Section 7.1 : Diagonalization of Symmetric Matrices

Chapter 7: Orthogonality and Least Squares

Math 1554 Linear Algebra

Topics and Objectives

Topics

1. Symmetric matrices
2. Orthogonal diagonalization

Learning Objectives

1. Construct an orthogonal diagonalization of a symmetric matrix, $A = PDP^T$.

Symmetric Matrices

should be square

Definition

Matrix A is **symmetric** if $A^T = A$.

Example. Which of the following matrices are symmetric? Symbols $*$ and $\star$ represent real numbers.

$$A = [*] \qquad B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = B^T \qquad C = \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix} = C^T$$

$$D = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \qquad E = \begin{bmatrix} 4 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad F = \begin{bmatrix} 4 & 2 & 0 & 1 \\ 2 & 0 & 7 & 4 \\ 0 & 7 & 6 & 0 \\ 1 & 4 & 0 & 3 \end{bmatrix}$$

$D^T = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ $\neq$

$E^T = \begin{bmatrix} 4 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix}$ $\neq$

Examples $A \in \mathbb{R}^{n \times n}$

- $A + A^T$: Symm. $\quad (A + A^T)^T = A^T + A$
- $A^T \cdot A$: Symm. $\quad (A^T A)^T = A^T \cdot A$.

A^TA is Symmetric for $A \in \mathbb{R}^{m \times n}$

A very common example: For **any** matrix A with columns $a_1, \ldots, a_n$,

$$A^TA = \begin{bmatrix} -- & a_1^T & -- \\ -- & a_2^T & -- \\ \vdots & \vdots & \vdots \\ -- & a_n^T & -- \end{bmatrix} \begin{bmatrix} | & | & \cdots & | \\ a_1 & a_2 & \cdots & a_n \\ | & | & \cdots & | \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} a_1^Ta_1 & a_1^Ta_2 & \cdots & a_1^Ta_n \\ a_2^Ta_1 & a_2^Ta_2 & \cdots & a_2^Ta_n \\ \vdots & \vdots & \ddots & \vdots \\ a_n^Ta_1 & a_n^Ta_2 & \cdots & a_n^Ta_n \end{bmatrix}}_{\text{Entries are the dot products of columns of } A}$$

Recall

matrix multi.

1. $u \cdot v = u^T \cdot v$ ← for real vectors

$u \cdot v = \overline{u^T} \cdot v$

$u = \begin{bmatrix} i \\ 1 \end{bmatrix}$ $\quad u \cdot u = \overline{u^T} \cdot u = \begin{bmatrix} -i & 1 \end{bmatrix} \begin{bmatrix} i \\ 1 \end{bmatrix}$

$= (-i) \cdot i + 1 \cdot 1 = 2$

2. $(Ax) \cdot y = (Ax)^T \cdot y = x^T \cdot (A^Ty)$

$= x \cdot (A^Ty)$

Symmetric Matrices and their Eigenspaces

Theorem

A is a symmetric matrix, with eigenvectors $\vec{v}_1$ and $\vec{v}_2$ corresponding to two distinct eigenvalues. Then $\vec{v}_1$ and $\vec{v}_2$ are orthogonal.

More generally, eigenspaces associated to distinct eigenvalues are orthogonal subspaces.

Proof:

$$A v_1 = \lambda_1 v_1 \qquad \lambda_1 \neq \lambda_2$$

$$A v_2 = \lambda_2 v_2$$

$$(A v_1) \cdot v_2 = \lambda_1 (v_1 \cdot v_2)$$

$$\| \qquad\qquad\qquad \|$$

$$v_1 \cdot (A v_2) = \lambda_2 (v_1 \cdot v_2)$$

$$\underbrace{(\lambda_1 - \lambda_2)}_{\neq 0} \cdot \underbrace{(v_1 \cdot v_2)}_{= 0} = 0$$

Example 1

Diagonalize A using an orthogonal matrix. Eigenvalues of A are given.

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \underline{\lambda = -1, 1}$$

Hint: Gram-Schmidt

$\lambda = -1$: $E_{-1} = \text{Nul}(A + I) = \text{Nul}\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

$= \text{Nul}\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\underline{v_1} = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

$\lambda = 1$: $E_1 = \text{Nul}(A - I) = \text{Nul}\begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{bmatrix}$

$= \text{Nul}\begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = v_3$

$P^T = P^{-1}$

$$A = \underbrace{\begin{bmatrix} | & | & | \\ v_1 & v_2 & v_3 \\ | & | & | \end{bmatrix}}_{P} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}}_{P^{-1}}$$

P has orthonormal columns, square

$\Rightarrow P^T \cdot P = I$

$P^T = P^{-1}$

Spectral Theorem

Recall: If P is an orthogonal $n \times n$ matrix, then $P^{-1} = P^T$, which implies $A = PDP^T$ is diagonalizable and symmetric.

Theorem: Spectral Theorem

An $n \times n$ symmetric matrix A real has the following properties.

1. All eigenvalues of A are real.
2. The dimenison of each eigenspace is full, that it's dimension is **equal to** it's algebraic multiplicity. Diagonalizable.
3. The eigenspaces are mutually orthogonal.
4. A can be diagonalized: $A = PDP^T$, where D is diagonal and P is orthogonal.

Proof (if time permits):

(1) $Av = \lambda v$, $\lambda \in \mathbb{C}$, $v \in \mathbb{C}^n$, $v \neq 0$

$\overline{v^T} \cdot v = v \cdot v \neq 0$

$\overline{(\lambda v)} \cdot v = \overline{Av} \cdot v = \overline{(Av)}^T \cdot v$

$= \overline{v^T}(\overline{A^T} \cdot v) = \overline{v^T} \cdot (A^T \cdot v)$

$= \overline{\lambda}(v \cdot v)$

$= v \cdot Av$

$= v \cdot (\lambda v) = \lambda (v \cdot v)$

$(\lambda - \overline{\lambda}) \cdot (v \cdot v) = 0$, $v \cdot v \neq 0$

$\lambda = \overline{\lambda} \Rightarrow \lambda$: real.

② E : eigenspace for λ

If $x \in E^\perp$ then $Ax \in E^\perp$

⌈ Why? Let $y \in E$ then $Ay = \lambda y$

$$0 = \lambda(x \cdot y) = x \cdot (\lambda y) = x \cdot Ay$$

$$= Ax \cdot y \Rightarrow Ax \in E^\perp$$ ⌋

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Spectral Theorem

A : $n \times n$ real symm.

(i) Every eigenvalue is real.

