MIDTERM EXAM 2

DATE : OCT 11 (WED)

TIME : 6:30 PM – 7:45PM

PLACE : SECTION A – BOGGS B5
(8:25)

SECTION E – HOWEY–PHYSICS L3
(11:00)

COVERAGE: UP TO FRI CLASS (5.2)

REVIEW : OCT 11 in CLASS
(SAMPLE EXAMS : S22, F22, S23)
↑
MASTER WEBPAGE.

OCT 9 NO CLASS

F22 : #9 T/F ①

S23 : #4-(c)

└ makeup #5

S22

└ makeup T/F ③

Math 1554 Linear Algebra Spring 2023

Midterm 2

PLEASE PRINT YOUR NAME CLEARLY IN ALL CAPITAL LETTERS

Name: ________________________ GTID Number: ___________________

Student GT Email Address: ________________________________@gatech.edu

Section Number (e.g. A3, G2, etc.) _______ TA Name ________________________

Circle your instructor:

Prof Kim Prof Barone Prof Schroeder Prof Kumar

Student Instructions

- **Show your work** and justify your answers for all questions unless stated otherwise.
- **Organize your work** in a reasonably neat and coherent way.
- **Simplify your answers** unless explicitly stated otherwise.
- **Fill in circles** completely. Do not use check marks, X's, or any other marks.
- Calculators, notes, cell phones, books are not allowed.
- Use dark and clear writing: your exam will be scanned into a digital system.
- Exam pages are double sided. Be sure to complete both sides.
- Leave a 1 inch border around the edges of exams.
- The last page is for scratch work. Please use it if you need extra space.
- This exam has 8 pages of questions.

Midterm 2. Your initials: ______________

You do not need to justify your reasoning for questions on this page.

1. (a) (8 points) Suppose A is a real $m \times n$ matrix and $\vec{b} \in \mathbb{R}^m$ unless otherwise stated. Select **true** if the statement is true for all choices of A and $\vec{b}$. Otherwise, select **false**.

true	false	
○	○	If $A, B \in \mathbb{R}^{n \times n}$ and $AB\vec{x} = \vec{0}$ has a non-trivial solution, then A is not invertible.
○	○	If A has LU-factorization $A = LU$, then $\det(L) = 1$.
○	○	If A and B share an eigenvector $\vec{x}$ corresponding to eigenvalue λ, so that λ is an eigenvalue of both A and B for the same eigenvector $\vec{x}$, then 2λ must be an eigenvalue of the matrix $A + B$.
○	○	If A is $m \times n$ and $A\vec{x} = b$ has a solution for every $\vec{b} \in \mathbb{R}^m$, then Col$(A) = \mathbb{R}^m$.
○	○	If $\det(A) = 1$ and $\det(B) = 0$, then $AB = BA$.
○	○	If A is $n \times n$ and 0 is an eigenvalue of A, then the transformation $T(\vec{x}) = A\vec{x}$ is not onto.
○	○	If $\vec{x}$ and $\vec{y}$ are probability vectors, then $\frac{1}{3}\vec{x} + \frac{2}{3}\vec{y}$ is a probability vector.
○	○	If A is 3×3, then $\det(2A) = 2\det(A)$.

You do not need to justify your reasoning for questions on this page.

(b) (4 points) Indicate whether the following situations are possible or impossible.

possible	impossible	
◯	◯	$A \in \mathbb{R}^{6\times 6}$, and rank$(A) = \dim$ Nul(A).
◯	◯	A 3×3 matrix whose nullspace is spanned by $\left\{ \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \\ 3 \end{pmatrix} \right\}$ and whose column space is spanned by $\left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \right\}$.
◯	◯	A 4×6 matrix A with a null space of dimension 5.
◯	◯	An $n \times n$ matrix A with $\det(AA^T) = -1$.

