4/27/22

Recall

Markov : $X \geq 0$, $a > 0$ $\quad \mathbb{P}(X \geq a) \leq \mathbb{E}X / a$.

Chebychev : $\mathbb{P}(|X-\mu| \geq a) \leq \dfrac{\sigma^2}{a^2}$.

Prop If $\mathrm{Var}(X) = 0$ then $\mathbb{P}(X = \mathbb{E}X) = 1$

Pf: By Chebyshev, $\mathbb{P}(|X - \mathbb{E}X| \geq a) \leq \dfrac{\sigma^2}{a^2} = 0$. $\forall a > 0$

Let $a = \frac{1}{n}$. $\quad \mathbb{P}(|X - \mathbb{E}X| > 0) = \mathbb{P}\left(\bigcap_{n=1}^{\infty} \{|X - \mathbb{E}X| \geq \tfrac{1}{n}\}\right)$

$$= \lim_{n\to\infty} \mathbb{P}(|X - \mathbb{E}X| \geq \tfrac{1}{n}) = 0 .$$

$\dashv$

One-sided Chebyshev : X w/ $\mu = 0$, σ^2 , $a > 0$

$$\mathbb{P}(X \geq a) \leq \frac{\sigma^2}{\sigma^2 + a^2} .$$

For $\mu \neq 0$,

$$\mathbb{P}(X \geq \mu + a) \leq \frac{\sigma^2}{\sigma^2 + a^2}$$

$$\mathbb{P}(X \leq \mu - a) \leq \frac{\sigma^2}{\sigma^2 + a^2} .$$

Chernoff : If X is a RV w/ MGF $M_X(t)$,

then $\forall a \in \mathbb{R}$, $\quad \mathbb{P}(X \geq a) \leq e^{-at} M_X(t)$ $\quad \forall t > 0$

$$\mathbb{P}(X \geq a) \leq e^{at} M_X(-t)$$

Proof $\quad \mathbb{P}(X \geq a) = \mathbb{P}(e^{tX} \geq e^{at}) \leq \dfrac{\mathbb{E}[e^{tX}]}{e^{at}}$

$\quad \mathbb{P}(X \leq a) = \mathbb{P}(e^{-tX} \geq e^{-at}) \leq \dfrac{\mathbb{E}[e^{-tX}]}{e^{-at}}$

$\blacksquare$

Example $X \sim Pois(\lambda)$ $(\Rightarrow M_X(t) = \exp(\lambda(e^t - 1)))$

$$P(X \geq a\lambda) \leq \frac{M_X(t)}{e^{a\lambda t}} = \exp(\lambda(\underbrace{e^t - 1 - at}_{=: f(t)})) \quad \text{minimize in } t.$$

$(a>1)$

$f'(t) = e^t - a = 0 \quad \therefore \quad t = \log a$

$$\therefore \quad P(X \geq a\lambda) \leq \exp(-\lambda(a\log a + 1 - a))$$

$$P\left(\frac{X}{EX} \geq a\right) \leq \exp(-\lambda(a \log a + 1 - a))$$

Example $X \sim N(0,1)$ $a>0$

$$P(X \geq a) \leq \frac{M_X(t)}{e^{at}} = \exp\left(\frac{t^2}{2} - at\right)$$

$$= \exp\left(\frac{1}{2}(t-a)^2 - \frac{a^2}{2}\right) \quad \text{for } t>0.$$

Let $t=a$ then $P(X \geq a) \leq e^{-a^2/2}$.

Similarly if $a<0$, $P(X \leq a) \leq e^{-a^2/2}$

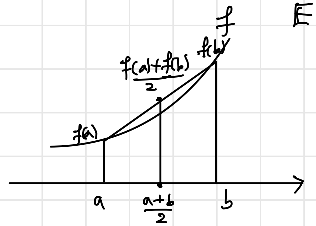
Jensen's Inequality

Def For a twice differentiable f, f is convex on I (Concave)
if $f''(x) \geq 0$ for $x \in I$.
$(f''(x) \leq 0)$

Example $f(x) = x^2,\ e^{ax},\ x\log x\ (x>0),\ |x|^p, \cdots$

Jensen's : If f is convex on I & $P(X \in I) = 1$ then

$$E[f(X)] \geq f(E[X])$$



<u>Example</u> $X = \begin{cases} a & \text{w/ prob. } \frac{1}{2} \\ b & \text{"} \end{cases}$ f : convex

$$\mathbb{E}[f(X)] = \frac{1}{2}(f(a) + f(b)) \geq f(\mathbb{E}X) = f\left(\frac{a+b}{2}\right)$$

<u>Example</u> $f(x) = |x|^p \quad (p \geq 1)$

$$f'(x) = \frac{p}{2}|x|^{p-2} \cdot 2x = p \cdot |x|^{p-2} \cdot x$$

$$f''(x) = p \cdot \left(\frac{p}{2} - 1\right) |x|^{p-4} 2x^2 + p|x|^{p-2}$$

$$= p(p-1)|x|^{p-2} \geq 0$$

$$\mathbb{E}[|X|^p] \geq |\mathbb{E}X|^p.$$