Practice Problems for Exam 1 with Answers

MATH 3215, Spring 2024

Chapter 1

1. Consider a thick coin with three possible outcomes of a toss (Heads, Tails, and Edge) for which Heads and Tails are equally likely, but Heads is five times as likely than Edge. What is the probability of Heads?

   ANS: $\frac{5}{11}$

2. Suppose A, B and C are events with $\mathbb{P}(A) = 0.43, \mathbb{P}(B) = 0.40, \mathbb{P}(C) = 0.32, \mathbb{P}(A \cap B) = 0.29, \mathbb{P}(A \cap C) = 0.22, \mathbb{P}(B \cap C) = 0.20$ and $\mathbb{P}(A \cap B \cap C) = 0.15$. Find $\mathbb{P}(A^c \cap B^c \cap C^c)$.

   ANS: 0.41

3. Suppose that the events $A_1, A_2, \cdots, A_6$ are mutually exclusive and exhaustive. If they satisfy

$$\mathbb{P}(A_1) = \frac{1}{2}\mathbb{P}(A_2) = \frac{1}{3}\mathbb{P}(A_3) = \cdots = \frac{1}{6}\mathbb{P}(A_6),$$

   then what is $\mathbb{P}(A_4)$?

   ANS: $\frac{4}{21}$

4. The World Series in baseball continues until either the American League team or the National League team wins four games. How many different orders are possible (e.g., ANNAAA means the American League team wins in six games) if the series goes

   (a) Four games?

   (b) Five games?

   (c) Six games?

   (d) Seven games?

   ANS: (a):2, (b):8, (c):20, (d):40

5. Consider an urn containing 12 balls, of which 5 are green and 7 are red. A sample of size 9 is to be drawn at random. Let A be the event that first 3 balls are green, and B the event that exactly 4 green balls are drawn.

   (a) If 9 balls are drawn **with replacement**, what are $\mathbb{P}(B)$ and $\mathbb{P}(A \cap B)$?

   ANS: $\mathbb{P}(B) = 126(5/12)^4(7/12)^5$, $\mathbb{P}(A \cap B) = 6(5/12)^4(7/12)^5$

   (b) If 9 balls are drawn **without replacement**, what are $\mathbb{P}(A)$, $\mathbb{P}(B)$, and $\mathbb{P}(B|A)$?

   ANS: $\mathbb{P}(A) = \frac{1}{22}$, $\mathbb{P}(B) = \frac{21}{44}$, and $\mathbb{P}(B|A) = \frac{1}{2}$

6. A single card is drawn at random from each of six well-shuffled decks of playing cards. Let A be the event that all six cards drawn are different. Find $\mathbb{P}(A)$. What is the probability that at least two of the drawn cards match?

   ANS: $\mathbb{P}(A) = 52!/(52^6 \cdot 46!)$, $1 - \mathbb{P}(A)$

7. If $\mathbb{P}(A) = 0.4, \mathbb{P}(B) = 0.5, \mathbb{P}(B \mid A) = 0.75$, find $\mathbb{P}(A \mid B)$ and $\mathbb{P}(A \mid B^c)$.

   ANS: $\mathbb{P}(A \mid B) = 0.6$, $\mathbb{P}(A \mid B^c) = 0.2$

8. Seventy percent of the light aircraft that disappear while in flight in Neverland are subsequently discovered. Of the aircraft that are discovered, 60% have an emergency locator, whereas 90% of the aircraft not discovered do not have such a locator. Suppose a light aircraft that has just disappeared has an emergency locator. What is the probability that it will not be discovered?

   ANS: $\frac{1}{15}$

9. Let $\mathbb{P}(A) = 0.3$ and $\mathbb{P}(B) = 0.6$.
   (a) Find $\mathbb{P}(A \cup B)$ when A and B are independent.

   ANS: 0.72

   (b) Find $\mathbb{P}(A|B)$ when A and B are mutually exclusive.

   ANS: 0

10. An urn contains two red balls and four white balls. Sample successively five times at random and with replacement, so that the trials are independent. Compute the probability that no two balls drawn consecutively have the same color. What if the sampling is without replacement?

    ANS: 12/243, 1/15

11. Let A and B be two events such that $\mathbb{P}(A) = 1/2$, $\mathbb{P}(B) = 1/4$, and $\mathbb{P}(A \cup B) = 7/12$. Is A and B independent?

    ANS: No

12. During the first week of the semester, 80% of customers at a local convenience store bought either beer or potato chips (or both). 60% bought potato chips. 30% of the customers bought both beer and potato chips. Are events a randomly selected customer bought potato chips and a randomly selected customer bought beer independent?