(c) (2 points) If A is the standard matrix for the transformation that projects vectors in $\mathbb{R}^3$ to the xy-plane, then what is the dimension of the null space of A? *Select only one.*

◯ 0
◯ 1
◯ 2
◯ 3

You do not need to justify your reasoning for questions on this page.

(d) (2 points) Suppose an 3×3 matrix A can be row reduced to reduced row echelon form (RREF) using only row replacement row operations (without any row swaps/scaling). Among the options listed below, which are possible values for $\det(A)$?
Select all that apply.

- ○ -1
- ○ 0
- ○ 1
- ○ 3

2. (3 points) Suppose B is a 2×5 matrix and C is 3×4 matrix. Find the dimensions of the matrices A, D, and M for the block matrix

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

A has		rows and		columns.
D has		rows and		columns.
M has		rows and		columns.

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You do not need to justify your reasoning for questions on this page.

3. (2 points) Find the dimension of the subspace S consisting of all vectors $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$ which satisfy the conditions that

$$\begin{aligned} x_1 + x_2 + x_3 - x_4 &= 0 \\ x_1 + 3x_2 - x_3 + 2x_4 &= 0 \\ 2x_1 + 4x_2 + 4x_3 + 3x_4 &= 0 \end{aligned}$$

$$\dim(S) = \boxed{}$$

4. (4 points) Suppose $\det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = 4$. Find the determinant of the matrices below.

$$A = \begin{pmatrix} g & h & i \\ a & b & c \\ d & e & f \end{pmatrix} \qquad B = \begin{pmatrix} a & b & c \\ 2d \cancel{+a} & 2e \cancel{+b} & 2f \cancel{+c} \\ g & h & i \end{pmatrix} \qquad C = \begin{pmatrix} a & a-c & c \\ d & d-f & f \\ g & g-i & i \end{pmatrix}$$

$$\det(A) = \boxed{4} \qquad \det(B) = \boxed{2 \cdot 4} \qquad \det(C) = \boxed{0}$$

$$\det(2A) = 2^n \cdot \det(A) \qquad A \in \mathbb{R}^{n \times n}$$

$$\det(-A) = (-1)^n \det(A)$$

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5. (3 points) Give a matrix A in RREF whose column space is spanned by $\{\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\}$ and whose null space is spanned by $\left\{\begin{pmatrix}-2\\1\\0\end{pmatrix}, \begin{pmatrix}3\\0\\1\end{pmatrix}\right\}$. If this is not possible, write `NP` in the box.

$A=$ $\boxed{}$

6. (2 points) Consider the transformation $T(\vec{x}) = A\vec{x}$ which reflects vectors in $\mathbb{R}^2$ across the line $x_1 = x_2$. List in the box the real eigenvalues of the matrix A, or write `NP` in the box if there are no real eigenvalues.

$\boxed{}$

7. (4 points) **Show all work for problems on this page.**

   Find an eigenvector $\vec{v}$ for the eigenvalue $\lambda = 3$ of A. *Hint: check your answer.*

$$A = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 4 & -1 \\ 1 & 0 & 4 \end{bmatrix}$$

$\vec{v}=$ $\boxed{}$

8. (6 points) Find the LU-factorization of

$$A = \begin{pmatrix} 1 & 5 & 6 \\ -1 & 1 & 2 \\ 2 & 7 & 8 \end{pmatrix}.$$

$L =$ $\boxed{}$

$U =$ $\boxed{}$

9. (4 points) **Show all work for problems on this page.**

   Find all possible values of k such that the matrix A is singular.
   Hint: use cofactor expansion to compute the determinant.