(ii) A is orthogonally diagonalizable.

$$A = P \cdot D \cdot P^T = \begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \ \lambda_n \end{bmatrix} \begin{bmatrix} — & v_1^T & — \\ — & v_2^T & — \\ & \vdots & \\ — & v_n^T & — \end{bmatrix}$$

P: orthogonal matrix

$$= \lambda_1 \cdot \underbrace{v_1 \cdot v_1^T}_{\in \mathbb{R}^{n \times n}} + \lambda_2 \cdot v_2 \cdot v_2^T + \cdots + \lambda_n \cdot v_n \cdot v_n^T$$

(Spectral Decomposition)

$v_i \cdot v_i^T$ ① $n \times n$, symmetric $\left((v_i \cdot v_i^T)^T = (v_i^T)^T \cdot v_i^T = v_i \cdot v_i^T \right)$

② $\text{Rank}(v_i \cdot v_i^T) = 1$

Ex

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$$

③ $$\underset{n \times n}{(v_i \cdot v_i^T)} \cdot y = v_i\,(\underbrace{v_i^T \cdot y}) = (y \cdot v_i) \cdot v_i$$

$$= \mathrm{proj}_{v_i}(y)$$

Spectral Decomposition of a Matrix

Spectral Decomposition

Suppose A can be orthogonally diagonalized as

$$A = PDP^T = \begin{bmatrix} \vec{u}_1 & \cdots & \vec{u}_n \end{bmatrix} \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} \vec{u}_1^T \\ \vdots \\ \vec{u}_n^T \end{bmatrix}$$

Then A has the decomposition

$$A = \lambda_1 \vec{u}_1 \vec{u}_1^T + \cdots + \lambda_n \vec{u}_n \vec{u}_n^T = \sum_{i=1}^{n} \lambda_i \vec{u}_i \vec{u}_i^T$$

Each term in the sum, $\lambda_i \vec{u}_i \vec{u}_i^T$, is an $n \times n$ matrix with rank 1.

Example 2

Construct a spectral decomposition for A whose orthogonal diagonalization is given.

$$A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} = PDP^T$$

$$= \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} \overset{\lambda_1 \\ \shortparallel}{4} & 0 \\ 0 & \underset{\shortparallel \\ \lambda_2}{2} \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

$\quad = v_1 \qquad = v_2$

$$= 4 \cdot \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} + 2 \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= 2 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Section 7.2 : Quadratic Forms

Chapter 7: Orthogonality and Least Squares

Math 1554 Linear Algebra

Topics and Objectives

Topics

1. Quadratic forms
2. Change of variables
3. Principle axes theorem
4. Classifying quadratic forms

Learning Objectives

1. Characterize and classify quadratic forms using eigenvalues and eigenvectors.
2. Express quadratic forms in the form $Q(\vec{x}) = \vec{x}^T A \vec{x}$.
3. Apply the principle axes theorem to express quadratic forms with no cross-product terms.

Motivating Question Does this inequality hold for all x, y?

$$\underbrace{x^2 - 6xy + 9y^2}_{=\,\underline{(x-3y)^2}} \geq 0$$

Diagonalization

Quadratic Forms

Definition

A **quadratic form** is a function $Q : \mathbb{R}^n \to \mathbb{R}$, given by

$$Q(\vec{x}) = \vec{x}^T A \vec{x} = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{12} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix}$$

Matrix A is $n \times n$ and symmetric.

In the above, $\vec{x}$ is a vector of variables.

Example 1

Compute the quadratic form $\vec{x}^T A \vec{x}$ for the matrices below.

$$A = \begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix}, \qquad B = \begin{bmatrix} 4 & 1 \\ 1 & -3 \end{bmatrix}$$

$$Q_A(x) = \underline{\begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= \begin{bmatrix} 4x_1 & 3x_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 4 \cdot x_1^2 + 3 \cdot x_2^2 \geq 0$$

$$Q_B(x) = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 4 & 1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= \begin{bmatrix} 4x_1 + x_2 & x_1 - 3x_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

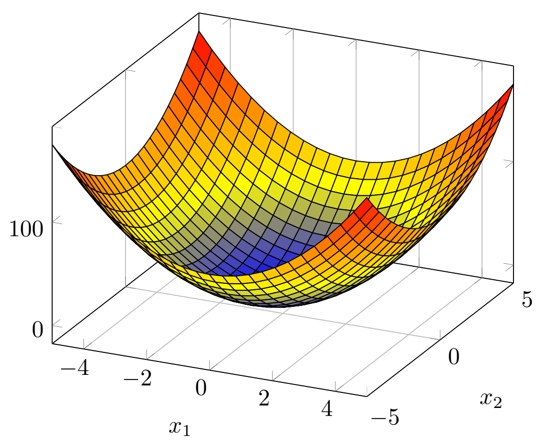
$$= (4x_1 + x_2)\, x_1 + (x_1 - 3x_2) \cdot x_2$$

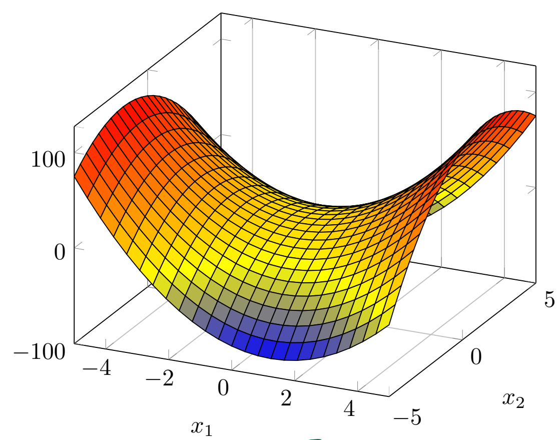
$$= 4x_1^2 + x_1 \cdot x_2 + x_1 \cdot x_2 - 3x_2^2$$

$$= 4x_1^2 + 2x_1 \cdot x_2 - 3x_2^2$$

Example 1 - Surface Plots

The surfaces for Example 1 are shown below.





$y = 4x_1^2 + 3x_2^2$

$y = 4x_1^2 + 2x_1 \cdot x_2 - 3x_2^2$

Students are not expected to be able to sketch quadratic surfaces, but it is helpful to see what they look like.