    ANS: Yes

13. Two processes of a company produce rolls of materials: The rolls of Process I are 3% defective and the rolls of Process II are 1% defective. Process I produces 60% of the company's output, Process II 40%. A roll is selected at random from the total output. Given that this roll is defective, what is the conditional probability that it is from Process I?

    ANS: $\frac{9}{11}$

14. In a particular sports event, it is known that 2% of the athletes consume a certain kind of drug. A blood test for detecting this drug is 96% effective, meaning that if the drug is present in the blood of the athlete, the test will return a positive result 96% of the time. However, the test also yields a false positive result (it gives a positive result when the drug is not present) for 1% of the drug-free persons tested. What is the probability that for a randomly selected athlete, the test is positive? What is the probability that a randomly selected athlete consumed the drug, given that the test is positive?

    ANS: 0.029, 192/290

15. Consider an urn containing 12 balls, of which 5 are green and 7 are red. A sample of size 9 is to be drawn at random. Let A be the event that first 3 balls are green, and B the event that exactly 4 green balls are drawn. If 9 balls are drawn **without replacement**, what is $\mathbb{P}(A|B)$?

    ANS: 1/21

16. A hat contains 10 coins, where 9 are fair but one is double-headed (always landing Heads). A coin is chosen uniformly at random. The chosen coin is flipped 4 times. Find the probability that the chosen coin is double-headed given that it lands Heads all 4 times.

    ANS: 16/25

Chapter 2

1. Let X be a random variable with PMF

$$f(x) = \frac{1}{x+1} - \frac{1}{x+2}, \qquad x = 0, 1, 2, \cdots.$$

   Find $\mathbb{P}(X \geq 4 | X \geq 1)$.

   ANS: 2/5

2. Let X be an integer-valued random variable with probability mass function given by $f(x) = \frac{C}{3^x}$ for $x = 2, 3, 4, \cdots$. Find C. Find the probability that X takes an odd value.

   ANS: $C = 6$, the probability is 1/4

3. Suppose you roll two 5 faced dice, with faces labeled 1,2,3,4,5, and each equally likely to appear on top. Let X denote the smaller of the two numbers that appear. If both dice show the same number, then X is equal to that common number. Find the PMF of X. Compute $\mathbb{P}(X \le 3|X > 1)$ and $\mathbb{E}[5X-2]$.

   ANS: $f(x) = (11-2x)/25$ for $x = 1, 2, 3, 4, 5$, $\mathbb{P}(X \le 3|X > 1) = 3//4$, and $\mathbb{E}[5X-2] = 9$.

4. Let X be a random variable with PMF $f(x) = \frac{1}{7}$ for $x = -3, -2, -1, 0, 1, 2, 3$. Define $Y = |X|$. Find the PMF of Y and $\mathbb{E}[Y]$.

   ANS: $f_Y(0) = 1/7, f_Y(y) = 2/7$ for $y = 1, 2, 3$. $\mathbb{E}[Y] = 12/7$.

5. Suppose that a school has 20 classes: 16 with 25 students in each, three with 100 students in each, and one with 300 students, for a total of 1000 students.

   (a) Select a class at random out of 20 classes. Define a random variable X by the size of the randomly chosen class. Find the PMF and the expectation of X.

   ANS: $f_X(25) = 16/20, f_X(100) = 3/20, f_X(300) = 1/20$, $\mathbb{E}[X] = 50$.

   (b) Select a student randomly out of the 1000 students. Let the random variable Y equal the size of the class to which this student belongs. Find the PMF and the expectation of Y.

   ANS: $f_Y(25) = 2/5, f_Y(100) = 3/10, f_Y(300) = 3/10$, $\mathbb{E}[Y] = 130$.

6. Let X equal the number of flips of a fair coin that are required to observe the same face on consecutive flips.

   (a) Find the PMF and MGF of X.

   ANS: $f_X(x) = (1/2)^{x-1}$ for $x = 2, 3, \cdots$. $M_X(t) = \frac{e^{2t}}{2-e^t}$

   (b) Use the MGF to find the expectation and the variance of X.

   ANS: $\mathbb{E}[X] = 3, \mathrm{Var}(X) = 2$

   (c) Find the $\mathbb{P}(X \le 3)$ and $\mathbb{P}(X \ge 5)$.

   ANS: 3/4, 1/8

7. Let X equal the larger outcome when two fair four-sided dice are rolled. Find the mean, variance, and standard deviation of X.