$$A = \begin{pmatrix} 1 & -3 & k \\ 7 & 2 & -3 \\ -1 & 2 & 5 \end{pmatrix}$$

$$k = \boxed{}$$

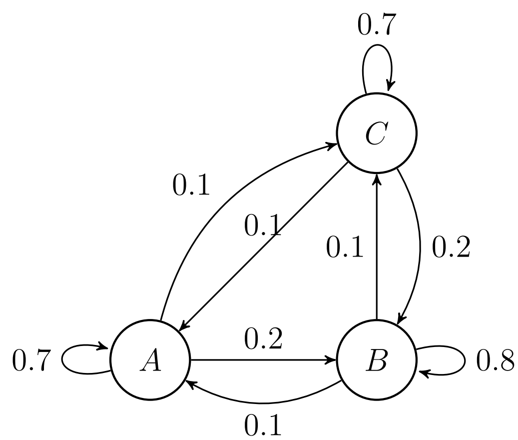
10. (6 points) **Show your work for part (c) on this page.**

Use the following Markov chain diagram to answer the questions.

(a) Find the stochastic matrix P of the Markov chain.

(b) Find the unique steady state probability vector $\vec{q}$ of P.

(c) What is $\det(P - I)$?



$P =$ $\boxed{}$

$\vec{q}=$ $\boxed{}$

$\det(P - I) =$ $\boxed{}$

Math 1554 Linear Algebra   Fall   2022

Midterm 2

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Section Number (e.g. A3, G2, etc.) ________   TA Name _________________________

Circle your instructor:

Prof Vilaca Da Rocha   Prof Kafer   Prof Barone   Prof Wheeler

Prof Blumenthal   Prof Sun   Prof Shirani

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$(A+2B)\vec{v} = \underbrace{A\vec{v}} + 2\cdot \underbrace{B\vec{v}} = \lambda\vec{v} + 2\mu\cdot\vec{v} = (\lambda+2\mu)\vec{v}$

Midterm 2. Your initials: _______________

You do not need to justify your reasoning for questions on this page.

F22

1. (a) (8 points) Suppose A is an $m \times n$ matrix and $\vec{b} \in \mathbb{R}^m$ unless otherwise stated. Select **true** if the statement is true for all choices of A and $\vec{b}$. Otherwise, select **false**.

true	false	
○	○	If A, B and C are $n \times n$ matrices, B is invertible and $\boxed{AC = B}$, then C is invertible.
		A
		$0 \neq \det(A)\cdot\underset{\neq 0}{\det(C)} = \det(AC) = \det(B) \neq 0$
○	○	If $A = LU$ is an LU-factorization of a square matrix A, then $\det(A) = \det(U)$
○	○	If $\vec{x}$ is a vector in $\mathbb{R}^3$ and B is a basis for $\mathbb{R}^3$, then $[\vec{x}]_B$ has 3 entries.
○	○	If A, B and C are $n \times n$ matrices, A is invertible and $AB = AC$, then $B = C$.
○	○	If $A \in \mathbb{R}^{m\times n}$ and $\vec{b} \in \mathbb{R}^m$, then the set of solutions $\vec{x}$ to the system $A\vec{x} = \vec{b}$ is a subspace of $\mathbb{R}^n$.
○	○	The set of all probability vectors in $\mathbb{R}^n$ is a subspace of $\mathbb{R}^n$.
○	○	If two matrices A, B share an eigenvector $\vec{v}$, with eigenvalue λ for matrix A and eigenvalue μ for the matrix B, then $\vec{v}$ is an eigenvector of the matrix $(A + 2B)$ with eigenvalue $\lambda + 2\mu$.
○	○	For any 2×2 real matrix A, we have $\det(-A) = -\det(A)$.

Null (A)
$\parallel$
subspace
$\parallel$
Solution set
of $A\vec{x} = \cancel{\vec{b}}\ 0$
$\neq$ subspace

(b) (4 points) Indicate whether the following situations are possible or impossible.

possible	impossible	
○	○	A matrix $A \in \mathbb{R}^{n\times n}$ such that A is invertible and A^T is singular.
○	○	A 3×3 matrix A with $\dim(\text{Null}(A)) = 0$ such that the system $A\vec{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ has no solution.
○	○	$T : \mathbb{R}^3 \to \mathbb{R}^3$ that is onto and its standard matrix has determinant equal to -1.
○	○	Two square matrices A, B with $\det(A)$ and $\det(B)$ both non-zero, and the matrix AB is singular.