Example 2

Write Q in the form $\vec{x}^T A \vec{x}$ for $\vec{x} \in \mathbb{R}^3$.

$$Q(x) = 5x_1^2 - x_2^2 + 3x_3^2 + 6x_1x_3 - 12x_2x_3 + 0 \cdot x_1 \cdot x_2$$

$$\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 5 & 0 & +3 \\ 0 & -1 & -6 \\ +3 & -6 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

In general,

$$\begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots \\ & & & \\ & & & \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$= \sum_{j=1}^{n} \sum_{i=1}^{n} a_{ij} x_i x_j$$

Change of Variable

If $\vec{x}$ is a variable vector in $\mathbb{R}^n$, then a **change of variable** can be represented as

$$\vec{x} = P\vec{y}, \quad \text{or} \quad \vec{y} = P^{-1}\vec{x}$$

With this change of variable, the quadratic form $\vec{x}^T A\vec{x}$ becomes:

Ex

$$x_1^2 - 6x_1 \cdot x_2 + \underbrace{10 \cdot x_2^2}_{9x_2^2 + x_2^2}$$

$$= (\underbrace{\boxed{x_1 - 3x_2}}_{= y_1})^2 + \underbrace{\boxed{x_2}}_{= y_2}^2$$

$$y_1 = x_1 - 3x_2$$

$$y_2 = 0 \cdot x_1 + 1 \cdot x_2$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \longrightarrow \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$Q_A(x) = x^T \cdot A \cdot x = \underbrace{x^T \cdot P}_{= y^T} \cdot D \cdot \underbrace{P^T x}_{= y}$$

$$y^T = (P^T \cdot x)^T$$

$$= x^T \cdot P$$

$$= y^T \cdot D \cdot y$$

$$= \lambda_1 y_1^2 + \lambda_2 \cdot y_2^2 + \cdots + \lambda_n \cdot y_n^2$$

Example 3

Make a change of variable $\vec{x} = P\vec{y}$ that transforms $Q = \vec{x}^T A\vec{x}$ so that it does not have cross terms. The orthogonal decomposition of A is given.

$$A = \begin{pmatrix} 3 & 2 \\ 2 & 6 \end{pmatrix} = PDP^T$$

$$P = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}$$

$$D = \begin{pmatrix} 2 & 0 \\ 0 & 7 \end{pmatrix}$$

$$y = P^T \cdot x$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

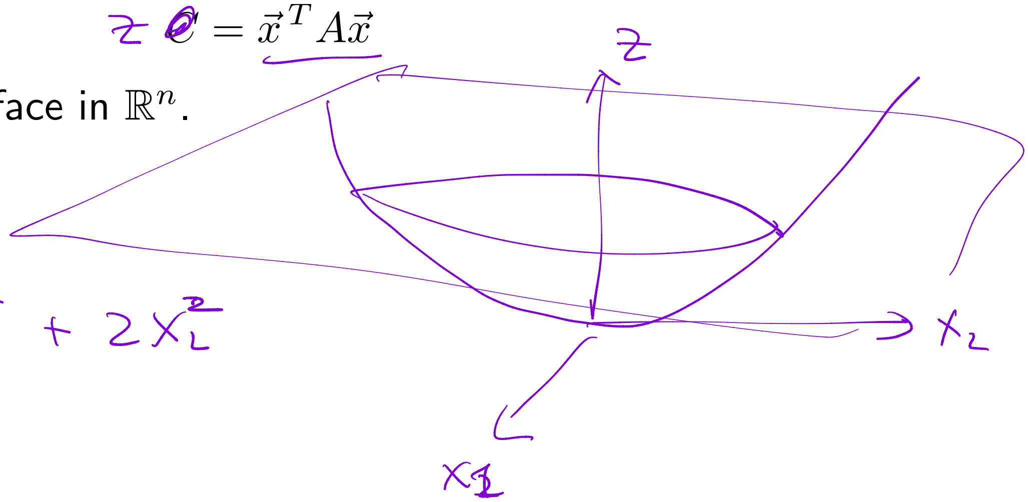
$$\begin{aligned} Q(x) &= 3x_1^2 + 4x_1x_2 + 6x_2^2 \\ &= y^T \cdot D \cdot y \\ &= 2y_1^2 + 7y_2^2 \\ &= 2 \cdot \left(\frac{1}{\sqrt{5}}(2x_1 - x_2)\right)^2 + 7 \cdot \left(\frac{1}{\sqrt{5}}(x_1 + 2x_2)\right)^2 \end{aligned}$$

Geometry

Suppose $Q(\vec{x}) = \vec{x}^T A \vec{x}$, where $A \in \mathbb{R}^{n\times n}$ is symmetric. Then the set of $\vec{x}$ that satisfies

$$z \cancel{C} = \underline{\vec{x}^T A \vec{x}}$$

defines a curve or surface in $\mathbb{R}^n$.



$$z = 4x_1^2 + 2x_2^2$$

$\{v_1, \dots, v_n\}$: ONB for $\mathbb{R}^n$

Note

$$\underbrace{(v_1 \cdot x)^2 + (v_2 \cdot x)^2 + \cdots + (v_n \cdot x)^2}_{= \|P^T \cdot x\|^2} = \|x\|^2$$

Phythagorean Thm.

$$P^T \cdot x = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} \text{---} & v_1^T & \text{---} \\ \text{---} & v_2^T & \text{---} \\ & \vdots & \\ \text{---} & v_n^T & \text{---} \end{bmatrix} \cdot x = \begin{bmatrix} v_1 \cdot x \\ v_2 \cdot x \\ \vdots \\ v_n \cdot x \end{bmatrix}$$

Recall $Q(x) = x^T \cdot A \cdot x = \sum_{j=1}^{n}\sum_{i=1}^{n} a_{ij} x_i x_j$

real $n \times n$ symm.

$= x^T \cdot (P \cdot D \cdot P^T)\, x = \underbrace{(x^T \cdot P)}_{y^T} \cdot D \cdot \overbrace{(P^T x)}^{y}$

$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$

$= y^T \cdot D \cdot y$

$= \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2$

$= \lambda_1 (v_1 \cdot x)^2 + \lambda_2 (v_2 \cdot x)^2 + \cdots + \lambda_n (v_n \cdot x)^2$

Principle Axes Theorem

Theorem

If A is a n×n real symm. matrix then there exists an orthogonal change of variable $\vec{x} = P\vec{y}$ that transforms $\vec{x}^T A\vec{x}$ to $\vec{y}^T D\vec{y}$ with no cross-product terms.

or $y = P^T \cdot x$

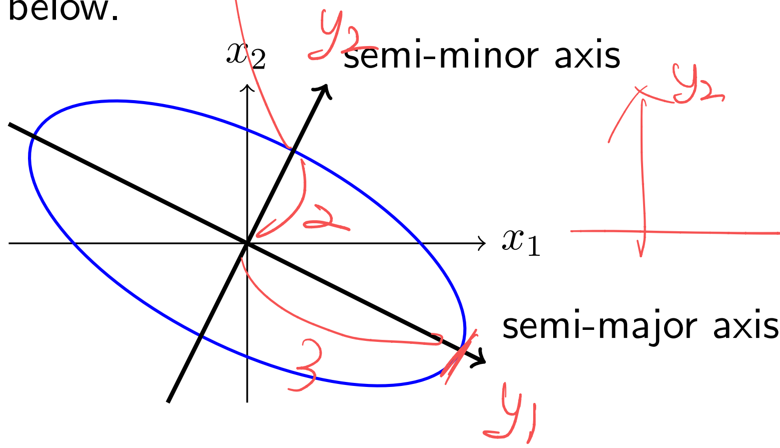
Proof (if time permits):

$y_1 = 0 \Rightarrow 2x_1 - x_2 = 0$

$x_2 = 2 \cdot x_1$

Example 5

Compute the quadratic form $Q = \vec{x}^T A \vec{x}$ for $A = \begin{pmatrix} 5 & 2 \\ 2 & 8 \end{pmatrix}$, and find a change of variable that removes the cross-product term. A sketch of Q is below.