   ANS: 50/16, 55/64, $\sqrt{55}/8$

8. Let X be a random variable with moment generating function given by $M(t) = c(1-2t)^{-5}$ for $t < \frac{1}{2}$ for some constant c. Find the constant c, the expectation of X, and the variance of X.

   ANS: $c = 1$, $\mathbb{E}[X] = 10$, $\mathrm{Var}(X) = 20$

9. It is claimed that 15% of the ducks in a particular region have patent schistosome infection. Suppose that seven ducks are selected at random. Let X equal the number of ducks that are infected. Assuming independence, how is Xdistributed? Find $\mathbb{P}(X > 2)$, $\mathbb{P}(X = 1)$, and $\mathbb{P}(X \le 3)$.

   ANS: $X \sim \mathrm{Bin}(7, 0.15)$, $\mathbb{P}(X > 2) = 0.0738$, $\mathbb{P}(X = 1) = 0.3960$, and $\mathbb{P}(X \le 3) = 0.9879$

10. An urn contains seven red and 1 white balls. Draw one ball at random from the urn. Let $X = 1$ if a red ball is drawn and let $X = -1$ if a white ball is drawn. Find the PMF, expectation, and variance of X.

    ANS: $f_X(1) = 7/8, f_X(-1) = 1/8, \mathbb{E}[X] = 3/4, \mathrm{Var}(X) = 7/16$

11. A certain basketball player knows that on average he will successfully make 78% of his free throw attempts. Assuming all throw attempts are independent. What are the expectation and the variance of the number of successful throws in 1020 attempts.

    ANS: 795.6, 175.032

12. An airline overbooks a flight, selling more tickets for the flight than there are seats on the plane (figuring that it's likely that some people won't show up). The plane has 100 seats, and 110 people have booked the flight. Each person will show up for the flight with probability 0.9, independently. Let X be the number of people showing up. Is X a binomial random variable? Justify your answer. If so, find the parameters. Find the approximate probability that exactly 100 passengers are shown up.

    ANS: It is a Binomial. $e^{-11} \cdot 11^{10}/10!$

13. When a customer buys a product at a supermarket, there is a concern about the item being underweight. Suppose there are 20 "one-pound" packages of frozen ground turkey on display and three of them are underweight. A consumer group buys five of the 20 packages at random. What is the probability that at least one

of the five is underweight?

ANS: 137/228

14. A particular brand of candy-coated chocolate comes in six different colors. Suppose 30% of all pieces are brown, 20% are blue, 15% are red, 15% are yellow, 10% are green, and 10% are orange. Thirty pieces are selected at random. What is the probability that 10 are brown, 8 are blue, 7 are red, 3 are yellow, 2 are green, and none are orange?

ANS: $\frac{30!}{10!8!7!2!3!}(0.3)^{10}(0.2)^{8}(0.15)^{10}(0.1)^{2}$

15. An urn contains 4 white and 5 black balls. We randomly choose 4 balls. If 2 of them are white and 2 are black, we stop. If not, we replace the balls in the urn and again randomly select 4 balls. This continues until exactly 2 of the 4 chosen are white. What is the probability that we shall make exactly 10 selections? What is the probability that we select 4 balls more than 5 times?

ANS: $(11/21)^9(10/21), (11/21)^5$

16. Suppose that a basketball player can make a free throw 60% of the time.

    (a) Let X equal the minimum number of free throws that this player must attempt to make one shot. Find $\mathbb{P}(X = 3)$, $\mathbb{E}[X]$, and $\text{Var}(X)$.

    ANS: $\mathbb{P}(X = 3) = (0.4)^2(0.6)$, $\mathbb{E}[X] = 5/3$, and $\text{Var}(X) = 10/9$

    (b) Let Y equal the minimum number of free throws that this player must attempt to make a total of five shots. Find $\mathbb{P}(Y = 10)$.

    ANS: $\mathbb{P}(Y = 10) = 126(0.4)^5(0.6)^5$

17. One of four different prizes was randomly put into each box of a cereal. If a family decided to buy this cereal until it obtained at least one of each of the four different prizes, what is the expected number of boxes of cereal that must be purchased?

ANS: 25/3

18. In a weekly lottery you have probability .05 of winning a prize with a single ticket. Suppose you buy 1 ticket per week for 20 weeks and let X be the number of winning tickets.

    (a) Find $\mathbb{E}[X]$.

    ANS: 1

    (b) What is the probability that among the twenty tickets you buy more than 3 winning tickets?

    ANS: 0.0159

    (c) Using a Poisson approximation, write down an expression for the probability that among the twenty tickets you buy more than 3 winning tickets.

    ANS: 0.019