You do not need to justify your reasoning for questions on this page.

(c) (3 points) If possible, fill in the missing elements of the matrices below with numbers so that each of the matrices are singular. If it is not possible write `NP` in the space.

$$\begin{pmatrix} 1 & & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 3 \\ 1 & & 4 \end{pmatrix} \qquad \begin{pmatrix} 2 & 3 & \\ 1 & 2 & 5 \\ 0 & 1 & 2 \end{pmatrix}$$

(d) (2 points) Let A be a 3×3 upper triangular matrix and assume that the volume of the parallelpiped determined by the columns of A is equal to 1. Which of the following statements is FALSE?

- ○ A is invertible.
- ○ The diagonal entries of A are either 1 or -1.
- ○ For every 3×3 matrix B we have $|\det(AB)| = |\det(B)|$.
- ○ If B is a matrix obtained by interchanging two rows of A, then the volume of the parallelepiped determined by the columns of B is equal to 1.

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You do not need to justify your reasoning for questions on this page.

2. (2 points) Suppose A and B are invertible $n \times n$ matrices. Find the inverse of the partitioned matrix

$$\begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}^{-1} = \begin{pmatrix} \_\_\_\_ & \_\_\_\_ \\ \_\_\_\_ & \_\_\_\_ \end{pmatrix}.$$

3. (2 points) Suppose A is a $m \times n$ matrix and B is $m \times 5$ matrix. Find the dimensions of the matrix C in the block matrix

$$\begin{pmatrix} A & B \\ I_n & C \end{pmatrix}.$$

C has $\boxed{}$ rows and $\boxed{}$ columns.

You do not need to justify your reasoning for questions on this page.

4. Fill in the blanks.

(a) (3 points) Give a matrix A whose column space is spanned by the vectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and whose null space is spanned by $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. If this is not possible, write `NP` in the box.

$$A = \boxed{}$$

(b) (3 points) Use the determinant to find all values of $\lambda \in \mathbb{R}$ such that the following matrix is singular.

$$\begin{pmatrix} 1 & 1 & 2 \\ 2 & 4 & 5 \\ \lambda & 2 & 3 \end{pmatrix}.$$

$$\lambda = \boxed{}$$

Midterm 2. Your initials: _______________

You do not need to justify your reasoning for questions on this page.

5. (3 points) Find the value of h such that the matrix

$$A = \begin{pmatrix} 5 & h \\ 1 & 3 \end{pmatrix}$$

has an eigenvalue with algebraic multiplicity 2.

$$h = \boxed{}$$

6. (3 points) Let $\mathcal{P}_B$ be a parallelogram that is determined by the columns of the matrix $B = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}$, and $\mathcal{P}_C$ be a parallelogram that is determined by the columns of the matrix $C = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$. Suppose A is the standard matrix of a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ that maps $\mathcal{P}_B$ to $\mathcal{P}_C$. What is the value of $|\det(A)|$?

$$|\det(A)| = \boxed{}$$

7. (5 points) **Show all work for problems on this page.**
Given that 4 is an eigenvalue of the matrix

$$A = \begin{pmatrix} 6 & -2 & 2 \\ 2 & 2 & -2 \\ 1 & -1 & 4 \end{pmatrix},$$

find an eigenvector $\vec{v}$ of A such that $A\vec{v} = 4\vec{v}$.