$4y_1^2 + 9y_2^2 = 36$

$\frac{y_1^2}{9} + \frac{y_2^2}{4} = 1$

Eigenvalue :

$$\text{Det}(A - \lambda I) = \lambda^2 - 13\lambda + 36 = 0$$

$$= (\lambda - 9)(\lambda - 4)$$

$\lambda = 4$: $\text{Nul}(A - 4I) = \text{Nul}\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} = \text{Nul}\begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$

$$v_1 = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$\lambda = 9$: $\text{Nul}(A - 9I) = \text{Nul}\begin{pmatrix} -4 & 2 \\ 2 & -1 \end{pmatrix} = \text{Nul}\begin{pmatrix} 1 & -\frac{1}{2} \\ 0 & 0 \end{pmatrix}$

$$v_2 = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

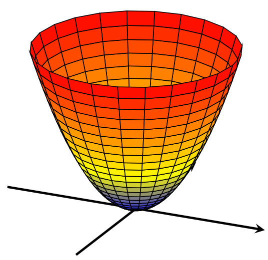
$$Q(x) = 4\cdot y_1^2 + 9\cdot y_2^2 = 4\cdot(v_1\cdot x)^2 + 9(v_2\cdot x)^2$$

$$= \underline{4\left(\frac{1}{\sqrt{5}}(2x_1 - x_2)\right)^2 + 9\left(\frac{1}{\sqrt{5}}(x_1+2x_2)\right)^2}$$

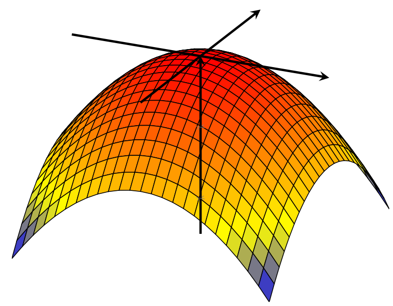
$$= 36$$

Classifying Quadratic Forms

$Q = x_1^2 + x_2^2$



$Q = -x_1^2 - x_2^2$



Definition

A quadratic form Q is ($Q(x) = 0$ if $x = 0$)

1. **positive definite** if $Q(x) > 0$ for all $\vec{x} \neq \vec{0}$.
2. **negative definite** if $Q(x) < 0$ for all $\vec{x} \neq \vec{0}$. ($Q(x) = 0$ if $x = 0$)
3. **positive semidefinite** if $Q(x) \geq 0$ for all $\vec{x}$.
4. **negative semidefinite** if $Q(x) \leq 0$ for all $\vec{x}$.
5. **indefinite** if there exist x_1, x_2 such that

$Q(x_1) > 0$, $Q(x_2) < 0$

Quadratic Forms and Eigenvalues

Theorem

If A is a n×n real symm. matrix with eigenvalues λ_i, then $Q = \vec{x}^T A \vec{x}$ is

1. **positive definite** iff λ_i > 0 for all $i = 1, 2, \ldots, n$
2. **negative definite** iff λ_i < 0 ″
3. **indefinite** iff λ_i ≥ 0 $\lambda_j < 0$ for some i, j.

Proof (if time permits):

Example 6

We can now return to our motivating question (from first slide): does this inequality hold for all x, y?

$$Q(x) = \underline{x^2 - 6xy + 9y^2 \geq 0}$$

$$A = \begin{bmatrix} 1 & -3 \\ -3 & 9 \end{bmatrix}$$

λ : $\quad \operatorname{Det}(A - \lambda I) = \lambda^2 - 10\lambda + 0 = 0$

$$\lambda = 0, 10 \geq 0$$

Q is <u>Positive Semi definite</u>

$$Q(x) \geq 0.$$

Section 7.3 : Constrained Optimization

Chapter 7: Orthogonality and Least Squares

Math 1554 Linear Algebra

Topics and Objectives

Topics

1. Constrained optimization as an eigenvalue problem
2. Distance and orthogonality constraints

Learning Objectives

1. Apply eigenvalues and eigenvectors to solve optimization problems that are subject to distance and orthogonality constraints.

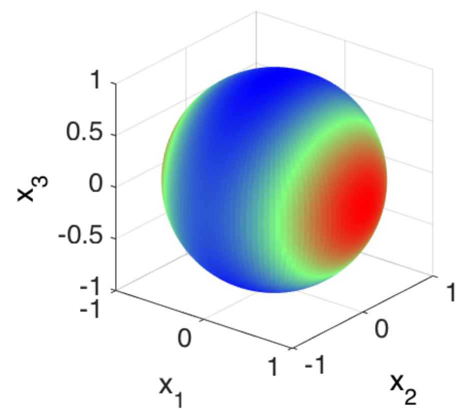
Example 1

The surface of a unit sphere in $\mathbb{R}^3$ is given by

$$1 = x_1^2 + x_2^2 + x_3^2 = ||\vec{x}||^2$$

Q is a quantity we want to optimize

$$Q(\vec{x}) = 9x_1^2 + 4x_2^2 + 3x_3^2$$



Find the largest and smallest values of Q on the surface of the sphere.

$$3(x_1^2 + x_2^2 + x_3^2) \leq Q(x) \leq 9 \cdot (x_1^2 + x_2^2 + x_3^2)$$

$$3 \leq Q(x) \leq 9$$

max?

$Q(x) = 9$ if $x = \pm \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$Q(x) = 3$ if $x = \pm \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

A Constrained Optimization Problem

Suppose we wish to find the maximum or minimum values of

$$Q(\vec{x}) = \vec{x}^T A \vec{x}$$

subject to

$$||\vec{x}|| = 1$$

That is, we want to find

$$m = \min\{Q(\vec{x}) : ||\vec{x}|| = 1\}$$
$$M = \max\{Q(\vec{x}) : ||\vec{x}|| = 1\}$$

This is an example of a **constrained optimization** problem. Note that we may also want to know where these extreme values are obtained.

Constrained Optimization and Eigenvalues

Theorem

If $Q = \vec{x}^T A\vec{x}$, A is a real $n \times n$ symmetric matrix, with eigenvalues

$$\lambda_1 \geq \lambda_2 \ldots \geq \lambda_n$$

and associated normalized eigenvectors

$$\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n$$

Then, subject to the constraint $||\vec{x}|| = 1$,

- the **maximum** value of $Q(\vec{x}) = \lambda_1$, attained at $\vec{x} = \pm\vec{u}_1$.
- the **minimum** value of $Q(\vec{x}) = \lambda_n$, attained at $\vec{x} = \pm\vec{u}_n$.

Proof:

11/18/24

$Q(x) = x^T \cdot A \cdot x$ max / min of Q under $\|x\| = 1$.

