac{1}{2}v_1 + 4\cdot 1\cdot v_2 = \begin{bmatrix}\frac{1}{2}\\0\end{bmatrix} + \begin{bmatrix}16\\4\end{bmatrix}$
2. A^{10}
3. $\lim_{k\to\infty} A^k(\vec{v}_1 + 4\vec{v}_2)$ $= \begin{bmatrix}0 & 4\\0 & 1\end{bmatrix}\cdot\begin{bmatrix}17\\4\end{bmatrix} = \begin{bmatrix}16\\4\end{bmatrix}$

$A^k = P\cdot D^k \cdot P^{-1}$

$= \underbrace{\begin{bmatrix}1 & 4\\0 & 1\end{bmatrix}\begin{bmatrix}(\frac{1}{2})^k & 0\\0 & 1^k\end{bmatrix}}\begin{bmatrix}1 & -4\\0 & 1\end{bmatrix}$

$= \begin{bmatrix}(\frac{1}{2})^k & 4\\0 & 1\end{bmatrix}\begin{bmatrix}1 & -4\\0 & 1\end{bmatrix} = \begin{bmatrix}(\frac{1}{2})^k & -4(\frac{1}{2})^k + 4\\0 & 1\end{bmatrix}$

$\lim_{k\to\infty} A^k = \begin{bmatrix}0 & 4\\0 & 1\end{bmatrix}$

$A^k(v_1 + 4v_2) = \underline{A^k\cdot v_1} + 4\cdot(A^k\cdot v_2)$

$= \cancel{(\frac{1}{2})^k\cdot v_1} \to 0 + 4\cdot 1^k\cdot v_2 = 4\begin{bmatrix}4\\1\end{bmatrix}$

$T : \mathbb{R}^n \to \mathbb{R}^m$ lin. transformation.

$T(x) = A \cdot x$

$$\underline{\underline{A}} = \begin{bmatrix} | & | & & | \\ T(e_1) & T(e_2) & \cdots & T(e_n) \\ | & | & & | \end{bmatrix}$$

$m \times n$

dim of codomain

dim of domain

Range of T

$= \mathrm{Col}(A)$

In-Class Final Exam Review Set C, Math 1554, Fall 2019

1. Indicate whether the statements are possible or impossible.

$\nearrow \vec{v} = 0$

possible	impossible	
○	⊗	$Q(\vec{x}) = \vec{x}^T A\vec{x}$ is a positive definite quadratic form, and $Q(\vec{v}) = 0$, where $\vec{v}$ is an eigenvector of A. $\neq 0$
○	⊗	The maximum value of $Q(\vec{x}) = ax_1^2 + bx_2^2 + cx_3^2$, where $a > b > c$, for $\vec{x} \in \mathbb{R}^3$, subject to $\|\|\vec{x}\|\| = 1$, is not unique.
⊗	○	The location of the maximum value of $Q(\vec{x}) = ax_1^2 + bx_2^2 + cx_3^2$, where $a > b > c$, for $\vec{x} \in \mathbb{R}^3$, subject to $\|\|\vec{x}\|\| = 1$, is not unique.
○	⊗	$\dim(\text{Nul}(A)) = 1$ A is 2×2, the algebraic multiplicity of eigenvalue $\lambda = 0$ is 1, and $\dim(\text{Col}(A)^{\perp})$ is equal to 0. $= 0$ $\dim(\text{Col}(A)) = 2$
○	○	Stochastic matrix P has zero entries and is regular.
○	○	A is a square matrix that is not diagonalizable, but A^2 is diagonalizable.
○	○	The map $T_A(\vec{x}) = A\vec{x}$ is one-to-one but not onto, A is $m \times n$, and $m < n$.

2. Transform $T_A = A\vec{x}$ reflects points in $\mathbb{R}^2$ through the line $y = 2 + x$. Construct a standard matrix for the transform using homogeneous coordinates. Leave your answer as a product of three matrices.

3. Fill in the blanks.

   (a) $T_A = A\vec{x}$, where $A \in \mathbb{R}^{2\times 2}$, is a linear transform that first rotates vectors in $\mathbb{R}^2$ clockwise by $\pi/2$ radians about the origin, then reflects them through the line $x_1 = x_2$. What is the value of $\det(A)$? $\boxed{}$

   (b) B and C are square matrices with $\det(BC) = -5$ and $\det(C) = 2$. What is the value of $\det(B)\det(C^4)$? $\boxed{}$

   (c) A is a 6×4 matrix in RREF, and $\text{rank}(A) = 4$. How many different matrices can you construct that meet these criteria? $\boxed{}$

   (d) $T_A = A\vec{x}$, where $A \in \mathbb{R}^{2\times 2}$, projects points onto the line $x_1 = x_2$. What is an eigenvalue of A equal to? $\boxed{}$

   (e) If an eigenvalue of A is $\frac{1}{3}$, what is one eigenvalue of A^{-1} equal to? $\boxed{}$

   (f) If A is 30×12 and $A\vec{x} = \vec{b}$ has a unique least squares solution $\hat{x}$ for every $\vec{b}$ in $\mathbb{R}^{30}$, the dimension of Null$A$ is $\boxed{}$.

4. A is a 2×2 matrix whose nullspace is the line $x_1 = x_2$, and $C = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$. Sketch the nullspace of $Y = AC$.

5. Construct an SVD of $A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$. Use your SVD to calculate the condition number of A.