$\vec{v}$= $\boxed{}$

8. (6 points) Find the LU-factorization of

$$A = \begin{pmatrix} 1 & 2 & 5 \\ 1 & -1 & 8 \\ 2 & 8 & 6 \end{pmatrix}.$$

$L =$ $\boxed{}$

$U =$ $\boxed{}$

v_1 eigenvector $\lambda = 1 \Rightarrow$ $P \cdot \vec{v}_1 = 1 \cdot \vec{v}_1$

$P^2 \vec{v}_1 = P \cdot (P\vec{v}_1) = P \cdot \vec{v}_1 = \vec{v}_1$

$P^3 \vec{v}_1 = \vec{v}_1$

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F22

9. (6 points) **Show all work for problems on this page.**
Consider the Markov chain $\vec{x}_{k+1} = P\vec{x}_k$, $k = 0, 1, 2, \ldots$.
Suppose P has eigenvalues $\lambda_1 = 1$, $\lambda_2 = 1/2$ and $\lambda_3 = 0$. Let $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$ be eigenvectors corresponding to λ_1, λ_2, and λ_3, respectively:

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, \quad \vec{v}_3 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}.$$

Note: you may leave your answers as linear combinations of the vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$.

(i) If $\vec{x}_0 = \frac{1}{2}\vec{v}_1 + \frac{1}{2}\vec{v}_2$, then what is $\vec{x}_3$?

$$\vec{x}_3 = P^3 \cdot \vec{x}_0$$
$$= P^3 \cdot \left(\tfrac{1}{2}\vec{v}_1 + \tfrac{1}{2}\vec{v}_2\right)$$
$$= \tfrac{1}{2} \underbrace{P^3 \cdot \vec{v}_1}_{\vec{v}_1} + \tfrac{1}{2} \cdot \underbrace{P^3 \vec{v}_2}_{\left(\frac{1}{2}\right)^3 \cdot \vec{v}_2}$$
$$= \tfrac{1}{2}\vec{v}_1 + \tfrac{1}{16}\vec{v}_2$$

$\vec{v}_2$: eigenvector $\lambda = \frac{1}{2}$

$P\vec{v}_2 = \frac{1}{2} \cdot \vec{v}_2$ $\quad P^2\vec{v}_2 = \left(\frac{1}{2}\right)^2 \vec{v}_2$

$\vec{x}_3 = \boxed{\begin{bmatrix} 8/16 \\ 7/16 \\ 1/16 \end{bmatrix}}$

(ii) If $\vec{x}_0 = \begin{pmatrix} 1/4 \\ 1/2 \\ 1/4 \end{pmatrix}$, then what is $\vec{x}_1$?
Hint: write $\vec{x}_0$ as a linear combination of $\vec{v}_1, \vec{v}_2, \vec{v}_3$.

$$\vec{x}_1 = P \cdot \vec{x}_0 = P \cdot (a_1 \vec{v}_1 + a_2 \vec{v}_2 + a_3 \cdot \vec{v}_3)$$
$$= a_1 \cdot P\vec{v}_1 + a_2 \cdot P\vec{v}_2 + a_3 \cdot P\vec{v}_3$$
$$= a_1 \cdot 1 \cdot \vec{v}_1 + a_2 \cdot \tfrac{1}{2} \cdot \vec{v}_2 + 0.$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 1/4 \\ 1 & -1 & 1 & 1/2 \\ 0 & 1 & 0 & 1/4 \end{array}\right] \xrightarrow[\text{operations}]{\text{row}} \ldots$$

$\vec{x}_1 = \boxed{}$

steady-state vector ?

(iii) If $\vec{x}_0 = \begin{pmatrix} 1/4 \\ 1/2 \\ 1/4 \end{pmatrix}$, then what is $\vec{x}_k$ as $k \to \infty$?

$$P\vec{q} = \vec{q}$$

$\lim_{k\to\infty} \vec{x}_k = \boxed{\frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}$

Math 1554 Linear Algebra Spring 2022

Midterm 2

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Name: ________________________ GTID Number: ___________________

Student GT Email Address: ______________________________@gatech.edu

Section Number (e.g. A3, G2, etc.) _______ TA Name ________________________

Circle your instructor:

Prof Barone Prof Shirani Prof Simone Prof Timko

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Midterm 2. Your initials: ______________