$= x^T \cdot P \cdot D \cdot P^T x$

$= (P^T x)^T \cdot D \cdot \underbrace{(P^T \cdot x)}_{=: y} = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2$

$= \lambda_1 (v_1 \cdot x)^2 + \lambda_2 (v_2 \cdot x)^2 + \cdots + \lambda_n (v_n \cdot x)^2$

$(v_1 \cdot x)^2 + (v_2 \cdot x)^2 + \cdots + (v_n \cdot x)^2 = \|x\|^2 = 1$

Example 2

Calculate the maximum and minimum values of $Q(\vec{x}) = \vec{x}^T A \vec{x}$, $\vec{x} \in \mathbb{R}^3$, subject to $||\vec{x}|| = 1$, and identify points where these values are obtained.

$$Q(\vec{x}) = x_1^2 + 2x_2x_3$$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\operatorname{Det}(A - \lambda I) = \operatorname{Det}\begin{bmatrix} 1-\lambda & 0 & 0 \\ 0 & -\lambda & 1 \\ 0 & 1 & -\lambda \end{bmatrix} = (1-\lambda) \cdot \det\begin{bmatrix} -\lambda & 1 \\ 1 & -\lambda \end{bmatrix}$$

$$= ((1-\lambda)(\lambda^2 - 1) = (1-\lambda)(\lambda+1)(\lambda-1)$$

$\lambda = 1, -1, 1$

$\lambda = 1$: $\quad \operatorname{Nul}(A - I) = \operatorname{Nul}\begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix}$

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad v_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$\lambda = -1$ $\quad \operatorname{Nul}(A + I) = \operatorname{Nul}\left(\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}\right)$

$$v_3 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

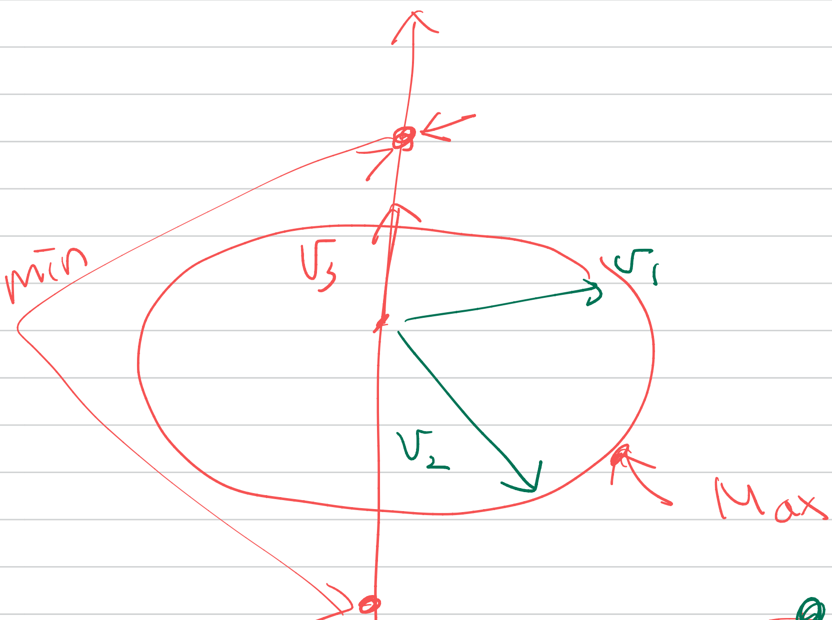
$$Q(x) = \underline{\underline{1}} \cdot (v_1 \cdot x)^2 + \underline{\underline{1}} \cdot (v_2 \cdot x)^2 + (-1)(v_3 \cdot x)^2$$

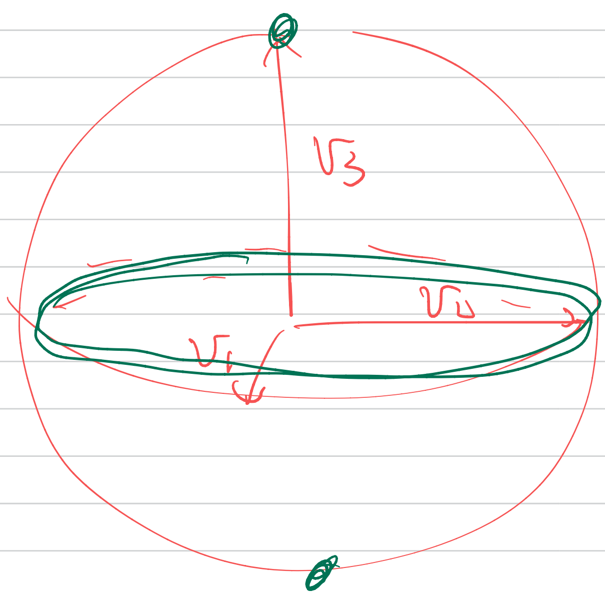
$$\|x\|^2 = (v_1 \cdot x)^2 + (v_2 \cdot x)^2 + (v_3 \cdot x)^2 = 1$$

Max : $Q(x) = 1$ when $x \cdot v_3 = 0$

Min : $Q(x) = -1$ when $v_1 \cdot x = 0$, $v_2 \cdot x = 0$

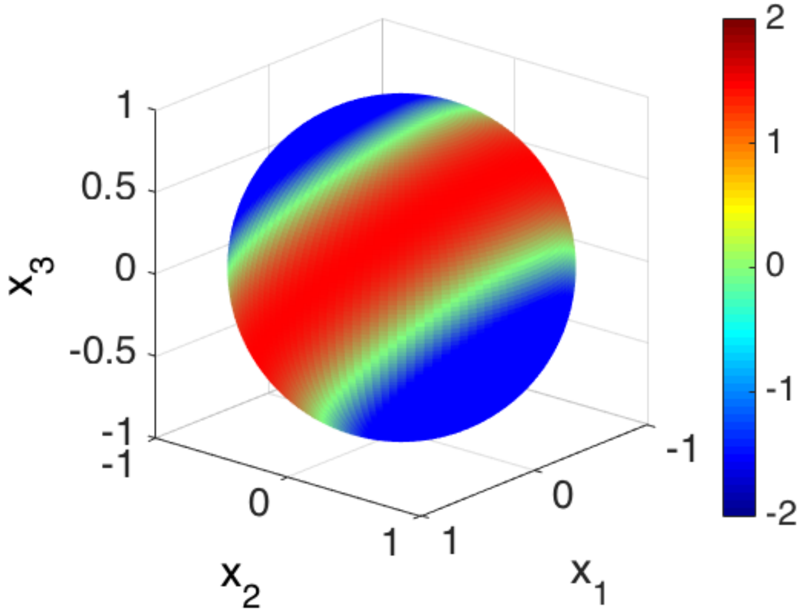
$x = v_3 \; , \; -v_3$





Example 2

The image below is the unit sphere whose surface is colored according to the quadratic from the previous example. Notice the agreement between our solution and the image.



