4/25/22

Chebyshev Inequality

<u>Markov Inequality</u> : If X is a nonnegative RV, then $\forall a > 0$

$$P(X \geq a) \leq \frac{\mathbb{E}X}{a}$$

<u>Proof</u>
$$\mathbb{E}[X] = \mathbb{E}[X \cdot \mathbb{1}_{\{X \geq a\}}] + \underbrace{\mathbb{E}[X \cdot \mathbb{1}_{\{X < a\}}]}_{\geq 0}$$
$$\geq a \cdot \mathbb{E}[\mathbb{1}_{\{X \geq a\}}] = a\,\mathbb{P}(X \geq a). \qquad \blacksquare$$

<u>Chebyshev Ineq</u> : If X is a RV w/ mean μ, variance σ^2,

then $\forall a > 0$, $$P(|X-\mu| \geq a) \leq \frac{\sigma^2}{a^2}.$$

<u>Proof</u> Let $Y = |X-\mu|^2$ then

$$P(|X-\mu| \geq a) = \mathbb{P}(Y \geq a^2) \leq \frac{\mathbb{E}[Y]}{a^2} = \frac{Var(X)}{a^2}. \qquad \blacksquare$$

<u>Example</u> X = # of items produced in a factory during a week. with $\mathbb{E}X = 50$.

- $\mathbb{P}(X > 75) \leq \dfrac{\mathbb{E}X}{75} = \dfrac{2}{3}$.

- If $\operatorname{Var} X = 25$,

$$P(40 \leq X \leq 60) = P(|X - 50| \leq 10) \leq \frac{25}{100} = \frac{1}{4}.$$

<u>One-sided Chebyshev</u>: If X is a RV w/ $\mathbb{E}X = 0$, $\operatorname{Var}(X) = \sigma^2$ then $\forall a > 0$, $\mathbb{P}(X \geq a) \leq \dfrac{\sigma^2}{\sigma^2 + a^2}$.

<u>Proof</u>

$$\mathbb{P}(X \geq a) = \mathbb{P}(X + t \geq a + t) \leq \mathbb{P}((X+t)^2 \geq (a+t)^2)$$

$$\leq \frac{\mathbb{E}[(X+t)^2]}{(a+t)^2} = \underline{\frac{\sigma^2 + t^2}{(a+t)^2}} =: f(t)$$

Optimize in t.

$$f'(t) = \frac{1}{(a+t)^4} \cdot \left(2t(a+t)^2 - 2(\sigma^2 + t^2)(a+t) \right) = 0$$

$$\Rightarrow \quad t(a+t) = \sigma^2 + t^2 \quad \Rightarrow \quad t = \frac{\sigma^2}{a}.$$

$$f\left(\frac{\sigma^2}{a}\right) = \frac{\sigma^2 + \sigma^4/a^2}{\left(a + \frac{\sigma^2}{a}\right)^2} = \frac{\sigma^2}{a} \cdot \frac{\left(a + \frac{\sigma^2}{a}\right)}{\left(\quad\right)^2} = \frac{\sigma^2/a}{a + \sigma^2/a} = \frac{\sigma^2}{a^2 + \sigma^2} \qquad \blacksquare$$

In general, if $a - \mathbb{E}X > 0$ then

$$\mathbb{P}(X > a) = \mathbb{P}(X - \mathbb{E}X > a - \mathbb{E}X)$$

$$\leq \frac{\operatorname{Var}(X)}{\operatorname{Var}(X) + (a - \mathbb{E}X)^2}.$$

<u>Example (revisit)</u> X with $\mu = 50$, $\sigma^2 = 25$

$$P(X \geq 75) \leq \mathbb{P}(|X - \mu| \geq 25) \leq \frac{\sigma^2}{25^2} = \frac{1}{25}.$$

$$\text{''} \qquad \leq \frac{25}{25 + (75 - 50)^2} = \frac{1}{26}.$$

Better Estimate!