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1. (a) (10 points) Suppose A is an $m \times n$ matrix and $\vec{b} \in \mathbb{R}^m$ unless otherwise stated. Select **true** if the statement is true for all choices of A and $\vec{b}$. Otherwise, select **false**.

true	false	
○	○	If $k > n$ and $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ spans $\mathbb{R}^n$, then $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ is a basis for $\mathbb{R}^n$.
○	○	If $A, B, C \in \mathbb{R}^{n\times n}$ and $AB = I_n = BC$, then $A = C$.
○	○	If $A \in \mathbb{R}^{n\times n}$ is invertible, then $A^T A$ is invertible.
○	○	If $A\vec{x} \neq A\vec{y}$ for all vectors $\vec{x} \neq \vec{y}$, then Null$(A) \neq \{\vec{0}\}$.
○	○	If LU is the LU factorization of a square matrix A, then A is invertible if and only if U is invertible.
○	○	If the rank of an $n \times n$ matrix A is equal to n, then all diagonal entries of a row echelon form of A are nonzero.
○	○	The set of all probability vectors in $\mathbb{R}^n$ is a subspace of $\mathbb{R}^n$.
○	○	If P is the stochastic matrix of a Markov chain, then any probability vector in Null$(P - I)$ is a steady-state vector for the Markov chain.
○	○	If M is an $n \times n$ matrix and $\det(M^{2022}) = 1$, then M has linearly independent columns.
○	○	If $A, B \in \mathbb{R}^{n\times n}$, $\det A = 2$, and $\det B = -3$, then the product AB is invertible.

(b) (4 points) Indicate whether the following situations are possible or impossible.

possible	impossible	
○	○	A is the standard matrix of an onto linear transformation $T : \mathbb{R}^5 \to \mathbb{R}^3$ with $\dim(\text{Null}(A)) = 3$.
○	○	A is the standard matrix of a one-to-one linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^5$ with rank$(A) = 2$.
○	○	A is a matrix whose columns do not form a basis for Col(A).
○	○	A is a 5×3 matrix with rank$(A) = 2\dim(\text{Null}(A))$.

Midterm 2. Your initials: ______________

You do not need to justify your reasoning for questions on this page.

(c) (2 points) The column space of a matrix A is spanned by the vector $\vec{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and the null space of A has dimension 2. Which one of the following statements is **false**? *Choose only one.*

- ◯ $\text{rank}(A) = 1$.
- ◯ A is a 2×3 matrix.
- ◯ If U is an echelon form of A, then $\{\vec{v}\}$ is a basis for $\text{Col}(U)$.
- ◯ The linear system $A\vec{x} = c\vec{v}$ is consistent for all values of $c \in \mathbb{R}$.

2. (2 points) Suppose $A, B \in \mathbb{R}^{n\times n}$ with $AB = -BA$ and $A^2 = B^2$. Fill in the blanks in the following equation **using only numbers** to make it true.

$$\begin{bmatrix} A & B \\ B & A \end{bmatrix}^2 = \begin{bmatrix} \_\_\_\_A^2 & \_\_\_\_I_n \\ \_\_\_\_I_n & \_\_\_\_A^2 \end{bmatrix}.$$

You do not need to justify your reasoning for questions on this page.

3. (2 points) Let $\mathcal{H}$ be a subspace of $\mathbb{R}^3$ that is composed of all vectors $\vec{x} = (x_1, x_2, x_3)$ that satisfy the following two equations:

$$x_1 + 3x_2 - x_3 = 0$$
$$2x_1 + 5x_2 + x_3 = 0$$

What is the dimension of $\mathcal{H}$?

$$\dim \mathcal{H} = \boxed{}$$

4. (2 points) Let $\mathcal{V}$ be a subspace of $\mathbb{R}^3$ that is spanned by the vectors

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ -3 \\ 0 \end{pmatrix} \right\}$$

What is the dimension of $\mathcal{V}$?

$$\dim \mathcal{V} = \boxed{}$$

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You do not need to justify your reasoning for questions on this page.