An Orthogonality Constraint

Theorem

Suppose $Q = \vec{x}^T A\vec{x}$, A is a real $n \times n$ symmetric matrix, with eigenvalues

$$\lambda_1 \geq \lambda_2 \ldots \geq \lambda_n$$

and associated eigenvectors

$$\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n$$

Subject to the constraints $||\vec{x}|| = 1$ and $\vec{x} \cdot \vec{u}_1 = 0$,

- The maximum value of $Q(\vec{x}) = \lambda_2$, attained at $\vec{x} = \vec{u}_2$.
- The minimum value of $Q(\vec{x}) = \lambda_n$, attained at $\vec{x} = \vec{u}_n$.

Note that λ_2 is the second largest eigenvalue of A.

$$\max_{\|x\|=1} x^T A \cdot x = \text{the largest eigenvalue}$$

$$\max_{\left(\substack{\|x\|=1 \\ x \cdot v_1 = 0}\right)} x^T \cdot A \cdot x = 2^{nd} \text{ largest eigenvalue.}$$

Example 3

Calculate the maximum value of $Q(\vec{x}) = \vec{x}^T A\vec{x}$, $\vec{x} \in \mathbb{R}^3$, subject to $||\vec{x}|| = 1$ and to $\vec{x} \cdot \vec{u}_1 = 0$, and identify a point where this maximum is obtained.

$$Q(\vec{x}) = x_1^2 + 2x_2x_3, \qquad \vec{u}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = v_1$$

$\lambda : \quad 1, \quad 1, \quad -1$

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = v_2 \quad \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = v_3$$

$$Q(x) = 1 \cdot (v_1 \cdot x)^2 + 1 \cdot (v_2 \cdot x)^2 + (-1) \cdot (v_3 \cdot x)^2$$

$$\begin{cases} (v_1 \cdot x)^2 + (v_2 \cdot x)^2 + (v_3 \cdot x)^2 = \|x\|^2 = 1 \\ (v_1 \cdot x) = 0 \end{cases}$$

Max : $Q(x) = \underline{\underline{1}}$ when $v_1 \cdot x = 0$, $v_3 \cdot x = 0$

second largest eigenvalue. $\Rightarrow x = v_2, -v_2$

Min : $Q(x) = -1$ when $x = \pm v_3$

Example 4 (if time permits)

Calculate the maximum value of $Q(\vec{x}) = \vec{x}^T A\vec{x}$, $\vec{x} \in \mathbb{R}^3$, subject to $||\vec{x}|| = 5$, and identify a point where this maximum is obtained.

$$Q(\vec{x}) = x_1^2 + 2x_2x_3$$

$$Q = x^T \cdot A \cdot x$$

$$= \lambda_1 (v_1 \cdot x)^2 + \cdots$$

$$(v_1 \cdot x)^2 + \cdots = \|x\|^2 = 25$$

Section 7.4 : The Singular Value Decomposition

Chapter 7: Orthogonality and Least Squares

Math 1554 Linear Algebra

Topics and Objectives

Topics

1. The Singular Value Decomposition (SVD) and some of its applications.

Learning Objectives

1. Compute the SVD for a rectangular matrix.
2. Apply the SVD to
   - estimate the rank and condition number of a matrix,
   - construct a basis for the four fundamental spaces of a matrix, and
   - construct a spectral decomposition of a matrix.

Recall (Spectral Decomposition)

A : $n\times n$ real symm.

$\Rightarrow$ (i) eigenvalues are real

(ii) Diagonalizable $A = P\cdot D\cdot P^T$

(iii) $A = \lambda_1 \cdot v_1 \cdot v_1^T + \lambda_2 v_2 \cdot v_2^T + \cdots + \lambda_n v_n \cdot v_n^T$

why (ii) $\Rightarrow$ (iii) :

From (ii) $\{v_1, \cdots, v_n\}$: ONB , eigenvectors

$P = [v_1 \cdots v_n]$, $P^T \cdot P = I = P\cdot P^T$

$$A\cdot I = A\left(v_1\cdot v_1^T + v_2\cdot v_2^T + \cdots + v_n\cdot v_n^T\right)$$

$$A = (Av_1)\cdot v_1^T + (Av_2)\cdot v_2^T + \cdots + (Av_n)\cdot v_n^T$$

$$= \lambda_1 v_1 \cdot v_1^T + \lambda_2 v_2 \cdot v_2^T + \cdots + \lambda_n v_n \cdot v_n^T.$$

A : $m\times n$ real. p.s.d.

$\Rightarrow$ $A^T\cdot A$: $n\times n$ real symm. positive semi-definite.

$$Q(x) = (x^T\cdot A^T)A\cdot x = (Ax)^T\cdot Ax = \|Ax\|^2 \geq 0$$

eigenvalues for $A^T\cdot A$

$\Rightarrow$ $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r > \lambda_{r+1} = \lambda_{r+2} = \cdots = \lambda_n = 0$

eigenvectors for $A^T\cdot A$ $\{v_1, v_2, \cdots, v_n\}$ ONB for $\mathbb{R}^n$

$v_1\cdot v_1^T + \cdots + v_n \cdot v_n^T = I_n$

$\Rightarrow$ $A = Av_1\cdot v_1^T + \cdots + Av_n \cdot v_n^T$

(1) $A^T \cdot A \cdot v_{r+1} = \overset{\overset{0}{\shortparallel}}{\lambda_{r+1}} \cdot v_{r+1} = 0 = A^TA\, v_{r+1} = \cdots = A^T \cdot A\, v_n$

$A^T \cdot A\, v_k = 0$ for $k = r+1, \cdots, n$.

$$Q(v_k) = v_k^T \underbrace{A^T \cdot A \cdot v_k}_{= 0} = 0 = \underbrace{\| A \cdot v_k \|^2}_{\Downarrow}$$

$A v_k = 0$, for $k = r+1, \cdots, n$.

$$\boxed{\mathrm{Nul}(A) = \mathrm{Nul}(A^T \cdot A).}$$

$$\Rightarrow \quad A = A v_1 \cdot v_1^T + \cdots + A v_r \cdot v_r^T$$

What is r? $\Rightarrow$ $r = \mathrm{Rank}(A)$

$\{v_1, \cdots, v_n\}$: basis

$$\big\downarrow A \times$$

$\{A v_1, \cdots, A v_r\}$ orthogonal basis for $\mathrm{Col}(A)$

$$\{ \underbrace{A v_1, A v_2, \cdots, A v_r}, 0, \cdots, 0 \}$$

$\Rightarrow$ orthogonal $\Rightarrow$ lin indep.

$$(A v_i) \cdot (A v_j) = (A v_i)^T \cdot (A v_j)$$
$$= v_i^T \cdot \boxed{A^T \cdot A \cdot v_j}$$
$$= v_i^T \cdot (\lambda_j v_j) = \lambda_j (v_i \cdot v_j)$$

If $i \neq j$, $A v_i \cdot A v_j = 0$

$$\{ \underbrace{\tfrac{1}{\sqrt{\lambda_1}} A v_1}_{\shortparallel \atop u_1}, \underbrace{\tfrac{1}{\sqrt{\lambda_2}} A v_2}_{\shortparallel \atop u_2}, \cdots, \underbrace{\tfrac{1}{\sqrt{\lambda_r}} A v_r}_{\shortparallel \atop u_r} \}$$

orthonormal ~~orthogonal~~ basis for $\mathrm{Col}(A)$

$$\|Av_1\|^2 = v_1^T \cdot \underline{A^T A v_1} = \lambda_1 \underbrace{v_1^T \cdot v_1}_{\|v_1\|^2} = \lambda_1$$

Def $\sigma_i = \sqrt{\lambda_i}$: the singular value for A.

$$A = \sigma_1 \cdot \frac{1}{\sigma_1} A v_1 \cdot v_1^T + \sigma_2 \cdot \frac{1}{\sigma_2} A v_2 \cdot v_2^T + \cdots + \sigma_r \cdot \frac{1}{\sigma_r} A v_r \cdot v_r^T$$

$$= \sigma_1 \cdot u_1 \cdot v_1^T + \sigma_2 u_2 \cdot v_2^T + \cdots + \sigma_r u_r v_r^T$$

: Singular value decomposition.

$\{v_1, v_2, \cdots, v_n\}$: ONB for $\mathbb{R}^n$

$\{v_1, \cdots, v_r\}$: ONB for $\underline{Nul(A)^\perp}$

$\{u_1, u_2, \cdots, u_r\}$: ONB for $Col(A) \subseteq \mathbb{R}^m$

$\{v_{r+1}, v_{r+2}, \cdots, v_n\}$: ONB for $Nul(A)$

$\{u_{r+1}, u_{r+2}, \cdots, u_m\}$ ONB for $Col(A)^\perp = Nul(A^T)$

From Gram-Schmidt

$\cancel{n}$? m?

$$\Rightarrow A = \overset{m\times m}{\begin{bmatrix} u_1 & \cdots & u_m \end{bmatrix}} \overset{m\times n}{\begin{bmatrix} \sigma_1 & & & 0 \\ & \ddots & & \\ & & \sigma_r & \\ 0 & & & 0 \ddots \end{bmatrix}} \begin{bmatrix} — & v_1^T & — \\ — & v_2^T & — \\ & \vdots & \\ — & v_n^T & — \end{bmatrix}$$

$$= U \cdot \Sigma \cdot V^T$$

U : $m \times m$ orthogonal matrix

V : $n \times n$ "

$\{v_1, \dots, v_n\}$

$$\mathrm{Col}(A) = \mathrm{Span}\{A \cdot e_1, Ae_2, \dots, Ae_n\}$$

$$= \mathrm{Span}\{Av_1, Av_2, \dots, Av_n\}$$

$$= \mathrm{Span}\underbrace{\{\tfrac{1}{\sigma_1}Av_1, \tfrac{1}{\sigma_2}Av_2, \dots, \tfrac{1}{\sigma_r}Av_r\}}_{\text{orthonormal}}$$

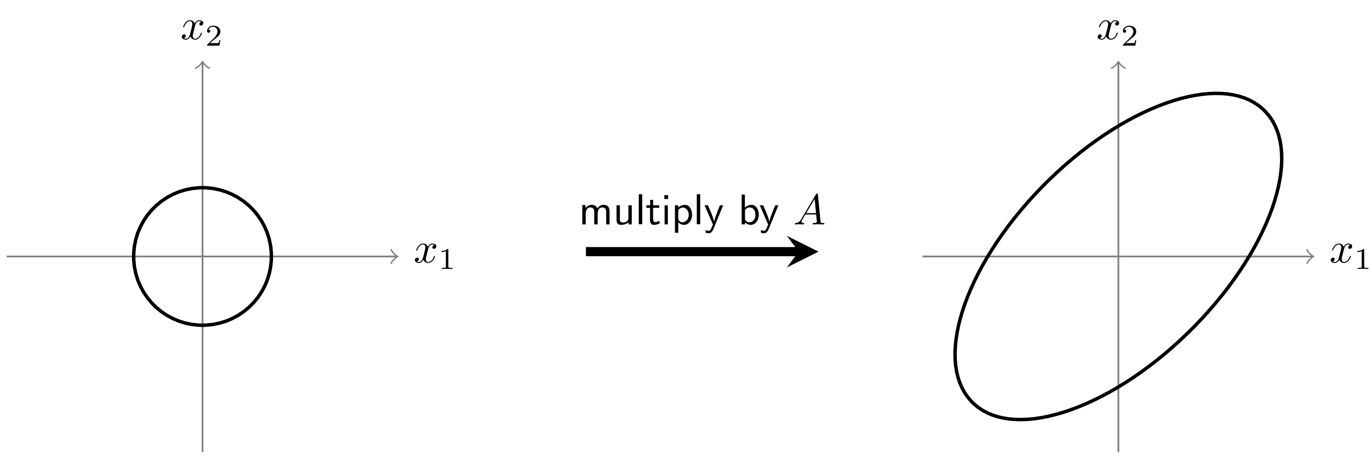
$$Ae_1 = c_1 Av_1 + c_2 Av_2 + \dots + c_n Av_n$$

Example 1

The linear transform whose standard matrix is

$$A = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2\sqrt{2} & 0 \\ 0 & \sqrt{2} \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 2 & 1 \end{pmatrix}$$

maps the unit circle in $\mathbb{R}^2$ to an ellipse, as shown below. Identify the unit vector $\vec{x}$ in which $||A\vec{x}||$ is maximized and compute this length.



Example 1 - Solution

Singular Values

Positive semidefinite.

The matrix $A^T A$ is always symmetric, with non-negative eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$. Let $\{\vec{v}_1, \ldots, \vec{v}_n\}$ be the associated orthonormal eigenvectors. Then

$$\|A\vec{v}_j\|^2 = \quad \lambda_j$$

If the A has rank r, then $\{A\vec{v}_1, \ldots, A\vec{v}_r\}$ is an $\boxed{\text{orthogonal basis}}$ for Col$A$:
For $1 \leq j < k \leq r$:

$$(A\vec{v}_j)^T A\vec{v}_k = \quad 0$$

$j \neq k$

Definition: $\sigma_1 = \sqrt{\lambda_1} \geq \sigma_2 = \sqrt{\lambda_2} \cdots \geq \sigma_n = \sqrt{\lambda_n}$ are the singular values of A.