5. (4 points) Find the LU factorization of A, where

$$A = \begin{bmatrix} 1 & 2 & 5 \\ 1 & -1 & 8 \\ 2 & 8 & 6 \end{bmatrix},$$

by filling in the blanks below.

$$L = \begin{bmatrix} 1 & 0 & 0 \\ \_\_\_ & 1 & 0 \\ \_\_\_ & \_\_\_ & 1 \end{bmatrix}, \quad U = \begin{bmatrix} \_\_\_ & \_\_\_ & \_\_\_ \\ 0 & \_\_\_ & \_\_\_ \\ 0 & 0 & \_\_\_ \end{bmatrix}.$$

6. (4 points) Find a basis for the $\lambda = -1$ eigenspace of the matrix A. *Hint: Check your answer.*

$$A = \begin{bmatrix} -7 & -6 & -6 \\ 6 & 5 & 6 \\ 3 & 3 & 2 \end{bmatrix}$$

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7. (6 points) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation defined by

$$T(x_1, x_2) = (x_1 + 3x_2, x_1 + x_2).$$

Let R be the rectangle in $\mathbb{R}^2$ with vertices $(0, 0)$, $(1, 0)$, $(0, 3)$, $(1, 3)$.

(i) What is the standard matrix of T?

(ii) What is the area of the rectangle R?

(iii) Find the area of the image of R under the linear transformation T.

8. (6 points) ***Show work*** *on this page with work under the problem, and* ***your answer in the box.***

Let

$$A = \begin{pmatrix} 1 & -1 & k \\ 1 & h & 2 \\ h & 1 & -2 \end{pmatrix}$$

(a) Find the value of h and the value of k such that $\dim(\text{Null}(A)) = 2$.

$h = \boxed{}$ $k = \boxed{}$

(b) Let $k = 4$. For what values of h is $\dim(\text{Col}(A)) = 2$.

$h = \boxed{}$

(c) Let $h = 0$ and $k = 0$. Is the vector $\vec{v} = \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$ in the null space of A?

Note: Compute $A\vec{v}$ and use this calculation to clearly justify your answer in a few words using the space below for full credit.

○ yes ○ no

9. (4 points) ***Show work*** *on this page with work under the problem, and* ***your answer in the box.***

Compute $\begin{bmatrix} 0 & -1 & 1 \\ 1 & 2 & 5 \\ 0 & 1 & -2 \end{bmatrix}^{-1}$. *Hint: Check your answer!*

10. (4 points) ***Show work*** *on this page with work under the problem, and* ***your answer in the box.***

Suppose

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 2 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}.$$

Solve $LU\vec{x} = \vec{b}$ for $\vec{x}$.

$$\vec{x} = \boxed{}$$

A: invertible $\Rightarrow \det(A) \neq 0$,

$$\det(A^T \cdot A) = \underline{\det(A^T)} \cdot \underline{\det(A)} = \underline{\underline{\det(A)^2}} > 0$$

Midterm 2 Make-up. Your initials: _______________

You do not need to justify your reasoning for questions on this page.

1. (a) (10 points) Suppose A is an $m \times n$ matrix and $\vec{b} \in \mathbb{R}^m$ unless otherwise stated. Select **true** if the statement is true for all choices of A and $\vec{b}$. Otherwise, select **false**.

true	false	
○	○	Any set of vectors in $\mathbb{R}^4$ which is linearly independent must be a basis for a subspace of $\mathbb{R}^4$. $\boxed{B = A^{-1}}$ $\boxed{B = C^{-1}}$ $\Rightarrow A^{-1} = C^{-1}$
○	○	If $A, B, C \in \mathbb{R}^{n\times n}$ and $AB = I_n = BC$, then $A = C^{-1}$. $A = C$
○	○	If $A \in \mathbb{R}^{n\times n}$ is invertible, then $\det(A^T A) \overset{= -2}{=} 1$.
○	○	If $A\vec{x} = A\vec{y}$ for all vectors $\vec{x}, \vec{y}$ in $\mathbb{R}^n$, then $\text{Null}(A) = \mathbb{R}^n$.
○	○	If LU is the LU factorization of a square matrix A, then $\det(A) = \det(U)$.
○	○	If the rank of an $n \times n$ matrix A is equal to $n - 1$, then all entries of some row of A are zero.
○	○	The set of all probability vectors in $\mathbb{R}^n$ is a subspace of $\mathbb{R}^n$.
○	○	If P is the stochastic matrix of a Markov chain, then any probability vector in $\text{Null}(P - I)$ is a steady-state vector for the Markov chain.
○	○	If M is an $n \times n$ matrix and $\overset{(\det(M))^2}{\det(M^2)} = 1$, then $\det(M) = \cancel{1}$ ± 1
○	○	If $A, B \in \mathbb{R}^{n\times n}$, $\det A = 1$, and $\det B = -1$, then the product AB is row equivalent to the identity matrix. $\Leftrightarrow$ $A \cdot B$ invertible $\Leftrightarrow$ $\det(A \cdot B) \neq 0$

(b) (4 points) Indicate whether the following situations are possible or impossible.

possible	impossible	
○	○	A is the standard matrix of an onto linear transformation $T : \mathbb{R}^5 \to \mathbb{R}^3$ with $\dim(\text{Null}(A)) = 3$.
○	○	A is the standard matrix of an onto linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ with $\dim \text{Nul}(A) = 1$.
○	○	A is a matrix whose columns do not form a basis for $\text{Col}(A)$.
○	○	A is a 5×4 matrix with $\text{rank}(A) = 2 \dim(\text{Null}(A))$.

Midterm 2 Make-up. Your initials: _______________

You do not need to justify your reasoning for questions on this page.

5. (3 points) Suppose $A = \begin{pmatrix} -1 & 3 & 0 \\ 2 & 0 & 1 \\ 0 & -2 & 0 \end{pmatrix}$ is a matrix with eigenvalue $\lambda_1 = 2$ with associated eigenbasis $\mathcal{B} = \{\vec{v}_1, \vec{v}_2\}$, and eigenvalue $\lambda_2 = 0$ with associated eigenvector $\vec{v}_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$. Find the coordinates of the vector $A\vec{x}$ in the basis $\mathcal{B} = \{\vec{v}_1, \vec{v}_2\}$ where

$$\vec{x} = \begin{pmatrix} 9 \\ 6 \\ -8 \end{pmatrix}, \quad \vec{v}_1 = \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} 3 \\ 0 \\ -2 \end{pmatrix}.$$

Find $\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$

$$\left[\begin{array}{ccc|c} -1 & 3 & 0 & 9 \\ 2 & 0 & 1 & 6 \\ 0 & -2 & 0 & -8 \end{array}\right]$$

$$[A\vec{x}]_{\mathcal{B}} = \boxed{\begin{bmatrix} 2a_1 \\ 2a_2 \end{bmatrix}}$$

$\vec{x} = a_1 \vec{v}_1 + a_2 \vec{v}_2 + a_3 \vec{v}_3$

$\mathcal{B} = \{\vec{v}_1, \vec{v}_2\}$

$A\vec{x} = a_1 A\vec{v}_1 + a_2 A\vec{v}_2 + a_3 A\vec{v}_3$

$= a_1 \cdot 2\vec{v}_1 + a_2 \cdot 2\vec{v}_2 + a_3 \cdot 0 \cdot \vec{v}_3 = 2a_1 \boxed{\vec{v}_1} + 2a_2 \boxed{\vec{v}_2}$

6. (2 points) Consider the transformation $T(\vec{x}) = A\vec{x}$ which rotates vectors in $\mathbb{R}^2$ by $90°$ counter-clockwise. List in the box the real eigenvalues of the matrix A, or write `NP` in the box if there are no real eigenvalues.

$\boxed{}$