$\sigma_{r+1} = \cdots = \sigma_n = 0$

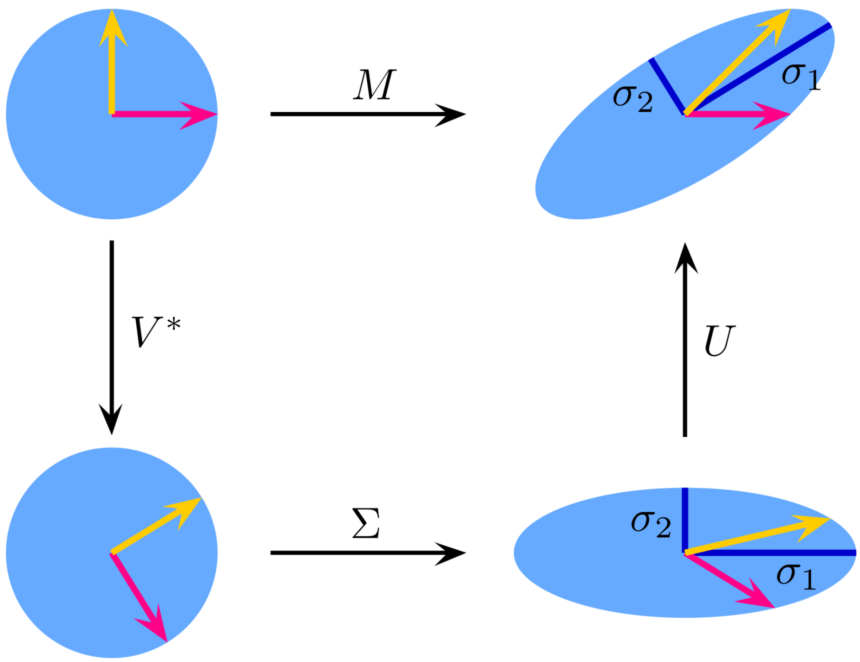
The SVD

Theorem: Singular Value Decomposition

A $m \times n$ matrix with rank r and non-zero singular values $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r$ has a decomposition $U\Sigma V^T$ where

$$\Sigma = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}_{m \times n} = \left[\begin{array}{cccc|c} \sigma_1 & 0 & \dots & 0 & \\ 0 & \sigma_2 & \dots & \vdots & \mathbf{0} \\ \vdots & \vdots & \ddots & & \\ 0 & 0 & \dots & \sigma_r & \\ & \mathbf{0} & & & \mathbf{0} \end{array}\right]$$

U is a $m \times m$ orthogonal matrix, and V is a $n \times n$ orthogonal matrix.



$$M = U \cdot \Sigma \cdot V^{\cancel{*}\,T}$$

Algorithm to find the SVD of A

Suppose A is $m \times n$ and has rank $r \leq n$.

1. Compute the squared singular values of $A^T A$, σ_i^2, and construct Σ.

2. Compute the unit singular vectors of $A^T A$, $\vec{v}_i$, use them to form V.

3. Compute an orthonormal basis for Col$A$ using

$$\vec{u}_i = \frac{1}{\sigma_i} A\vec{v}_i, \quad i = 1, 2, \ldots r$$

   Extend the set $\{\vec{u}_i\}$ to form an orthonomal basis for $\mathbb{R}^m$, use the basis for form U.

Example 2: Write down the singular value decomposition for

$$\begin{bmatrix} 2 & 0 \\ 0 & -3 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} =$$

Example 3: Construct the singular value decomposition of

$$A = \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix}.$$

(It has rank 1.)

Applications of the SVD

The SVD has been applied to many modern applications in CS, engineering, and mathematics (our textbook mentions the first four).

- Estimating the rank and condition number of a matrix
- Constructing bases for the four fundamental spaces
- Computing the pseudoinverse of a matrix
- Linear least squares problems
- Non-linear least-squares
  https://en.wikipedia.org/wiki/Non-linear_least_squares
- Machine learning and data mining
  https://en.wikipedia.org/wiki/K-SVD
- Facial recognition
  https://en.wikipedia.org/wiki/Eigenface
- Principle component analysis
  https://en.wikipedia.org/wiki/Principal_component_analysis
- Image compression

Students are expected to be familiar with the 1^{st} two items in the list.

The Condition Number of a Matrix

If A is an invertible $n \times n$ matrix, the ratio

$$\frac{\sigma_1}{\sigma_n}$$

is the **condition number** of A.

Note that:

- The condition number of a matrix describes the sensitivity of a solution to $A\vec{x} = \vec{b}$ is to errors in A.
- We could define the condition number for a rectangular matrix, but that would go beyond the scope of this course.

Example 4

For $A = U\Sigma V^*$, determine the rank of A, and orthonormal bases for Null$A$ and $(\text{Col}A)^\perp$.

$$\mathbf{U} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{\Sigma} = \begin{bmatrix} 4 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{5} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{V}^* = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \sqrt{0.2} & 0 & 0 & 0 & \sqrt{0.8} \\ 0 & 0 & 0 & 1 & 0 \\ -\sqrt{0.8} & 0 & 0 & 0 & \sqrt{0.2} \end{bmatrix}$$

Example 4 - Solution

The Four Fundamental Spaces

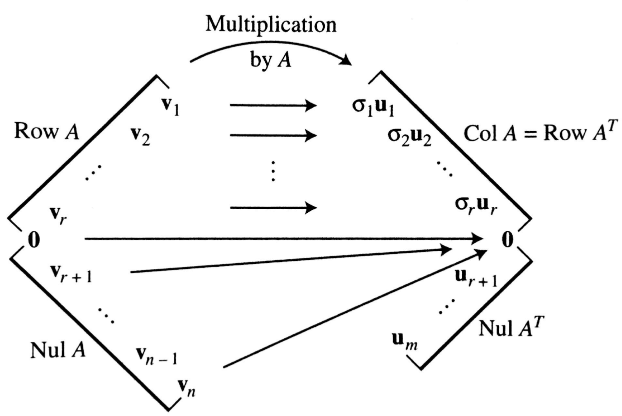


FIGURE 4 The four fundamental subspaces and the action of A.

1. $A\vec{v}_s = \sigma_s \vec{u}_s$.
2. $\vec{v}_1, \ldots, \vec{v}_r$ is an orthonormal basis for Row$A$.
3. $\vec{u}_1, \ldots, \vec{u}_r$ is an orthonormal basis for Col$A$.
4. $\vec{v}_{r+1}, \ldots, \vec{v}_n$ is an orthonormal basis for Null$A$.
5. $\vec{u}_{r+1}, \ldots, \vec{u}_n$ is an orthonormal basis for NullA^T.

The Spectral Decomposition of a Matrix

The SVD can also be used to construct the spectral decomposition for any matrix with rank r

$$A = \sum_{s=1}^{r} \sigma_s \vec{u}_s \vec{v}_s^T,$$

where $\vec{u}_s, \vec{v}_s$ are the s^{th} columns of U and V respectively.

For the case when $A = A^T$, we obtain the same spectral decomposition that we encountered in Section 7.2.